

# Universal correlations of trapped one-dimensional impenetrable bosons

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## Abstract

We calculate the asymptotic behaviour of the one-body density matrix of one-dimensional impenetrable bosons in finite size geometries. Our approach is based on a modification of the replica method from the theory of disordered systems. We obtain explicit expressions for oscillating terms, similar to fermionic Friedel oscillations. These terms are universal and originate from the strong short-range correlations between bosons in one dimension.

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## 1. Introduction

Recently, one-dimensional (1D) Bose gases have been created in long cylindrical traps by tightly confining the transverse motion of particles to zero-point oscillations [1]. These experiments revived an interest in exactly solvable one-dimensional models of statistical physics, in particular in the Lieb–Liniger model [2] for one-dimensional bosons interacting via a delta-function potential. In this case, the Bethe ansatz solution accounts for the ground state properties, spectrum of elementary excitations [3] and thermodynamics [4]. In contrast to these properties, following from the solution of relatively simple integral equations, the correlation functions are not easily obtained from the Bethe ansatz due to the extremely complicated form of the wavefunctions. In the limiting cases of weak and strong interactions, closed analytical results can be found as perturbative expansions [5, 7]. In a recent series of papers [6], progress has been made in calculating the correlation properties of finite spin chains, closely related to the one-dimensional bosons.

Another drawback of the Bethe ansatz is the requirement for periodic boundary conditions and thermodynamic limit. This precludes the study of finite size effects, except for a number of cases in which a special symmetry of the confining potential [8] is available. In most cases [9, 10], the finite geometry was treated relying on the local density approximation, where the macroscopic length scale induced by the confining potential is assumed to be well separated from the microscopic correlation length emerging from the Lieb–Liniger solution.

A special case that allows one to go beyond the local density approximation is the limiting case of an infinite coupling constant. Then the interactions are taken into account by mapping the system onto free fermions as shown by Girardeau [11]. The resulting system of impenetrable bosons, or ‘Tonks gas’, shares many of its properties with free fermions. In the present stage, achieving this strong-coupling Tonks–Girardeau regime is one of the main experimental goals [12–14]. In this respect, it is desirable to have results for finite systems beyond the local density approximation and to compare them directly with experimental data.

From a theoretical point of view, the fermionic mapping leads to a considerable simplification of the general Bethe ansatz expression for the wavefunctions. Historically, even before the Bethe ansatz solution became available, Girardeau [11] was the first to note the boson–fermion correspondence. He found a simple expression for the ground state wavefunction of  $N$  bosons in arbitrary potential  $V(x)$  in the form of an absolute value of the fermionic Slater determinant:

$$\Phi(x_1, \dots, x_N) = |\det_{k,l}[\varphi_k(x_l)]|, \quad (1)$$

where  $\varphi_k(x)$  are one particle wavefunctions in the potential  $V(x)$ .

The equivalence of impenetrable bosons and free fermions can be stated as the equivalence of correlation functions. Indeed, any correlation function of the density is given by the corresponding expression for fermions, since it is diagonal in field operators and does not involve phase correlations. The same holds for the ground state energy and the spectrum of elementary excitations. However, off-diagonal correlation functions of impenetrable bosons are different from those of fermions, due to the presence of the absolute value in (1). This drastically changes phase correlations. The simplest and well-studied example of such off-diagonal correlation functions is the one-body density matrix

$$g_1(t, t') = N \int dx_2 \dots dx_N \Phi^*(t, x_2, \dots, x_N) \Phi(t', x_2, \dots, x_N). \quad (2)$$

This quantity is of major importance for bosonic systems, since the eigenvalues of  $g_1$  show the presence or absence of Bose–Einstein condensation according to the criterion of Penrose and Onsager [15]. In the translationally invariant case,  $g_1$  depends only on the relative distance  $x = t - t'$  and its Fourier transform with respect to  $x$  is the momentum distribution of particles in the ground state. In this case, the condensation would manifest itself as a macroscopic occupation of the zero momentum state.

The problem of calculating the one-body density matrix, or equivalently the momentum distribution of impenetrable bosons, has a long history in mathematical physics. First considered in 1963 by Schultz [16], the one-body density matrix was found in the form of a determinant with special properties, that is a Toeplitz determinant [17]. Using the known asymptotics of the Toeplitz determinants it was possible to prove the absence of Bose–Einstein condensation by showing the power-law decay of the one-body density matrix at large distance. The precise form of this power law was obtained later by Lenard in [18]. His calculation resulted in the following long distance behaviour:

$$\frac{\bar{g}_1(x)}{n} = \frac{\rho_\infty}{|k_F x|^{1/2}}, \quad (3)$$

where  $n = k_F/\pi$  is the density of particles. In their *tour de force*, Vaidya and Tracy [19] calculated from first principles the asymptotic long-distance behaviour of the one-body density matrix (the short distance behaviour has also been calculated) and found the expression in (3) as the leading term with  $\rho_\infty = \pi e^{1/2} 2^{-1/3} A^{-6} = 0.92418\dots$  where  $A = \exp(1/12 - \zeta'(-1)) = 1.2824\dots$  is Glaisher's constant, related to the Riemann zeta function  $\zeta(z)$ . They also succeeded in obtaining the sub-leading terms. This work has been extended to higher order terms by Jimbo *et al* [20] who related  $g_1(x)$  to the solution of a certain nonlinear differential equation, the Painlevé equation of the fifth kind. The general structure of the large-distance expansion of the one-body density matrix consists of trigonometric functions, cosines or sines of multiples of  $2k_F x$ ; each such trigonometric term is multiplied by series of even or odd powers of  $1/k_F x$ , respectively. Using a hydrodynamic approach, Haldane [21] has shown that this structure is a general property of any one-dimensional compressible liquid, bosonic or fermionic. This method is unable to predict exact coefficients of the  $2k_F x$  harmonics, which should be calculated using exact methods. For example, according to [19, 20] the first oscillatory correction to equation (3) is given by

$$\frac{g_1(x)}{\bar{g}_1(x)} - 1 = +\frac{1}{8} \frac{\cos 2k_F x}{(k_F x)^2}. \quad (4)$$

Together with  $\rho_\infty$ , the coefficients of sub-leading oscillatory terms provide full information on the long distance asymptotics of the one-body density matrix. It is then desirable to have these coefficients in a simple analytical form or to be able to calculate them by a perturbation theory.

Experimental conditions for obtaining the Tonks–Girardeau regime require a small number of particles, which in an isolated system can be as small as  $N \sim 100$  atoms [13]. It is then important to extend the results of equations (3) and (4) to finite size geometries. The case of harmonic confinement is directly related to experiments. It was recently studied analytically in [10] by using Haldane's hydrodynamical approach and, independently, in [22] by the Coulomb gas analogy (for numerical results, see the work [23] and references therein). The expression for the leading term in the one-body density matrix was obtained, generalizing the expression in equation (3) to a non-uniform density profile. The case of a circular geometry has been considered by Lenard [24] who conjectured the main smooth contribution, analogous to (3) in the form:

$$\bar{g}_1(\alpha) = \frac{N\rho_\infty}{|N \sin \pi \alpha|^{1/2}}, \quad (5)$$

where  $2\pi\alpha$  is the angle between two points on the circle. This result was justified rigorously by Widom [25] using the theory of Toeplitz determinants. It is worth mentioning here that this result follows straightforwardly from the conformal field theory (see e.g. [26]), which to some extent is equivalent to the hydrodynamic [21] and Coulomb gas [22, 27] approaches. These methods are capable of predicting long-wavelength, large scale behaviour of correlation functions, while giving only qualitative answers for details on a scale of mean inter-particle distance. In both geometries, harmonic and circular, the finite size corrections, analogous to the expression in equation (4), remain unknown.

Here we present the first calculation of finite size corrections to the one-body density matrix for both harmonic confinement and circular geometry. We compare our analytical expressions with numerical calculations based on the exact representation of the one-body density matrix as a Toeplitz determinant and find excellent agreement. We also take on the task of reproducing Vaidya and Tracy asymptotic expansion in the thermodynamic limit using both systems as a starting point and increasing the number of particles and the system size in such a way as to preserve a constant density. Our calculations are the first to resolve

analytically the sign ambiguity of the oscillating terms, similar to that in equation (4) which appears in the studies of the Painlevé representation for the one-body density matrix [20, 28]. We find that the signs of all oscillating terms, such as (4) in the Vaidya and Tracy expansion [19], should be reversed<sup>1</sup>.

Our calculations are based on a modification of the replica method developed in the theory of disordered systems [29] and applied recently to random matrices [30–32] and Calogero–Sutherland models [33]. Recent progress [34, 35] in calculations based on the replica method has elucidated its intimate relation to the theory of nonlinear integrable systems. The present work gives yet another example of this interconnection.

The paper is organized as follows. In section 2, we present our method and illustrate it by calculating the amplitude of removing a particle from the ground state of harmonically trapped impenetrable bosons. The one-body density matrix for harmonically confined bosons is calculated in section 3. Section 4 describes the calculation of the one-body density matrix in circular geometry. We calculate sub-leading terms using perturbation theory in section 5. The conclusions are presented in section 6 and mathematical details are given in the two appendices.

