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Two-Dimensional Ising Correlations: Convergence of the Scaling Limit*

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INTRODUCTION

In this paper we will establish formulas for the correlations of the two-dimensional Ising model in the absence of a magnetic field and prove the convergence of the scaling limit from above and below the critical temperature.

The theoretical developments which lead up to our results begin with Onsager's calculation of the free energy for this model in a classic 1944 paper [52]. Statistical mechanics in the infinite-volume limit is expected to exhibit phase transitions through nonanalytic behavior in thermodynamic quantities; the Onsager formula for the free energy as a function of temperature was the first explicit example of such behavior. In a sequel to Onsager's paper, Kaufman [34] simplified the analysis by emphasizing the role of the spin representations of the orthogonal group; Kaufman and Onsager [35] subsequently used this idea to study the short-range order. By 1949 Onsager [53] knew the formula for the spontaneous magnetization, and Yang gave an independent derivation of this result in 1952 [74].

In [28] Kac and Ward and later in [32] Kasteleyn pioneered a combinatorial attack on the Ising model. Montroll, Potts, and Ward [49] used this method to give formulas for the correlations as Pfaffians. The size of the Pfaffians in these formulas grows with the separation of the sites in the correlations and the asymtotic behavior at large separation (clustering) is far from evident. To go beyond the spontaneous magnetization in the analysis of the clustering of correlations, corrections to the Szegö formula were devised. This problem has a long history, and we mention in connection with the Ising model the fundamental papers by Wu [72] and by Kadanoff [29] in 1966, and by Cheng and Wu in 1967 [13], and refer the reader to the book by McCoy and Wu [40] for further details up to 1972.

In Fisher [18] and Kadanoff [30] a notion of scaling for statistical systems near a critical point was proposed. To understand the scaling limit for the

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Ising model it proved important to have formulas for the lattice correlations which manifested clustering explicitly. In 1973 the calculation of the two-point scaling function was announced in [11, 68] with details appearing in [73]. Somewhat later several groups announced series expansion formulas for the scaled n-point functions [4, 10, 43, 57]. McCoy *et al.* [43] employed Pfaffian techniques which evolved from the combinatorial approach to the Ising model (see also [41]). The work of Sato *et al.* [57], of Abraham [2–5], and of Bariev [9, 10] is more directly descended from the original algebraic approach of Onsager and Kaufman; an approach which, incidently, received further stimulus in the papers of Schultz *et al.* [65] and Kadanoff [29].

In the passage to the scaling limit, the correlations become singular at points of coincidence. For example, the critical exponent specifying this singularity in the two-point function is "known" from the large-scale behavior at the critical temperature [40] (the two-point scaling function interpolates between the behavior at large separation at the critical temperature and the behavior at large separation away from the critical temperature). However, the precise asymptotics at short distance has never been directly computed from the known series expansions. This is not too surprising since these series were developed specifically to exhibit the behavior at large separation in the scaled distance. In [73] Wu, McCoy, Tracy, and Barouch found the precise short-distance asymptotics for the scaled two-point function by first showing that this function was expressible in terms of a Painlevé transcendent. Part of this analysis was put on a firmer footing in a later paper [42].

The deeper reason for the occurrence of the Painlevé transcendent was first understood by Sato, Miwa, and Jimbo (S.M.J.) [58–63]. They were aware that Painlevé transcendents occur naturally in the integration of Schlesinger's equations [64] for monodromy-preserving deformations of linear differential equations (oddly, the extensive work of Garnier [19] on this connection is not mentioned in the principal English reference, Ince [25]). In a remarkable series of papers, they developed new techniques in the theory of Clifford algebras [59], generalized the monodromy idea to a partial differential equation (the Euclidean Dirac equation) [61], showed that the scaled n-point functions were the coefficients in the local expansion of a basis of multivalued solutions to the Euclidean Dirac equation [62], and finally used this to demonstrate that the scaled n-point functions satisfy a nonlinear Pfaffian system of differential equations (every derivative is specified) [62, 63]. In the case of the two point function, the Pfaffian system is integrable in terms of the particular Painlevé transcendent appearing in [42, 73]. A review of this work can be found in [26].