## 2. Description of the method

The task of reproducing and extending the original calculations of Vaidya and Tracy to finite systems is obscured by the technical complexity of their method which consist in the asymptotic expansion of a Toeplitz determinant assuming the size of the determinant tending to infinity in order to reproduce the continuous limit. It is therefore important to have an alternative way of representing the one-body density matrix. One starts with the representation of the one-body density matrix as an  $N$ -dimensional integral, first proposed by Lenard [18]:

$$g_1(\alpha) = \frac{1}{N!} \int_0^1 d^N \theta |\Delta_N(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_N})|^2 \prod_{l=1}^N |1 - e^{2\pi i \theta_l}| |e^{2\pi i \alpha} - e^{2\pi i \theta_l}|$$

$$\equiv \left\langle \prod_{l=1}^N |1 - z_l| |e^{2\pi i \alpha} - z_l| \right\rangle \quad (6)$$

which follows immediately from definition (2). We have chosen here a circular geometry of  $N + 1$  particles described by cyclic coordinates  $0 < 2\pi \theta_l < 2\pi$  with periodic boundary conditions. The ground state wavefunction of  $N + 1$  particles is given by the absolute value of the Slater determinant (1) composed of plane waves  $\varphi_{k+1}(z_l) = \exp(2\pi i k \theta_l) = z_l^k$ . It is then identified with the absolute value of the Vandermonde determinant

$$\det_{1 \leq k, l \leq N+1} [\varphi_k(z_l)] = \Delta_{N+1}(z) = \Delta(z_1, \dots, z_{N+1}) = \prod_{1 \leq j < k \leq N+1} (z_i - z_j). \quad (7)$$

The above expression is factorized straightforwardly to yield the expression being integrated in (6). Consider now a positive integer  $n$  and correlation function

$$Z_{2n}(\alpha) = \left\langle \prod_{l=1}^N (1 - z_l)^{2n} (e^{2\pi i \alpha} - z_l)^{2n} \right\rangle. \quad (8)$$

Our method is based on the fact that the one-body density matrix (6) can be obtained from  $Z_{2n}$  by suitable analytical continuation to  $n = 1/2$ . It happens that  $Z_{2n}$  can be

<sup>1</sup> A similar observation has been communicated to the author by V Dunjko, who matched numerically short-distance and long-distance asymptotics of the one-body density matrix.

evaluated straightforwardly in the asymptotic large  $N$  limit. In this respect, expression (8) supplemented with the proper procedure for analytic continuation  $n \rightarrow 1/2$  is the desired alternative representation of the one-body density matrix.

The idea of calculating the averages of the absolute value of a non-positive definite function was put forward by Kurchan [36] and was named a modification of the replica trick. In the present context replica means the following. In [30–33], the correlation function (8) was expressed using a dual representation involving an integral over components of a  $n$ -dimensional field, with the action symmetric under rotations in the space of the components. This zero-dimensional field theory was considered in the  $n \rightarrow 0$  limit to obtain the density–density correlation function. In the present work, a different limit  $n \rightarrow 1/2$  is taken to reproduce the off-diagonal correlation function (6). However, the idea of the analytical continuation in  $n$  is common to the above-mentioned works and we use heavily the techniques introduced in [31]. For example, we use the same dual representation of (8) and evaluate it in the asymptotic large  $N$  limit.

For  $n$  integer, the result consists of a main smooth contribution and exactly  $n$  oscillatory corrections. This is the expected structure of one-dimensional correlation functions conjectured by Haldane [21] from his hydrodynamic approach. The sensible analytic continuation  $n \rightarrow 1/2$  is then performed in such a way as to preserve this structure and the resulting expression is believed to represent the large  $N$  limit of the (unknown) analytical continuation in  $n$ . The lack of rigour in this approach is shared by most replica calculations and is justified *a posteriori* by remarkably transparent resulting expressions which are in full agreement with the numerics as well as the known analytical results.

To demonstrate the method in detail and set up notation, we first calculate the ground state amplitude  $A(t)$  of impenetrable bosons in a harmonic potential:

$$A(t) = {}_N\langle \Psi(t) \rangle_{N+1}. \quad (9)$$

This quantity describes the probability amplitude to remove a particle at position  $t$  by acting with annihilation operator  $\Psi(t)$  from the ground state of  $N + 1$  particles and leave the system in the ground state of  $N$  particles. It can be considered as a many-body wavefunction of the removed particle. We consider a geometry different from circular, since for the latter the ground state amplitude is just the square root of mean density independent of the position due to the translational invariance. We consider the system confined by harmonic potential  $V(x) = m\omega^2 x^2/2$  for which  $A(t)$  has a non-trivial position dependence. The one particle orbitals are given by eigenfunctions of the harmonic oscillator:

$$\phi_m(x) = \frac{1}{\sqrt{c_m}} H_m \left( x \sqrt{\frac{N}{2}} \right) e^{-\frac{N}{4} x^2}, \quad c_m = \left( \frac{2\pi}{N} \right)^{\frac{1}{2}} 2^m m!, \quad (10)$$

and  $H_m$  are Hermite polynomials. We measure the distances in units of half of the Thomas–Fermi radius  $R = \sqrt{2\hbar N/m\omega}$ , corresponding to half the size of the particle cloud in the large  $N$  limit. The ground state wavefunction is obtained similarly to the uniform case taking the absolute value of the fermionic Slater determinant:

$$\Phi(x_1, \dots, x_N) = |\det_{l,m}[\phi_{m-1}(x_l)]| = \frac{1}{\sqrt{S_N(N)}} |\Delta_N(x)| e^{-\frac{N}{4} \sum x_j^2}. \quad (11)$$

The last identity follows from the linearity of the determinant with respect to its columns, which enables us to write the determinant of Hermite polynomials  $H_m(x)$  as a determinant of their leading monomials  $H_m(x) \sim x^m$ . The normalization constant is given by the Selberg integral [37, 38] of Hermite type

$$S_N(\lambda) = \int_{-\infty}^{\infty} d^N x \Delta_N^2(x) e^{-\frac{\lambda}{2} \sum x_j^2} = \lambda^{-N^2/2} (2\pi)^{N/2} \prod_{j=1}^N \Gamma(1+j). \quad (12)$$

Using definition (9) of the ground state amplitude leads to the expression

$$A(t) = \frac{e^{-\frac{N}{4}t^2}}{\sqrt{S_N(N)S_{N+1}(N)}} \int_{-\infty}^{\infty} d^N y \Delta_N^2(y) e^{-\frac{N}{2} \sum_{j=1}^N y_j^2} \prod_{j=1}^N |t - y_j|. \quad (13)$$

To deal with absolute value in this expression we use the identity

$$|t - y_j|^{2n} = (t - y_j)^{2n}, \quad (14)$$

valid for integer  $n$ . Performing the analytical continuation  $n \rightarrow 1/2$  at the end of the calculations, one recovers expression (13). To treat the ground state amplitude and one-body density matrix on the same footing it is convenient to consider a more general quantity

$$Z_m(t_1, \dots, t_m) = \frac{1}{S_N(N)} \int_{-\infty}^{\infty} d^N y \Delta_N^2(y) e^{-\frac{N}{2} \sum_{j=1}^N y_j^2} \prod_{j=1}^N \prod_{a=1}^m (t_a - y_j). \quad (15)$$

The variables  $t_a$ , different in general, are later taken equal to a single value  $t$ ,

$$Z(t) = \lim_{n \rightarrow 1/2} Z_{2n}(t) \equiv \lim_{n \rightarrow 1/2} Z_{2n}(\underbrace{t, \dots, t}_{2n}) \quad (16)$$

to recover equation (13) up to a normalization:

$$A(t) = \sqrt{\frac{S_N(N)}{S_{N+1}(N)}} e^{-\frac{N}{4}t^2} Z(t). \quad (17)$$

It is crucial that one cannot set  $2n = 1$  directly in (15), which means that  $Z(t)$  in (16) is different from  $Z_1(t)$ . The latter is nothing but a fermionic ground state amplitude obtained by removing the absolute value in (13). It is given by the wavefunction (10) of the  $(N + 1)$ th particle removed from the system and results in an expression that oscillates rapidly around zero with period equal to the mean inter-particle separation. In contrast, the bosonic ground state amplitude, obtained in the large  $N$  limit, is expected to contain a smooth positive leading term which is a direct consequence of positivity of the ground state wavefunction (11). Our claim is that it can be obtained from  $Z(t)$ .

Recently the leading smooth contribution to  $Z_{2n}(t)$  for integer  $n$  was evaluated in the large  $N$  limit by Brézin and Hikami [39] and it was shown to survive the analytical continuation in  $n$  off the integers. This suggests that we deal with two functions of variable  $n$ , one for fermions and the other for bosons, or equivalently one function with two branches, a fermionic one and a bosonic one which coincide at integer  $n$  but become different as  $n$  is moved away from integers. The goal of the present calculation is the generalization of the bosonic analytic continuation considered in [39] to the whole asymptotic expression for the ground state amplitude.

To illustrate the ideas above we start with a remarkable duality transformation, which represents the  $N$ -fold integral  $Z_{2n}$  in equations (15), (16) as an integral over  $m = 2n$  variables:

$$Z_{2n}(t) = \frac{1}{S_{2n}(N)} \int_{-\infty}^{\infty} d^{2n} x \Delta_{2n}^2(x) \exp \left( -N \sum_a S(x_a, t) \right) \quad (18)$$

with an effective action

$$S(x, t) = \frac{(x - it)^2}{2} - \ln x + \frac{\pi i}{2}. \quad (19)$$

This representation is exact and its proof can be found in the mathematical literature (see [40] and references therein). For the case of harmonic confinement the proof was presented in [31] using the random matrix theory. This proof has the advantage of being readily extended to

deal with two-point correlation functions in a harmonic potential. In appendix A, we present yet another proof using second quantization for fermions.