In the work we have described on the Ising model the level of mathematical rigor fluctuates considerably. In much of the work on the correlations, the subtleties of the boundary conditions for the infinite volume limit are side stepped. In all the work we are aware of there are "holes" of positive measure in the known regions of convergence for the series representations of the scaled n-point functions. In particular the important S.M.J. [62] analysis of the scaled correlations introduces (multivalued) continuum order–disorder correlations through complicated infinite series expansions whose known region of convergence has large gaps. The coefficients in the local expansions of these order–disorder correlations are identified as n-point functions again only at the level of the series expansions. One of the principal motivations for our paper is to lay the foundation for a treatment of the S.M.J. analysis in which the multivalued order–disorder correlations and the n-point functions appear as well-controlled limits of simply defined lattice analogs, and in which the local expansions are computed rigorously. We shall present this analysis in a forthcoming paper. Another important consideration for our work was to establish some of the expected probabilistic and field theoretic properties for the scaled n-point functions. Our contribution to these matters is presented in the final section of this paper.

In the first three sections of this paper, we will prove (regularized) determinant formulas for the infinite-volume correlations (Theorems 2.1 and 3.2). The transfer matrix formalism in Section 1 permits us to express the correlations (with "plus" boundary conditions) for a semi-infinite box as the Fock expectation of a product in a finite-dimensional Clifford algebra. We apply results from [56] to give determinant formulas in this finite-dimensional situation and then prove the convergence of these determinants to their infinite-volume counterparts directly. Our proof is valid only below the critical temperature. Above the critical temperature we use a variant of Kramers–Wannier duality to relate the correlations with "open" boundary conditions to correlations of disorder variables (see Kadanoff and Ceva [31]) with "plus" boundary conditions below the critical temperature. This effectively reduces the convergence proof to the previous case and incidently identifies a natural disorder variable on the lattice. Once the determinant formulas are established, the infinite-dimensional results in [56] then give simple "abstract" characterizations of the infinite-volume correlations as Fock expectations (Theorems 2.2 and 3.3).

The use of "plus" boundary conditions permits us to use convergence results [39] which show that the correlations obtained in the two step infinite-volume limit natural for the transfer matrix approach are the same as the correlations which result from letting the sides of a square box tend simultaneously to infinity. This coincidence of limits establishes dihedral group invariance and that the correlations are the expectations of products of random fields [14, 15], neither of which properties are manifest in our explicit formulas.

In the fourth section we prove convergence of the scaling limit from above and below the critical temperature. Our formulas are not valid everywhere but the exceptional sets are measure zero. The resulting n-point scaling functions (below T_c) are given by formulas $\det_2(1 + G)$, where G is a Schmidt class operator.

In Section 5 we use Gaussian domination [51] and some integrability estimates for the two point function to conclude that the correlations are locally integrable functions. We then use the Bochner–Minlos theorem to demonstrate that we have computed the n-point functions of a generalized random field [16, 20]. The Osterwalder–Schrader axioms [54] are all direct consequences of the convergence of the scaling limit with the exception of rotational invariance. We do not prove rotational invariance; however, we note that McCoy and Wu have an unpublished demonstration of this property. Of particular interest in this last section are new formulas for the lattice two-point functions which we use to establish dominated convergence.

We conclude this introduction by mentioning some work of McCoy and Wu [44–46] in which they analyze the decay properties of the n-point functions and relate the non-tree-like decay below T_c to analytic properties of the two-point function in the complex H plane (H is the magnetic field). We believe our formulas are well suited to such analysis.

1.0. Let $\Lambda_{M,N} = \{i \in \mathbf{Z}^2 \colon |i_1| \le M, |i_2| \le N\}$ and write $\sigma(i)$ for the spin at site $i \in \Lambda_{M,N}$. A configuration of $\Lambda_{M,N}$ is an assignment of a value (either $+1$ or -1) for the spin $\sigma(i)$ at each site $i \in \Lambda_{M,N}$. Let C^+ denote the set of configurations such that $\sigma(m, n) = 1$ if either $|m| = M$ or $|n| = N$. The partition function with $+$ boundary conditions is

$$Z_{M,N} = \sum_{C^+} \exp \sum_{\langle i,j\rangle} K(i, j)\sigma(i)\sigma(j),$$

where $K(i, j) = J/kT$ if i and j are nearest neighbors and is zero otherwise, T is temperature, and k is the Boltzmann constant. The transfer matrix used by Onsager and Kaufman is for periodic boundary conditions. Abraham and Martin-Löf [6] were the first to adapt this method to plus boundary conditions. To establish notation and for the reader's convenience, we summarize the results we need from [6].

Let A denote a subset of $\Lambda_{M,N}$ and define $\sigma_A = \Pi_{i\in A}\sigma(i)$. The correlation $\langle\sigma_A\rangle^+_{M,N}$ is defined by

$$\langle\sigma_A\rangle^+_{M,N} = Z^{-1}_{M,N} \sum_{C^+} \sigma_A \exp \sum_{\langle i,j\rangle} K(i, j)\sigma(i)\sigma(j).$$

The transfer matrix formalism splits the sum over all configurations into a multiple sum over the configurations of the rows. Let X denote the set of maps from $\{-M, \ldots, M\}$ into $\{-1, 1\}$. Then X is naturally identified with the configurations of a row. We will write $+$ for the element in X whose values are all $+1$.

Suppose $b \in \mathbb{R}$ and $\alpha_n \in X$ ($n = -N, \ldots, N$). We define

$$V_1(\alpha_{n+1}, \alpha_n) = e^{-2b} \prod_{m=-M}^{M} \exp K(m)\alpha_{n+1}(m)\alpha_n(m),$$

$$V_2(\alpha_n) = \prod_{m=-M}^{M-1} \exp K\alpha_n(m+1)\alpha_n(m), \tag{1.1}$$

where

$$\begin{aligned} K(m) &= b, \qquad |m| = M, \\ &= K, \qquad |m| \neq M. \end{aligned}$$

Let A_n denote the intersection of A with the nth row of $\Lambda_{M,N}$ and write σ_{A_n} for the product of the spin values in A_n (with $\sigma_\varnothing = 1$). We define

$$\begin{aligned} Z_{M,N}(b) = \prod_{n=-N+1}^{N-1} \sum_{\alpha_n \in X} & V_1(+, \alpha_{N-1})V_2(\alpha_{N-1}) \\ & \times \cdots V_2(\alpha_{-N+1})V_1(\alpha_{-N+1}, +). \end{aligned} \tag{1.2}$$

Then

$$\begin{aligned} \langle \sigma_A \rangle^+_{M,N} = \lim_{b\to\infty} Z_{M,N}(b)^{-1} \prod_{n=-N+1}^{N-1} \sum_{\alpha_n \in X} & V_1(+, \alpha_{N-1})V_2(\alpha_{N-1})\sigma_{A_{N-1}} \\ & \times \cdots \sigma_{A_{-N+1}} V_1(\alpha_{-N+1}, +). \end{aligned} \tag{1.3}$$

To see this, observe first that the terms in (1.2) and (1.3) which involve b may be rewritten $\exp b(\alpha_{n+1}(-M)\alpha_n(-M) - 1)\exp b(\alpha_{n+1}(M)\alpha_n(M) - 1)$. In the limit $b \to \infty$ only the configurations for which $\alpha_{n+1}(-M)\alpha_n(-M) = \alpha_{n+1}(M)\alpha_n(M) = 1$ will survive. Since $\alpha_{-N}(k) = \alpha_N(k) = 1$ ($k = -M, \ldots, M$) it follows that only configurations in C^+ survive the limit $b \to \infty$ (this construction is used by Abraham and Martin-Löf [6]).

It is natural to regard $V_1(\alpha_n)$ and $V_2(\alpha_{n+1}, \alpha_n)$ as matrices of operators on $\mathbf{C}^X$. We identify $\mathbf{C}^X$ with $\prod_{m=-M}^{M} \otimes\, \mathbf{C}^2_m$ ($\mathbf{C}^2_m \equiv \mathbf{C}^2$) by the map which takes $\delta(\alpha, \cdot)$ (the function on X which is 1 on α and 0 for the other configurations) to

$$a_\alpha(M) \otimes a_\alpha(M-1) \cdots \otimes a_\alpha(-M), \qquad \text{where } 2a_\alpha(m) = \begin{bmatrix} 1 + \alpha(m) \\ 1 - \alpha(m) \end{bmatrix}.$$