The dual representation (18) provides us with an alternative representation of the  $N$ -fold integral (13) and is the starting point of the large  $N$  asymptotic expansion. The dominant contribution comes from the saddle points of the action (19), situated at

$$x_{\pm} = \frac{it}{2} \pm \sqrt{1 - \frac{t^2}{4}} = \pm e^{\pm i\phi}, \quad \sin \phi = \frac{t}{2} \quad (20)$$

for  $t$  inside the ground state density support  $-2 < t < 2$ . The stationary value of the action and its second derivative at these points read

$$S(x_{\pm}, t) \equiv S_{\pm} = \frac{e^{\mp 2i\phi}}{2} \mp i\phi \pm \frac{\pi i}{2}, \quad (21)$$

$$S''(x_{\pm}, t) \equiv S''_{\pm} = 2e^{\mp i\phi} \cos \phi. \quad (22)$$

To take into account all possible saddle points we put  $l$  variables  $x_a$  in the vicinity of  $x_-$  and  $2n - l$  variables in the vicinity of  $x_+$ , such that

$$\begin{aligned} x_a &= x_- + \xi_a / \sqrt{N}, & a &= 1, \dots, l \\ x_b &= x_+ + \xi_b / \sqrt{N}, & b &= l + 1, \dots, 2n. \end{aligned} \quad (23)$$

In what follows, the saddle points with  $l = 0$  or  $2n$  will be referred to as *replica symmetric* for obvious reasons. The other saddle points break the replica symmetry to some degree; the maximal degree of replica symmetry breaking happens for  $l = n$ . We shall see that the analytical continuation of the contribution of this saddle point leads to the result of [39]. On the other hand, in the calculation of the fermionic ground state amplitude, the replica symmetry is preserved, since in this case the contribution comes from the replica symmetric saddle points.

The pre-exponential factor in (18) given by the Vandermonde determinant vanishes at each saddle point (23) and should be expanded as

$$\Delta_{2n}^2(x) = \left( \frac{1}{\sqrt{N}} \right)^{l(l-1)+(2n-l)(2n-l-1)} (x_- - x_+)^{2l(2n-l)} \Delta_l^2(\xi_a) \Delta_{2n-l}^2(\xi_b) \quad (24)$$

to yield a non-zero result. Using this expansion the integration of the fluctuations near each saddle point is performed by using the Selberg integral (12). Combining the results of integration with the main saddle point contribution and summing over all saddle points we represent the original integral (18) as a sum of  $2n + 1$  terms

$$Z_{2n}(t) = \sum_{l=0}^{2n} F_{2n}^l N^{l(2n-l)} \frac{(x_+ - x_-)^{2l(2n-l)}}{(\sqrt{S''_-})^{l^2} (\sqrt{S''_+})^{(2n-l)^2}} e^{-NlS_- - N(2n-l)S_+}. \quad (25)$$

The factors

$$F_{2n}^l = \binom{2n}{l} \frac{\prod_{a=1}^l \Gamma(a+1) \prod_{b=1}^{2n-l} \Gamma(b+1)}{\prod_{c=1}^{2n} \Gamma(c+1)} = \prod_{a=1}^l \frac{\Gamma(a)}{\Gamma(2n+1-a)} \quad (26)$$

combine the numerical factors originating from the Selberg integral and the binomial coefficient equal to the number of ways to choose  $l$  variables  $x_a$  in the vicinity of  $x_-$  out of total  $2n$  variables.

As  $|S_+| = |S_-|$ , the  $N$  dependence of each term in (25) enters only through the factor  $N^{l(2n-l)}$ . Therefore, the central term of the sum (25) with  $l = n$  corresponding to the maximal degree of the replica symmetry breaking gives the dominant non-oscillating contribution to  $Z_{2n}(t)$ . The other terms, including the replica symmetric terms  $l = 0, 2n$ , are at least



$1/N$  times smaller and oscillate rapidly. This is to be contrasted with the replica approach in [31, 33] where in the limit  $n \rightarrow 0$  the dominant contribution comes from the replica symmetric points and the role of the saddle points with broken replica symmetry is to provide oscillatory corrections.

Observing the behaviour of (25) for integer  $n$  we are led to the conjecture that the right analytical continuation of the integer  $n$  following the bosonic branch would preserve this form: one smooth term plus oscillatory corrections. We now prepare expression (25) for analytical continuation. To this end, it is convenient to change the summation variable  $l = k + n$  and factorize the amplitude (26) as  $F_{2n}^l = A_n D_k^{(n)}$ , where the first factor is given by the product

$$A_n = \prod_{a=1}^n \frac{\Gamma(a)}{\Gamma(2n+1-a)}, \quad (27)$$

where  $n$  enters explicitly as the range of the product. The analytical continuation of this factor to non-integer values of  $n$  is described, for example in [22, 39]. We reproduce it in appendix B, where it is shown that  $\rho_\infty = A_{1/2}^2 / \sqrt{2}$ .

The analytic continuation of the second factor

$$D_k^{(n)} = \prod_{a=1}^k \frac{\Gamma(n+a)}{\Gamma(n+1-a)} = \frac{\Gamma(n+k)}{\Gamma(n+1-k)} D_{k-1}^{(n)}, \quad k > 0; \quad D_0^{(n)} = 1; \quad D_k^{(n)} = D_{-k}^{(n)} \quad (28)$$

to non-integer values of  $n$  is straightforward. For integer  $n$  the coefficients  $D_k^{(n)}$  vanish for  $|k| > n$  due to the divergence of the gamma function in the denominator, so for integer  $n$  the sum in (25) can be formally extended to  $-\infty < k < +\infty$ . For general values of  $n$ , the factor  $D_k^{(n)}$  has a non-zero value for any  $k$ , which results in genuine infinite series. These series are asymptotic, rather than convergent, but it can be shown by using the Stirling formula that the coefficients decrease very fast for small values of  $k$ . Hence, for large enough  $N$ , a few terms around  $k = 0$  provide an excellent approximation for the sum. We rewrite the series (25) using the new summation variable:

$$Z_{2n}(t) = A_n (2N \cos \phi)^{n^2} e^{-Nn \cos 2\phi} \sum_{k=-\infty}^{\infty} \frac{D_k^{(n)}}{(8N \cos^3 \phi)^{k^2}} e^{-iNk\Theta - 2ink\phi}, \quad (29)$$

where  $\Theta(t) = 2\phi + \sin 2\phi + \pi = 2\pi \int_{-2}^t \rho(s) ds$  is expressed as an integral of the mean density of particles  $\rho(t) = (1/\pi) \cos \phi$ , given by the celebrated Wigner semi-circle law:

$$\rho(t) = \frac{1}{\pi} \sqrt{1 - \frac{t^2}{4}}. \quad (30)$$

The result has the expected form consisting of a sum of smooth and oscillating parts:

$$Z_{2n}(t) = A_n (2\pi N)^{n^2} e^{-Nn(1-t^2/2)} \times [\rho(t)]^{n^2} \left[ 1 + 2 \sum_{k=1}^{\infty} \frac{D_k^{(n)}}{[8N\pi^3 \rho^3(t)]^{k^2}} \cos(kN\Theta(t) + 2nk\phi(t)) \right]. \quad (31)$$

This is the desired asymptotic representation of  $Z_n$ . For an integer  $n$  it has a simple general structure: a smooth main contribution plus oscillatory corrections. This structure is preserved if analytical continuation  $n \rightarrow 1/2$  is applied term by term. The validity of the change of summation variable  $l = n + k$  for a non-integer  $n$  is explained as follows: we assume that in fact the infinite asymptotic series (29) is the correct expression of the large  $N$  asymptotics for the analytical continuation of  $Z_{2n}(t)$  to arbitrary  $n$ . For integer  $n$  the sum terminates and can be rewritten as (25) after the corresponding change of the summation index.



Multiplying (31) by the normalization factors defined in (36) and using the asymptotic expansion

$$\frac{S_N(N)}{S_{N+1}(N)} = \frac{1}{\sqrt{2\pi}} \frac{N^{N+\frac{1}{2}}}{(N+1)!} \sim \frac{e^N}{2\pi N}, \quad (32)$$

we obtain the desired asymptotic expression for the ground state amplitude

$$A(t) = \sqrt{\rho_\infty} \left( \frac{\rho(t)}{\pi N} \right)^{1/4} \left[ 1 + 2 \sum_{k=1}^{\infty} \frac{D_k^{(1/2)}}{[8N\pi^3 \rho^3(t)]^{k^2}} \cos[kN\Theta(t) + k\phi(t)] \right]. \quad (33)$$

$$D_1^{(1/2)} = 1/2, \quad D_2^{(1/2)} = -3/8, \dots, \quad D_{k+1}^{(1/2)} = D_k^{(1/2)} \frac{\Gamma(k + \frac{1}{2})}{\Gamma(\frac{3}{2} - k)} \quad (34)$$

The explicit values of the coefficients of oscillatory terms are obtained from (28).

In recent papers [34, 35], the functions  $Z_m(t)$  were shown to satisfy a remarkable recursion relation, a Toda lattice hierarchy, extensively studied in the theory of nonlinear integrable systems [41]. The same Toda lattice equations relate the solutions of Painlevé equations, where  $m$  enters as a parameter not restricted to integer values [42]. This observation was crucial for *exact* evaluation of the replica limit  $m \rightarrow 0$  without relying on the large  $N$  asymptotics.

In the present framework, the quantity of interest is  $Z_m(t)$  for  $m \rightarrow 1$ , so according to [34, 42] it can be related straightforwardly to the solution of the Painlevé IV equation. The boundary conditions, obtained from the small  $t$  expansion of the ground state amplitude (13), distinguish between bosonic and fermionic branches of  $Z_1(t)$ . Our analytic continuation with broken replica symmetry chooses automatically the correct boundary conditions for bosons.

To justify numerically the result (34) we have calculated the ground state amplitude as a determinant

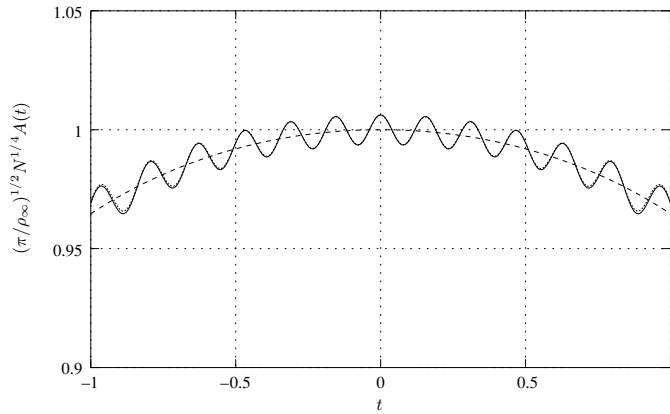
$$A(t) = \frac{N! e^{-Nt^2/4}}{\sqrt{S_N(N)S_{N+1}(N)}} \det_{1 \leq j, k \leq N} A_{j+k-2}(t), \quad (35)$$

where the matrix elements can be expressed by the gamma function and incomplete gamma function  $\gamma(\alpha, x)$  as

$$A_m(t) = \int_{-\infty}^{\infty} dy |t-y| y^m e^{-Ny^2/2} = \left( \frac{2}{N} \right)^{m/2+1} \left[ f\left(m+1, \frac{Nt^2}{2}\right) - \sqrt{\frac{Nt^2}{2}} f\left(m, \frac{Nt^2}{2}\right) \right]$$

$$f(m, x) = \frac{1 - (-1)^m}{2} \Gamma\left(\frac{m+1}{2}\right) - \gamma\left(\frac{m+1}{2}, \frac{Nt^2}{2}\right).$$

The results are presented in figure 1 for  $N = 20$  particles and  $-1 < t < 1$ . In the central part of the cloud, the ground state amplitude is indeed well approximated by a few terms in the expansion (34). Further improvement can be achieved using the perturbation theory around each saddle point similar to the perturbation theory described in section 5 in the case of circular geometry. Close to the edges of the atomic cloud, which in our units correspond to  $t = \pm 2$ , the saddle point approximation we used to derive (34) breaks down since for  $t$  approaching one of the edges the saddle points (20) coalesce at  $\pm i$  and the second derivative of the action (22) vanishes. In this case, analysis of the higher order expansion of action (22) is required and is beyond the scope of this work.



**Figure 1.** Comparison between asymptotic expression for the ground state amplitude (34) (solid line) of  $N = 20$  particles in harmonic potential evaluated up to the  $k = 1$  terms and exact results (dotted line) based on numerical evaluation of the determinant (35). The contribution given by the smooth term  $k = 0$  in (34) is represented by the dashed line. The ground state amplitude is normalized in such a way that the smooth part equals unity for  $t = 0$ .

Thus, we see that our method of calculation provides the exact large  $N$  asymptotic of the ground state amplitude. We extend it to the calculation of the one-body density matrix in the subsequent sections for the case of particles in harmonic potential and circular geometry.

### 3. One-body density matrix in harmonic potential

The one-body density matrix can be written using definition (2) together with expression (1) for the ground state function

$$g_1(t, t') = (N + 1) \frac{S_N(N)}{S_{N+1}(N)} e^{-\frac{N}{4}(t^2 + t'^2)} Z(t, t'), \quad (36)$$

where  $Z(t, t')$  is obtained from (15) according to the following rule:

$$Z(t, t') = \lim_{n \rightarrow 1/2} Z_{4n}(t, t') \equiv \lim_{n \rightarrow 1/2} Z_{4n}(\underbrace{t, \dots, t}_{2n}, \underbrace{t', \dots, t'}_{2n}). \quad (37)$$

It is again important to calculate  $Z_{4n}(t, t')$  in the large  $N$  limit before taking analytical continuation  $n \rightarrow 1/2$ , otherwise the result will be just the one-body density matrix for one-dimensional fermions.

The duality transformation can be worked out in the case of two variables as explained in appendix A and yields

$$Z_{4n}(t, t') = \frac{1}{S_{2n}^2(N)} \int_{-\infty}^{\infty} d^{2n}x d^{2n}x' \Delta_{2n}^2(x) \Delta_{2n}^2(x') \times \frac{\prod_{a,a'=1}^{2n} (x_a - x'_{a'})}{[i(t - t')]^{4n^2}} e^{-N \sum S(x_a, t)} e^{-N \sum S(x'_{a'}, t')}, \quad (38)$$

with the same definition (19) for the effective action,  $S(x, t)$ .

Before calculating the one-body density matrix by the saddle point method, we would like to remark that this correlation function depends on the scaling of the distance  $t - t'$  when  $N$  goes to infinity. We distinguish two limits: macroscopic, when  $t, t'$  are of order of the cloud

size and microscopic, or, more precisely, the mesoscopic limit, when  $t, t'$  remains finite on the scale of mean inter-particle distance  $1/N\rho(t)$  equal to  $\pi/N$  in the centre of the cloud. The second limit is called mesoscopic, since though  $t, t'$  are small on the scale of the cloud size, we are interested in the asymptotic behaviour of correlations  $t - t' \gg 1/N$ , so there is still a very large number of particles between  $t$  and  $t'$ .

In the macroscopic limit, the variables are changed in the following way:

$$\begin{aligned} x_a &= x_- + \xi_a/\sqrt{N}, & a &= 1, \dots, n+k \\ x_b &= x_+ + \xi_b/\sqrt{N}, & b &= n+k+1, \dots, 2n \\ x'_{a'} &= x'_+ + \xi'_{a'}/\sqrt{N}, & a' &= 1, \dots, n+k' \\ x'_{b'} &= x'_- + \xi'_{b'}/\sqrt{N}, & b' &= n+k'+1, \dots, 2n, \end{aligned} \quad (39)$$

where  $x_{\pm} = x_{\pm}(t)$  and  $x'_{\pm} = x_{\pm}(t')$  are defined in equation (20). The stationary value of the action and fluctuation integrals are done exactly as in the last section. The only new factor is the double product in the integral, calculated at the saddle point  $(k, k')$  with the result

$$\prod_{a=1}^{2n} \prod_{a'=1}^{2n} (x_a - x'_{a'}) = i^{4n^2} i^{2n(k+k')} |t - t'|^{n^2} \left[ \frac{\cos^2 \frac{\phi+\phi'}{2}}{\sin^2 \frac{\phi-\phi'}{2}} \right]^{kk'} e^{-2in(k\phi - k'\phi')}. \quad (40)$$

This factor represents the interaction between saddle points in  $x_a$  and  $x'_{b'}$  and as we shall see it is crucial for obtaining the correct expression for  $g_1$  in the mesoscopic limit. Combining this factor with each  $(k, k')$  saddle point contribution and multiplying by the number of ways to distribute  $x, x'$  among different stationary values, we get the integral (38) as a double sum

$$\begin{aligned} Z_{4n}(t, t') &= A_n^2 (2\pi N)^{2n^2} e^{-Nn(2-(t^2+t'^2)/2)} \frac{[\rho(t)\rho(t')]^{n^2}}{|t - t'|^{2n^2}} \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \left| \frac{\cos \frac{\phi+\phi'}{2}}{\sin \frac{\phi-\phi'}{2}} \right|^{2kk'} \\ &\times \frac{(-1)^{nk} D_k^{(n)}}{[8N\pi^3 \rho^3(t)]^{k^2}} e^{-iNk\Theta - 4ink\phi} \frac{(-1)^{nk'} D_{k'}^{(n)}}{[8N\pi^3 \rho^3(t)]^{k'^2}} e^{iNk'\Theta' + 4ink'\phi'}, \end{aligned} \quad (41)$$

where the summations are extended to infinity, relying on the factors  $D_k^{(n)}$  which cut off the finite number of terms in the sum.