In this representation, the products in (1.1) are tensor products of matrices. In particular $\lim_{b\to\infty}\exp b(\alpha_{n+1}(\pm M)\alpha_n(\pm M)-1)$, is the matrix of the identity on $\mathbf{C}^2_{\pm M}$ and $\exp K\alpha_{n+1}(m)\alpha_n(m)$ is the matrix $\begin{bmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{bmatrix}$ on $\mathbf{C}^2_m$. We write $\begin{bmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{bmatrix} = (2sh2K)^{1/2}\exp K^*\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, where K^* is defined by $sh2Ksh2K^*=1$ and introduce the notation

$$\sigma_i = I\otimes\cdots\otimes\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_i\otimes\cdots\otimes I,$$

$$C_j = I\otimes\cdots\otimes\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_j\otimes\cdots\otimes I,$$

$$\sigma_{A_n} = \prod_{(j,n)\in A_n}\sigma_j,$$

$$V_1 = \exp K^*\sum_{m=-M+1}^{M-1} C_m,$$

$$V_2 = \exp K\sum_{m=-M}^{M-1}\sigma_{m+1}\sigma_m.$$

If we let $V_M' = V_2^{1/2}V_1V_2^{1/2}$ and $+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\otimes\cdots\otimes\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then upon canceling common factors such as $(2sh2K)^{M+1/2}$ from both the numerator and denominator of (1.3) it follows that

$$\langle\sigma_A\rangle^+_{M,N} = \frac{\langle V_M'\sigma_{A_{N-1}}V_M'\sigma_{A_{N-2}}\cdots\sigma_{A_{-N+1}}V_M'+,+\rangle}{\langle (V_M')^{2N+1}+,+\rangle}, \tag{1.4}$$

where the inner product on $\prod_{m=-M}^{M}\otimes\mathbf{C}^2_m$ is the one derived from $\langle x,y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2$ on $\mathbf{C}^2$.

We wish to fix A and let $N\to\infty$ in (1.4). Further information about V_M' is useful and following Onsager and Kaufman we introduce a Clifford algebra which facilitates the analysis of V_M'. It is natural (for reasons that will appear later) to index the vector space of the Clifford algebra on the half-integer lattice (see Kadanoff and Ceva [31]). Let $I_M = \{-M,\ldots,M\}$ and define $W_M' = l^2(I_M - 1/2,\mathbf{C})\oplus l^2(I_M+1/2,\mathbf{C})$. Then an orthonormal basis for W_M' is given by $e_1(k) = \delta(k,\cdot)\oplus 0$, $k\in I_M - 1/2$ and $e_2(k) = 0\oplus\delta(k,\cdot)$, $k\in I_M+1/2$.

On $\prod_{m\in I_M}\mathbf{C}^2_m$ the Brauer–Weyl [12] representation of the Clifford relations is

$$p_k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\otimes\cdots\otimes\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{k-1/2}\otimes\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{k+1/2}\otimes I\cdots\otimes I,$$
$$k\in I_M - 1/2,$$

$$q_k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{k-3/2} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}_{k-1/2} \otimes I \otimes \cdots \otimes I,$$
$$k \in I_M + 1/2.$$

The map $e_1(k) \to p_k/\sqrt{2}$, $e_2(k) \to q_k/\sqrt{2}$ extends to a representation of the Clifford algebra, $\mathcal{C}(W'_M, P)$, on $\amalg_{m=-M}^{M} \mathbf{C}^2_m$. The conjugation P on W'_M is the standard one associated with the real subspace $l^2(I_m - 1/2, \mathbf{R}) \oplus l^2(I_M + 1/2, \mathbf{R})$ of W'_M. If $W'_M \ni w = \sum_{k \in I_M - 1/2} w_{1k} e_1(k) + \sum_{k \in I_M + 1/2} \cdot w_{2k} e_2(k)$ then we write $\sqrt{2}F(w) = \sum_{k \in I_M - 1/2} w_{1k} p_k + \sum_{k \in I_M + 1/2} w_{2k} q_k$. If g is an operator on $\amalg_{m=-M}^{M} \otimes \mathbf{C}^2_m$ such that $gF(w)g^{-1} = F(T(g)w)$ for some linear operator $T(g)$ on W'_M then we shall say g is an element of the Clifford group. Since $F(w_1)F(w_2) + F(w_2)F(w_1) = \langle w_1, Pw_2 \rangle$ it is easy to see that $T(g)$ must preserve the complex bilinear form $\langle \cdot, P \cdot \rangle$, i.e., $T(g)$ is P-orthogonal. Furthermore since $F(w), w \in W'_M$, generates an irreducible algebra on $\amalg_{m \in I_M} \mathbf{C}^2_m$ [12] it follows that g is determined up to multiplication by a constant by $T(g)$. The introduction of the Clifford algebra in the analysis of the Ising model proves useful precisely because both the spin operators σ_m and the transfer matrix V'_M are elements of the Clifford group. A considerable simplification is achieved by working with $T(\sigma_m)$ and $T(V'_M)$ rather than with σ_m and V'_M directly. We refer the reader to [12] for the results we require concerning Clifford algebras.