We see immediately that again the most replica asymmetric saddle point  $(k, k') = 0$  in each set of variables  $x_a$  and  $x'_{a'}$  provides the dominant smooth contribution, which after analytic continuation  $n \rightarrow 1/2$  and normalization given in (36) yields the macroscopic one-body density matrix

$$\bar{g}_1(t, t') = \rho_{\infty} \left( \frac{N}{\pi} \right)^{\frac{1}{2}} \frac{[\rho(t)\rho(t')]^{\frac{1}{4}}}{|t - t'|^{\frac{1}{2}}}. \quad (42)$$

This result is identical to that of [22] and agrees completely with the functional form deduced in [10]. The finite size correction to (42) is obtained by taking  $n = 1/2$  in each term summed up in (41) with the result

$$\begin{aligned} \frac{g_1(t, t')}{\bar{g}_1(t, t')} - 1 &= \sum_{(k, k') \neq (0, 0)} (-1)^{(k+k')/2} D_k^{(1/2)} D_{k'}^{(1/2)} \\ &\times \left| \frac{\cos \frac{\phi+\phi'}{2}}{\sin \frac{\phi-\phi'}{2}} \right|^{2kk'} \frac{\exp(-iNk\Theta - 2ik\phi + iNk'\Theta' + 2ik'\phi')}{[8N\pi^3 \rho^3(t)]^{k^2} [8N\pi^3 \rho^3(t)]^{k'^2}}. \end{aligned} \quad (43)$$

Going to the mesoscopic limit, we focus on the centre of the potential  $t + t' = 0$  and define the scaling variable  $x$  such that  $t - t' = x/N$ . The factor in (41) becomes

$$\left| \frac{\cos \frac{\phi + \phi'}{2}}{\sin \frac{\phi - \phi'}{2}} \right|^{2kk'} \simeq \left( \frac{4N}{x} \right)^{2kk'}. \quad (44)$$

Adding powers of  $N$  in this expression with those in (41) we see that the diagonal elements  $k = k'$  in the sum are of order 1, while the off-diagonal elements are smaller by at least a factor  $1/N$ . Therefore, we anticipate that the mesoscopic limit is given by the diagonal part of the sum (41).

In principle, one should reconsider the asymptotic expansion (41) in the mesoscopic limit, since in this case the distance between saddle points is of order  $1/N$ , which can change the contribution of the fluctuations. These calculations can be done straightforwardly along the lines of [31]. Somewhat surprisingly, the conclusion is that the asymptotic limit  $N \rightarrow \infty$ , and the mesoscopic limit  $t - t' \rightarrow 0$  commute, so one can change the variables to  $x$  directly in the sum (41). Apart from the factor (44), the other factors simplify as  $\rho(t) = \rho(t') \rightarrow 1/\pi$ ,  $\Theta - \Theta' = 4\phi - 4\phi' = 2x/N$  and one gets

$$Z_{4n}(x) = A_n^2 e^{-Nn(2-(t^2+t'^2)/2)} \left( \frac{\sqrt{2}N}{|x|^{1/2}} \right)^{4n^2} \left[ 1 + 2 \sum_{k=1}^{\infty} (-1)^{2nk} [D_k^{(n)}]^2 \frac{\cos 2kx}{(2x)^{2k^2}} \right]. \quad (45)$$

Normalizing and putting  $n = 1/2$  one recovers the dominant term:

$$\bar{g}_1(x) = \left( \frac{N}{\pi} \right) \frac{\rho_{\infty}}{x^{1/2}} \quad (46)$$

which is identical to expression (3) of Vaidya and Tracy if one identifies  $k_F = \pi N \rho(0) = N$  and normalizes to the density  $N \rho(0) = N/\pi$  in the centre of the cloud. By the same procedure we obtain the oscillatory corrections

$$\frac{g_1(x)}{\bar{g}_1(x)} - 1 = 2 \sum_{k=1}^{\infty} (-1)^k [D_k^{(1/2)}]^2 \frac{\cos 2kx}{(2x)^{2k^2}}. \quad (47)$$

Using the explicit value  $D_1^{(1/2)} = 1/2$  we see that the  $k = 1$  term is identical to the leading term (4) of the Vaidya and Tracy expansion [19] apart from the sign. To our knowledge, it is the first analytical justification of the sign inconsistency in the Vaidya and Tracy asymptotic expression first noticed in [28] by using numerical solution of the Painlevé equation. The same sign change appears in the thermodynamical limit of circular geometry which we consider in the next section.

The expressions (43) and (47) together with expressions (34) of the coefficients  $D_k^{(1/2)}$  provide only the leading non-perturbative contribution of each saddle point and do not contain the sub-leading terms, which arise, for example, from the deviation of the action from its second-order Taylor expansion near a saddle point or additional terms in the large  $N$  expansion of the double product (44). These terms can be treated by a perturbation theory near each saddle point and we show in section 5 how such perturbation theory can be constructed for the case of circular geometry.

#### 4. One-body density matrix in circular geometry

Our method is also able to give the one-body density matrix in circular geometry, i.e. for  $N + 1$  particles on a ring of length  $L$  with periodic boundary conditions. The one-body density matrix is given by expression (6). The corrections to the leading smooth term (5) in the long-distance

expansion of the one-body density matrix remained unknown to the best of our knowledge (see discussion in [22]). We now show how they can be obtained with our method. Let us consider the quantity (8) rewritten explicitly as an  $N$ -dimensional integral

$$Z_{2n}(t) = \frac{1}{M_N(2n, 2n, 1)} \int_{-1/2}^{1/2} d^N \theta |\Delta_N(e^{2\pi i \theta})|^2 \prod_{l=1}^N |1 + e^{2\pi i \theta_l}|^{2n} |t + e^{2\pi i \theta_l}|^{2n}, \quad (48)$$

where  $t = \exp(2\pi i \alpha)$  and we have normalized  $Z_{2n}(1) = 1$ , introducing the normalization constant  $M_N(2n, 2n, 1)$ . Its value is given by the Morris integral of random matrix theory [38], but its precise value is not needed. Taking  $n \rightarrow 1/2$  in  $Z_{2n}$  gives the density matrix normalized to the density  $g_1/n$ . Now we write

$$|1 + e^{2\pi i \theta_l}|^{2n} = e^{-2\pi n i \theta_l} (1 + e^{2\pi i \theta_l})^{2n} \quad (49)$$

and change the integration variables  $\theta_l \rightarrow \theta + \alpha$  to obtain

$$Z_{2n}(t) = \frac{t^{-Nn}}{M_N(2n, 2n, 1)} \int_{-1/2}^{1/2} d^N \theta |\Delta_N(e^{2\pi i \theta})|^2 \prod_{l=1}^N e^{-2\pi n i \theta_l} |1 + e^{2\pi i \theta_l}|^{2n} (1 + t e^{2\pi i \theta_l})^{2n}. \quad (50)$$

One observes that the transformation from (48) to the last expression is possible only when  $n$  is integer, otherwise the change of variables is not permitted due to discontinuity of the phase in (49). It is parallel to the representation (14) we used for the harmonic confinement. As in the last section we proceed with the representation (50) and assume that it remains valid for any  $n$ .

The integral (50) has a remarkable dual representation (see equation (3.41) in [40]) by an integral over  $n$  variables

$$Z_{2n}(t) = \frac{t^{-Nn}}{S_{2n}(0, 0, 1)} \int_0^1 d^{2n} x \Delta_{2n}^2(x) \prod_{a=1}^{2n} (1 - (1-t)x_a)^N, \quad (51)$$

where the normalization constant is given by Selberg integral

$$S_{2n}(0, 0, 1) = \int_0^1 d^{2n} x \Delta_{2n}^2(x) = \prod_{a=1}^{2n} \frac{\Gamma^2(a) \Gamma(1+a)}{\Gamma(2n+a)}. \quad (52)$$

On the right-hand side of (51), the number of particles  $N$  appears only as a parameter. This representation is a direct analogy of (18) and (38) and allows us to obtain the asymptotic expression for  $Z_{2n}$  suitable for analytic continuation in  $n$ . In the large  $N$  limit, the integrand in (51) oscillates rapidly and the main contribution comes from the endpoints  $x_{\pm} = 1, 0$  which are the only stationary points of the phase. We change variables near each endpoint

$$\begin{aligned} x_a &= x_- + \frac{\xi_a}{N(1-t)}, & a &= 1, \dots, l \\ x_b &= x_+ - \frac{\xi_b}{N(1-t^{-1})}, & b &= l+1, \dots, 2n. \end{aligned} \quad (53)$$

The integrand in (51) simplifies in the large  $N$  limit:

$$(1 - (1-t)x_a)^N \simeq \begin{cases} e^{-\xi_a}, & a = 1, \dots, l \\ t^N e^{-\xi_a}, & a = l+1, \dots, 2n \end{cases} \quad (54)$$

and the integration measure including the Vandermonde determinant is factorized as

$$d^{2n} x \Delta_{2n}^2(x) = \left( \frac{1}{N(1-t)} \right)^{l^2} \left( \frac{1}{N(1-t^{-1})} \right)^{(2n-l)^2} d^l \xi_a \Delta_l^2(\xi_a) d^{2n-l} \xi_b \Delta_{2n-l}^2(\xi_b). \quad (55)$$

The remaining integrals are calculated using the Laguerre variant of the Selberg formula

$$I_l(\lambda) = \int_0^\infty d^l \xi_a \Delta_l^2(\xi_a) \prod_{a=1}^m e^{-\lambda \xi_a} = \lambda^{-l^2} \prod_{a=1}^l \Gamma(a) \Gamma(1+a). \quad (56)$$

Multiplying the contribution of each saddle point by the number of ways to distribute variables, we obtain the asymptotics of the integral (51) as a sum of  $2n+1$  terms:

$$Z_{2n}(t) = \prod_{c=1}^{2n} \Gamma(2n+c) \sum_l (-1)^{2n(n-l)} \frac{[F_{2n}^l]^2 t^{(N+2n)(n-l)}}{(2X)^{l^2+(2n-l)^2}}, \quad (57)$$

where we have introduced  $X = N \sin \pi \alpha$  and the factors  $F_{2n}^l$  are defined in (26). We note that due to the translation invariance, the resulting expansion is given by a simple sum over saddle points and not a double sum as in the case (41) of a harmonic potential.

Changing the summation variable to  $k = l - n$  and factorizing the amplitudes  $F_{2n}^l$  according to definitions (27), (28) we obtain finally

$$Z_{2n}(t) = \frac{A_n^2 \prod_{c=1}^{2n} \Gamma(2n+c)}{(2N |\sin \pi \alpha|)^{2n^2}} \left( 1 + 2 \sum_{k=1}^{\infty} (-1)^{2nk} [D_k^{(n)}]^2 \frac{\cos[2k\pi(N+2n)\alpha]}{(4N^2 \sin^2 \pi \alpha)^{k^2}} \right). \quad (58)$$

Now we are in a position to take the limit  $n \rightarrow 1/2$  which results in the desired corrections to the smooth part (5) of the one-body density matrix:

$$\frac{g_1(\alpha)}{\bar{g}_1(\alpha)} - 1 = 2 \sum_{k=1}^{\infty} (-1)^k [D_k^{(1/2)}]^2 \frac{\cos 2k\pi(N+1)\alpha}{(4N^2 \sin^2 \pi \alpha)^{k^2}}. \quad (59)$$

In the thermodynamic limit  $N\pi\alpha = k_F x$ , when  $N \rightarrow \infty$  we recover the result (47). Again we note the sign difference between the  $k=1$  term of (59) and that of Vaidya and Tracy (4). As in the case of harmonic confinement only the leading contribution of each saddle point is included in the expansion (59). The circular geometry is particularly suitable for discussing the perturbative corrections, which are calculated in the next section.

## 5. Perturbation theory

Up to now, we have considered only the leading non-perturbative contribution of each saddle point in the expansion (58) of  $Z_{2n}$ . To provide all the corrections to a given order in  $1/N$  we have to deal with the sub-leading terms by perturbation theory. This amounts to multiplying the contribution of each saddle point in (57) by the following average (here  $l = n+k$ ):

$$\langle F(\xi, \xi') \rangle \equiv \frac{1}{I_{n+k}(1) I_{n-k}(1)} \int_0^\infty d^{n+k} \xi d^{n-k} \xi' \Delta_{n+k}^2(\xi) \Delta_{n-k}^2(\xi') F(\xi, \xi') e^{-\sum \xi_a - \sum \xi'_b} \quad (60)$$

of the function

$$F(\xi_1, \dots, \xi_{n+k}; \xi'_1, \dots, \xi'_{n-k}) = \prod_{a=1}^{n+k} \prod_{b=1}^{n-k} \left( 1 - \frac{\Lambda_{ab}}{2X} \right)^2 \times \exp \left( - \sum_{a=1}^{n+k} \left( \frac{\xi_a^2}{2N} + \frac{\xi_a^3}{3N^2} + \dots \right) \right) \exp \left( - \sum_{b=1}^{n-k} \left( \frac{\xi_b'^2}{2N} + \frac{\xi_b'^3}{3N^2} + \dots \right) \right), \quad (61)$$

where we have defined  $\xi'_b \equiv \xi_{b+n+k}$  for  $b = 1, \dots, n-k$  and

$$\Lambda_{(ab)} = \frac{i}{2} (e^{\pi i \alpha} \xi'_b - e^{-\pi i \alpha} \xi_a). \quad (62)$$

In (61) the product comes from the neglected terms in the factorization of the Vandermonde determinant, while the exponents represent the corrections to the leading term (54) in the  $1/N$  expansion of the action.

The perturbation theory consists in expanding  $F(\xi, \xi')$  up to the desired order in  $1/N$  (recall that  $1/X = 1/N \sin \pi\alpha$ ) and averaging (60) term by term with the unperturbed action of each saddle point  $k$ . It follows from the very essence of our method, reflected in the structure of (60), that the sets of variables  $\xi_a$  and  $\xi'_b$  are independent, so that their averages factorize

$$\langle f(\xi)g(\xi') \rangle = \langle f(\xi) \rangle_{n+k} \langle g(\xi') \rangle_{n-k}, \quad \langle f(\xi) \rangle_m \equiv \frac{1}{I_m(1)} \int_0^\infty d^n \xi \Delta_m^2(\xi) f(\xi) \prod_{a=1}^m e^{-\xi_a}, \quad (63)$$

where the subscript reflects the number of integration variables in each factor. The remaining averages are performed using known results from the theory of Selberg integrals (see chapter 17 of [37]). In the following we shall need the following results:

$$\begin{aligned} \langle \xi_1 \rangle_m &= m, \\ \langle \xi_1^2 \rangle_m &= 2m^2, \\ \langle \xi_1 \xi_2 \rangle_m &= m(m-1), \\ \langle \xi_1^3 \rangle_m &= m(5m^2+1), & \langle \xi_1^2 \xi_2 \rangle_m &= m(m-1)(2m-1), \\ \langle \xi_1 \xi_2 \xi_3 \rangle_m &= m(m-1)(m-2), \\ \langle \xi_1^4 \rangle_m &= m(8m^3+15m^2-2m+3), & \langle \xi_1^3 \xi_2 \rangle_m &= m(m-1)(5m^2-4m+3), \\ \langle \xi_1^2 \xi_2 \xi_3 \rangle_m &= 2m(m-1)^2(m-2), & \langle \xi_1^2 \xi_2^2 \rangle_m &= m(m-1)(2m-1)^2, \\ \langle \xi_1 \xi_2 \xi_3 \xi_4 \rangle_m &= m(m-1)(m-2)(m-3). \end{aligned} \quad (64)$$

In first order we have

$$\begin{aligned} \frac{2}{X} \sum_{ab} \langle \Lambda_{ab} \rangle - \frac{1}{2N} \sum_a \langle \xi_a^2 \rangle - \frac{1}{2N} \sum_b \langle \xi'_b{}^2 \rangle \\ = \frac{1}{X} \sum_{a=1}^{n+k} \sum_{b=1}^{n-k} [e^{\pi i \alpha} \langle \xi'_b \rangle_{n-k} - e^{-\pi i \alpha} \langle \xi_a \rangle_{n+k}] - \frac{1}{2N} \sum_{a=1}^{n+k} \langle \xi_a^2 \rangle_{n+k} - \frac{1}{2N} \sum_{b=1}^{n-k} \langle \xi'_b{}^2 \rangle_{n-k} \\ = \frac{i}{X} (n+k)(n-k) [(n-k) e^{\pi i \alpha} - (n+k) e^{-\pi i \alpha}] - \frac{1}{N} [(n+k)^3 + (n-k)^3], \end{aligned} \quad (65)$$

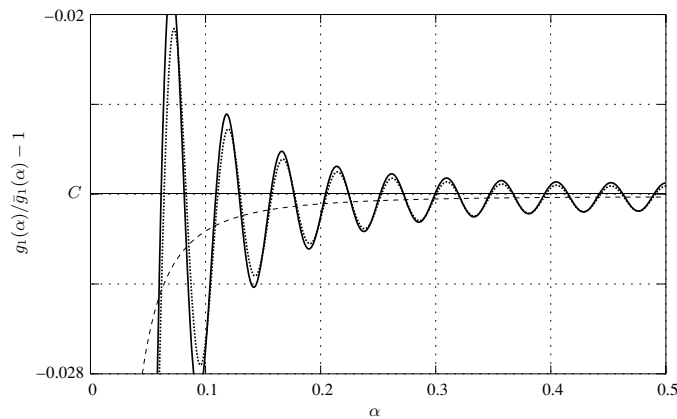
where we have used the fact that the averages are independent of the index  $a$  or  $b$  and the corresponding sums yield a factor  $n+k$  or  $n-k$ , respectively. The calculation of the next order correction to this term is performed similarly to the first order. One only has to pay attention to the appearance of identical indices in the sums. For instance we have a sum of diagonal terms:

$$\frac{1}{X^2} \sum_{ab} \langle \Lambda_{ab}^2 \rangle = -\frac{(n+k)(n-k)}{4X^2} \left[ e^{2\pi i \alpha} \langle \xi_1'^2 \rangle_{n-k} + e^{-2\pi i \alpha} \langle \xi_1^2 \rangle_{n+k} - 2 \langle \xi \rangle_{n+k} \langle \xi' \rangle_{n-k} \right] \quad (66)$$

and non-diagonal ones:

$$\begin{aligned} \frac{2}{X^2} \sum_{ab \neq a'b'} \langle \Lambda_{ab} \Lambda_{a'b'} \rangle = -\frac{(n+k)(n-k)}{2X^2} (e^{2\pi i \alpha} [(n+k-1) \langle \xi_1'^2 \rangle_{n-k} \\ + (n+k)(n-k-1) \langle \xi_1' \xi_2' \rangle_{n-k}] + e^{-2\pi i \alpha} [(n-k-1) \langle \xi_1^2 \rangle_{n+k} \\ + (n-k)(n+k-1) \langle \xi_1 \xi_2 \rangle_{n+k}] + 2((n+k)(n-k) - 1) \langle \xi_1 \rangle_{n+k} \langle \xi'_1 \rangle_{n-k}). \end{aligned} \quad (67)$$





**Figure 2.** Comparison between asymptotic expression (69) for the one-body density matrix (solid line) of  $N = 20$  particles in circular geometry and exact results (dotted line) based on numerical evaluation of Toeplitz determinant. The correction due to the finite number of particles  $C = -1/2N + 13/32N^2 = -0.023\,984\,375$  is shown and the dashed line represents the mean contribution of the oscillatory terms.