It is a straightforward consequence of the definitions of σ_m and p_k and q_k that

$$\begin{aligned} \sigma_m p_k \sigma_m^{-1} &= \operatorname{sgn}(m-k) p_k, \qquad k \in I_M - 1/2, m \in I_M, \\ \sigma_m q_k \sigma_m^{-1} &= \operatorname{sgn}(m-k) q_k, \qquad k \in I_M + 1/2, m \in I_M. \end{aligned} \tag{1.5}$$

For reasons that will be apparent momentarily, we let $H_M = \{-M + 1/2, \ldots, M - 1/2\}$ and define $W_M = l^2(H_M, \mathbf{C}^2)$. The vectors $e_1 = e_1(-M - 1/2)$ and $e_2 = e_2(M + 1/2)$ will play a special role and to distinguish them we write $W'_M = \mathbf{C}^2 \oplus W_M$ with $\mathbf{C}^2$ spanned by (e_1, e_2). Next we define a family of real orthogonal maps $s(m)$ on W_M by

$$\begin{aligned} s(m)e_1(k) &= \operatorname{sgn}(m-k)e_1(k), \qquad k \in H_M, m \in I_M, \\ s(m)e_2(k) &= \operatorname{sgn}(m-k)e_2(k), \qquad k \in H_M, m \in I_M. \end{aligned} \tag{1.6}$$

Consulting (1.5) one sees that $T(\sigma_m) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus s(m)$ for $m \in I_M$. The operators σ_m are determined by (1.5) and the additional requirement $\sigma_m^2 = I$ up to a sign. For plus boundary conditions below T_c we will resolve this ambiguity by insisting that $\langle \sigma_m \rangle^+ > 0$. Above T_c the sign ambiguity will be of no consequence since the odd correlations vanish.

The P-orthogonal $T'_M \stackrel{\text{def}}{=} T(V'_M)$ is more complicated. It may be computed by first calculating $T(V_2^{1/2})$ and $T(V_1)$ and then multiplying the results to get $T'_M = T(V_2^{1/2})T(V_1)T(V_2^{1/2})$. An easy way to see what $T(V_2^{1/2})$ and $T(V_1)$ are is to observe

$$V_1 = \exp K^* \sum_{m=-M+1}^{M-1} C_m = \exp iK^* \sum_{k=-M+1/2}^{M-3/2} p_k q_{k+1},$$
$$V_2 = \exp K \sum_{m=-M}^{M-1} \sigma_{m+1}\sigma_m = \exp -iK \sum_{k=-M+1/2}^{M-1/2} p_k q_k. \tag{1.7}$$

Performing the indicated calculations one finds that $T'_M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus T_M$, where T_M is the P-orthogonal on W_M defined by