Restricting ourselves to the terms up to second order in perturbation theory we see that only the case  $k = 0$  has to be considered, since the leading contribution of  $k \neq 0$  saddle points is already at least of second order in  $1/X$ . In this case, the first-order terms (65) simplify to  $-4n^3/N$ , while calculating the contribution of all the second-order terms in (61), including that of (66) and (67), yields

$$\langle F(\xi, \xi') \rangle = -\frac{n^4}{2X^2} + \frac{n^2}{N^2} \left( 8n^4 - n^3 + \frac{23}{3}n^2 - 2n + \frac{1}{3} \right). \quad (68)$$

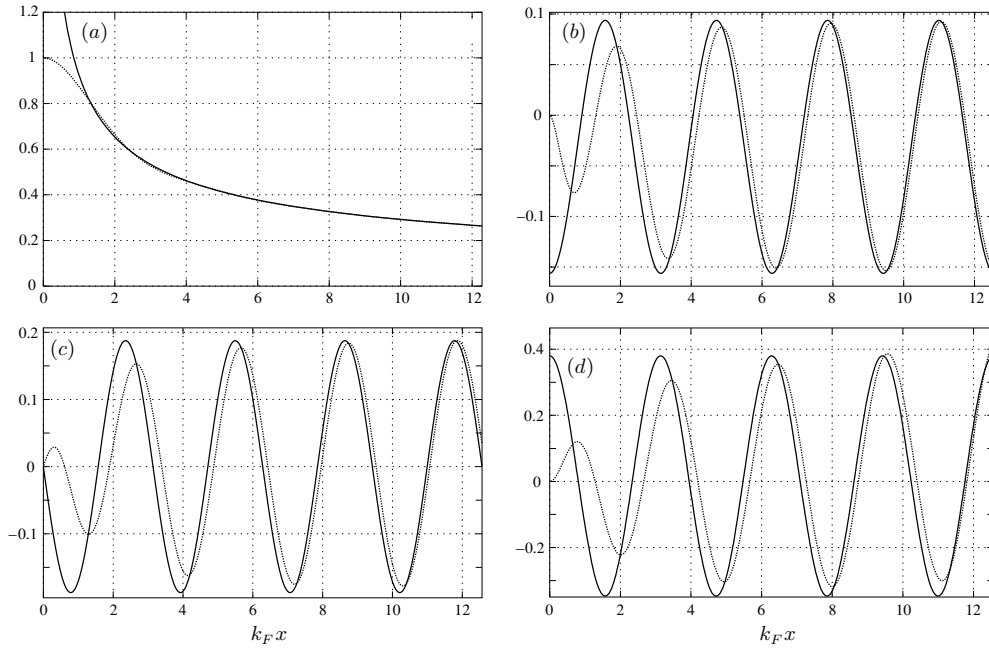
In every order in perturbation theory, the coefficient is given by a polynomial in  $n$ , so its analytical continuation to  $n = 1/2$  is straightforward. A peculiar feature, familiar in the replica method, of such perturbation theory of this  $n = 1/2$  component field is that averages of positive quantities  $\xi, \xi'$  vanish or become negative.

Adding up the second-order contribution (68) with that of first order (65) and setting  $n = 1/2$ , we obtain the factor multiplying the contribution of the  $k = 0$  saddle point in the expansion (58). Combining it with the leading contribution of the  $k = \pm 1$  saddle point, we get the finite size correction up to the second order:

$$g_1(\alpha) = \frac{N\rho_\infty}{|N \sin \pi\alpha|^{1/2}} \left[ 1 - \frac{1}{2N} + \frac{13}{32N^2} - \frac{1}{32N^2 \sin^2 \pi\alpha} - \frac{\cos 2\pi(N+1)\alpha}{8N^2 \sin^2 \pi\alpha} \right]. \quad (69)$$

We have compared this result with the exact calculation based on a numerical evaluation of the Toeplitz determinant representation [18, 22] of  $g_1(\alpha)$ . The result of the comparison is presented in figure 2 for  $N = 20$  particles. The agreement of the two expressions is remarkable.

Higher order corrections can be calculated in similar way. However, the number of averages proliferates quickly and the computations become very tedious. The simplification occurs in the thermodynamic limit, where terms proportional to just inverse powers of  $N$  in (61) can be neglected. In this limit only the double product contributes in (61) and the exponential factors in (62) can be set to 1 from the very beginning. In this case we were able to proceed up to the terms of order  $1/X^4$ , reproducing the results of the asymptotic expansion of Vaidya



**Figure 3.** Comparison between different orders in  $1/k_F x$  in the asymptotic expansion (70) and results based on numerical calculation of the one-body density matrix  $g_1(k_F x)$  on the lattice with filling factor  $\nu = 0.005$ . (a) Dotted line: the Toeplitz determinant evaluation of  $g_1(k_F x)$ , solid line:  $\bar{g}_1(k_F x) = \rho_\infty/|k_F x|^{1/2}$ . (b) Dotted line:  $G^{(1)}(k_F x) = (k_F x)^2(g_1(k_F x)/\bar{g}_1(k_F x) - 1)$ , solid line:  $G_{th}^{(1)} = -1/32 - \cos 2k_F x/8$ . (c) Dotted line:  $G^{(2)}(k_F x) = k_F x(G^{(1)}(k_F x) - G_{th}^{(1)}(k_F x))$ , solid line:  $G_{th}^{(2)}(k_F x) = -(3/16) \sin 2k_F x$ . (d) Dotted line:  $G^{(3)}(k_F x) = k_F x(G^{(2)}(k_F x) - G_{th}^{(2)}(k_F x))$ , solid line:  $G^{(3)}(k_F x) = 33/2048 + (93/256) \cos 2k_F x$ .

and Tracy. Identifying in the thermodynamic limit  $X = N \sin \pi \alpha = N \pi \alpha = k_F x$  we get

$$g_1(k_F x) = \frac{\rho_\infty}{|k_F x|^{1/2}} \left[ 1 - \frac{1}{32} \frac{1}{(k_F x)^2} - \frac{1}{8} \frac{\cos(2k_F x)}{(k_F x)^2} - \frac{3}{16} \frac{\sin(2k_F x)}{(k_F x)^3} + \frac{33}{2048} \frac{1}{(k_F x)^4} + \frac{93}{256} \frac{\cos(2k_F x)}{(k_F x)^4} \right]. \quad (70)$$

To compare this result with the numerics we use the representation for  $g_1$  given in the work of Schultz [16] as a continuum limit of the one-body density matrix on a lattice which was a starting point for Vaidya and Tracy calculations [19]. It allows for the direct study of  $g_1$  in the thermodynamical limit if the distance on the lattice  $d$  is related to the dimensionless distance in (70) as  $\pi \nu d = k_F x$ . The filling factor  $\nu$  approaches zero while  $k_F x$  is held fixed in order to reproduce the continuous limit. The results of the comparison are shown in figure 3 and the agreement is good order by order. In particular, our result (70) implies that the sign of all the trigonometric terms in Vaidya and Tracy expansion should be reversed as in (70), while the values of numerical coefficients remain unaffected.

## 6. Conclusions

The main results of our calculations are expressions for the one-body density matrix accounting for finite size effects in the cases of harmonic confinement and circular geometry. These

expressions consist of a sum of the leading smooth term and sub-leading oscillating terms. These terms involve universal amplitudes  $D_k^{(1/2)}$  given explicitly by equation (34). The oscillations occur with periods  $2k_F, 4k_F, \dots$  and are similar to Friedel oscillations in fermionic systems. Their physical origin lies in particle correlations on scales of mean inter-particle separation and they represent a generic property of all strongly correlated one-dimensional systems. For fermions their existence is due to the Pauli principle and is not restricted to 1D, while for bosons the Friedel oscillations are due to the effective Pauli principle induced by strong interactions, a situation that is possible only in the one-dimensional world. Our results are in agreement with the general structure of the asymptotic expansion of  $g_1$  given in [19] and with Haldane's hydrodynamic approach [21] or conformal field theory [26].

Using semiclassical methods, such as the local density approximation, this universal behaviour of correlations can be extended for a wide class of external confining potentials with sufficiently smooth behaviour. Our calculations provide *ab initio* justification of the local density approximation in the bulk of a bosonic cloud in a harmonic trap which is important to current experiments with cold atoms. For this system our method is capable of considering the edges of a thermodynamical density profile, thus going beyond the local density approximation. Another system of experimental interest is the lattice model of impenetrable bosons, produced recently [14] by confining atoms in optical lattices. Given an exact expression for the one-body density matrix in terms of Toeplitz determinants [16, 18], it is highly desirable to obtain its asymptotics without going to the continuum limit, thus revealing the interplay between the short-range correlations and the effects of the lattice.

The results were obtained using a novel modification of the replica method which in the context of exactly solvable models consists of an alternative representation of correlation functions. The phenomenon of replica symmetry breaking serves here as a tool to single out the bosonic branch of the one-body density matrix out of various possible analytic continuations in the replica index  $n$ . It can be considered as a fresh insight in quantum statistics in one dimensions [43], a question to explore in the future. The obvious candidate for this study is the Calogero–Sutherland model. Indeed, the ground state wavefunction of the Calogero–Sutherland model is proportional to the Vandermonde determinant (7) taken to a power  $\lambda$ , which characterizes the statistical interactions between the particles [44]. Based on equivalence of Calogero–Sutherland models and random matrices for particular values of statistical interaction parameter  $\lambda$ , the replica method also provides an efficient way to study correlations of spectral determinants of random matrices directly related to averages of the form (8). These objects play an important role in statistical physics, mathematical physics and modern combinatorics (see [39, 45] and references therein). A more distant, but certainly tempting perspective is the application of the replica method to other integrable models and calculation of their correlation properties.

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## Appendix A. Derivation of the duality formula

Using the obvious identity among the Vandermonde determinants

$$\prod_{j=1}^N \prod_{a=1}^m (t_a - y_j) = \frac{\Delta_{N+m}(t_1, \dots, t_m, y_1, \dots, y_N)}{\Delta_m(t_1, \dots, t_m) \Delta_N(y_1, \dots, y_N)}, \quad (\text{A.1})$$

expression (15) is represented in the second quantization

$$Z_m(t_1, \dots, t_m) = \sqrt{\frac{S_{N+m}}{S_N}} e^{\frac{N}{4} \sum_a t_a^2} \frac{1}{\Delta_m(t)} {}^N \langle \psi(t_1) \psi(t_2) \dots \psi(t_m) \rangle_{N+m}, \quad (\text{A.2})$$

using the fermionic annihilation operators  $\psi(x)$  acting between the ground state of  $N$  and  $N + m$  fermions. One then uses the Wick theorem to calculate this matrix element:

$${}_N \langle \psi(t_1) \psi(t_2) \dots \psi(t_m) \rangle_{N+m} = \det_{k,l} [\phi_{N+k-1}(t_l)]. \quad (\text{A.3})$$

The resulting determinant on the right-hand side of (A.3) is constructed using the one-particle wavefunctions (10), which have an integral representation

$$\phi_k(t) = e^{-\frac{Nt^2}{4}} \sqrt{\frac{2^k N^{k+1}}{2\pi c_k}} \frac{1}{i^k} \int_{-\infty}^{\infty} dx x^k e^{-\frac{N}{2}(x-it)^2}. \quad (\text{A.4})$$

Representing in this way the one-particle wavefunctions in the expansion of the determinant (A.3), we have the following result:

$$Z_m(t_1, \dots, t_m) = \left(\frac{N}{2\pi}\right)^{\frac{m}{2}} (-i)^{Nm} \int_{-\infty}^{\infty} d^m x e^{-\frac{N}{2} \sum (x_a - it_a)^2} \frac{\Delta_m(x_1, \dots, x_m)}{\Delta_m(it_1, \dots, it_m)} \prod_{a=1}^m x_a^N. \quad (\text{A.5})$$

In order to calculate various correlation functions, such as density matrix (36) or ground state amplitude (13) we need to take the limit where several variables  $t_a$  become equal to each other. This limit is finite, despite the apparent singularity in the last integral representation of  $Z_m$ . We demonstrate this in the simpler case of ground state amplitude obtained from (16). Shift the variables  $t_a = t + \eta_a$ ,  $\eta_a \rightarrow 0$  and rewrite expression (A.5) as

$$Z_m = \left(\frac{N}{2\pi}\right)^{\frac{m}{2}} (-i)^{Nm} \int_{-\infty}^{\infty} d^m x \Delta_m(x) \frac{e^{iN \sum x_a \eta_a}}{\Delta_m(i\eta)} \prod_{a=1}^m x_a^N e^{-\frac{N}{2} [(x_a - it)^2 - \eta_a^2]}, \quad (\text{A.6})$$

Due to the presence of totally antisymmetric function  $\Delta_m(x)$  only the totally antisymmetric part of  $e^{iN \sum x_a \eta_a}$  survives the integration. Using this and the fact that

$$\lim_{\eta \rightarrow 0} \frac{\det_{a,b} e^{iN x_a \eta_b}}{\Delta_m(i\eta)} = \frac{N^{m(m-1)/2}}{\prod_{a=0}^{m-1} \Gamma(a+1)} \Delta_m(x), \quad (\text{A.7})$$

we arrive at the dual representation (18).

The case of two variables  $t, t'$  is similar. One starts with expression (15) and shifts the variables

$$\begin{aligned} t_a &= t + \eta_a, & a &= 1, \dots, m/2 \\ t_b &= t' + \eta_b, & b &= m/2 + 1, \dots, m, \end{aligned}$$

where  $\eta_a$  and  $\eta_b$  go to zero independently. One uses the fact (A.1) that

$$\Delta_m(it_1, \dots, it_m) = \Delta_{m/2}(i\eta_a) \Delta_{m/2}(i\eta_b) [i(t - t')]^{m^2/4}$$

to leading non-vanishing order in  $\eta_a, \eta_b$  and rewrites (A.5) as

$$Z_m(t, t') = \left(\frac{N}{2\pi}\right)^{m/2} (-i)^{mN} \int_{-\infty}^{\infty} d^m x \frac{\Delta_m(x_1, \dots, x_m)}{[i(t-t')]^{m/4}} \frac{e^{iN \sum x_a \eta_a}}{\Delta_n(i\eta_a)} \frac{e^{iN \sum x_b \eta_b}}{\Delta_{m/2}(i\eta_b)} \\ \times \prod_{a=1}^{m/2} x_a^N e^{-\frac{N}{2}[(x_a - it)^2 - \eta_a^2]} \prod_{b=1}^{m/2} x_b^N e^{-\frac{N}{2}[(x_b - it')^2 - \eta_b^2]}.$$

Anti-symmetrizing the numerators in each set of variables  $x_a$  and  $x_b$  independently and using (A.7) one gets (38).

## Appendix B. Analytical continuation of $A_n$

We use the following integral representation [46] for the logarithm of Euler's gamma function:

$$\ln \Gamma(z) = \int_0^{\infty} \frac{dt}{t} \left( \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1)e^{-t} \right) \quad (\text{B.1})$$

to represent the logarithm of  $A_n$  as an integral

$$\ln A_n = \sum_{a=1}^n (\ln \Gamma(a) - \ln \Gamma(2n+1-a)) = \int_0^{\infty} \frac{dt}{t} e^{-t} \left[ \left( \frac{1 - e^{-nt}}{1 - e^{-t}} \right)^2 - n^2 \right], \quad (\text{B.2})$$

where we have summed finite geometric series under the integral. The integral representation defines  $A_n$  for any value of  $n$ . In particular for  $n = 1/2$  we get

$$\ln A_{1/2} = \int_0^{\infty} \frac{dt}{t} e^{-t} \left[ \left( \frac{1 - e^{-t/2}}{1 - e^{-t}} \right)^2 - \frac{1}{4} \right] = \frac{1}{4} \int_0^{\infty} \frac{dt}{t} \frac{3e^t - 8e^{t/2} + 6 - e^{-t}}{(e^t - 1)^2}. \quad (\text{B.3})$$

In order to calculate the last integral, we regularize the divergence at  $t = 0$  in the following way:

$$\ln A_{1/2} = \lim_{\nu \rightarrow 0} C(\nu), \quad C(\nu) = \frac{1}{4} \int_0^{\infty} dt t^{\nu-1} \frac{3e^t - 8e^{t/2} + 6 - e^{-t}}{(e^t - 1)^2} \quad (\text{B.4})$$

and calculate the integral term by term using the formula

$$\int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x} dx}{(e^x - 1)^2} = \Gamma(\nu) [\zeta(\nu-1, \mu+2) - (\mu+1)\zeta(\nu, \mu+2)], \quad (\text{B.5})$$

where  $\zeta(z, q)$  is Riemann's zeta function

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z}, \quad \zeta(z, 0) = \zeta(z). \quad (\text{B.6})$$

The result of integration can be represented as

$$C(\nu) = 2\Gamma(\nu)[2(1-2^{\nu-2})\zeta(\nu-1) - (1-2^{\nu-1})\zeta(\nu) - 1]. \quad (\text{B.7})$$

Taking the limit by the l'Hôpital rule and using the fact that  $\zeta(0) = -1/2$ ,  $\zeta(-1) = -1/12$ ,  $\zeta'(0) = -\ln \sqrt{2\pi}$  we arrive at

$$\ln A_{1/2} = \lim_{\nu \rightarrow 0} C(\nu) = 3\zeta'(-1) + \frac{1}{12} \ln 2 + \frac{1}{2} \ln \pi, \quad (\text{B.8})$$

which relates  $A_{1/2}$  to Glaisher's constant  $A = \exp(1/12 - \zeta'(-1))$  as

$$\rho_{\infty} = A_{1/2}^2 / \sqrt{2} = \pi e^{1/2} 2^{-1/3} A^{-6}. \quad (\text{B.9})$$

An alternative method of analytical continuation to that described above is to relate the constants  $A_n$  to Barnes  $G$  function [38]. Definition (27) yields

$$A_n = \frac{\prod_{a=1}^n \Gamma(a)}{\prod_{a=1}^n \Gamma(n+a)} = \frac{G(n+1)}{G(1)} \frac{G(n+1)}{G(2n+1)} = \frac{G^2(n+1)}{G(2n+1)}, \quad (\text{B.10})$$

where we have used  $G(1) = 1$  and the functional relation  $G(z+1) = \Gamma(z)G(z)$ . The analytical continuation leads to identity  $A_{1/2} = G^2(3/2)$ .

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