$$\begin{aligned}
T_M e_1(k) &= -(1/2)e_1(k-1) + c_1c_2e_1(k) - (1/2)e_1(k+1) \\
&\quad - (i/2)(c_1 - s_1)e_2(k-1) + ic_2e_2(k) \\
&\quad - (i/2)(c_1 + s_1)e_2(k+1) \qquad k \in H_{M-1} \\
T_M e_1(-M+1/2) &= (1/2)((c_2+1)c_1 + (c_2-1))e_1(-M+1/2) \\
&\quad - (1/2)e_1(-M+3/2) + (i/2)(c_2+s_2)\times \\
&\quad e_2(-M+1/2) - (i/2)(c_1+s_1)e_2(-M+3/2) \\
T_M e_1(M-1/2) &= (1/2)((c_2+1) + (c_2-1)c_1)e_1(M-1/2) \\
&\quad - (1/2)e_1(M-3/2) + (i/2)(c_2+s_2)e_2(M-1/2) \\
&\quad - (i/2)(c_1-s_1)e_2(M-3/2) \qquad (1.8) \\
T_M e_2(k) &= -(1/2)e_2(k-1) + c_1c_2e_2(k) - (1/2)e_2(k+1) \\
&\quad + (i/2)(c_1+s_1)e_1(k-1) - ic_2e_1(k) \\
&\quad + (i/2)(c_1-s_1)e_1(k+1) \qquad k \in H_{M-1}, \\
T_M e_2(-M+1/2) &= (1/2)((c_2+1) + (c_2-1)c_1)e_2(-M+1/2) \\
&\quad - (1/2)e_2(-M+3/2) - (i/2)(c_2+s_2)\times \\
&\quad e_1(-M+1/2) + (i/2)(c_1-s_1)e_1(-M+3/2) \\
T_M e_2(M-1/2) &= (1/2)((c_2+1)c_1 + (c_2-1))e_2(M-1/2) \\
&\quad - (1/2)e_2(M-3/2) - (i/2)(c_2+s_2)e_1(M-1/2) \\
&\quad + (i/2)(c_1+s_1)e_1(M-3/2),
\end{aligned}$$

where $c_1 = ch2K^*$, $c_2 = ch2K$, $s_1 = sh2K^*$, and $s_2 = sh2K$ and we used $sh2Ksh2K^* = 1$.

Our first result is the identification of the $N \to \infty$ limit in (1.4) with a Fock expectation. Suppose W is a complex Hilbert space (even or infinite dimensional) with a distinguished conjugation P. The Fock representations of $\mathcal{C}(W, P)$ are parametrized by self-adjoint idempotents Q on W such that $QP + PQ = 0$. Let $Q_{\pm} = \frac{1}{2}(1 \pm Q)$, $Pw = \bar{w}$, and $W_{\pm} = Q_{\pm} W$. We write $A(W_+)$ for the complex alternating tensor algebra $\mathbf{C} \oplus W_+ \oplus (W_+ \otimes W_+) + \cdots$. On $A(W_+)$ there exist annihilation and creation operators $a(w)$, $a^*(v)$ for $w, v \in W_+$ which anticommute among themselves, satisfy $a^*(v)a(w) + a(w)a^*(v) = \langle v, w\rangle I$ and such that $a(v)1 = 0$, where $1 = 1 \oplus 0 \oplus 0 \cdots$ is the vacuum vector. The Q-Fock representation of $\mathcal{C}(W, P)$ is generated by $F(w) = a^*(Q_+ w) + a(\overline{Q_- w})$, $w \in W$. The Q-Fock state on $\mathcal{C}(W, P)$ is given by $\mathcal{C}(W, P) \ni g \to \langle F(g)1, 1\rangle$. We would like to identify the eigenvector for V'_M which has the largest eigenvalue with the vacuum vector for a Fock representation. Modulo a degeneracy in the spectrum of V'_M this proves possible.

To deal with this degeneracy, it is convenient to think of (1.4) in representation dependent terms. For finite-dimensional W, all Fock representations are unitarily equivalent [12]. Using (1.8) and (1.6) to characterize V'_M and σ_m, we may work in a representation which simplifies V'_M. Of course, (1.8) only determines V'_M up to multiplication by a constant. However, since the factor V'_M occurs the same number of times in the numerator and the denominator of (1.4) this ambiguity will not affect the correlations. If T is a P-orthogonal on W we will write $\Gamma(T)$ for an element of the Clifford group such that $\Gamma(T)F(w)\Gamma(T)^{-1} = F(Tw)$.

We next describe the representation (i.e., the choice of Q) in which the action of V'_M is simple. Since $T'_M = T(V_2^{1/2})T(V_1)T(V_2^{1/2})$ with $T(V_1)$ a positive self-adjoint operator and $T(V_2^{1/2})$ self-adjoint it follows that T'_M is also a positive self-adjoint operator. It is a result of Abraham and Martin-Löf [6] that T_M (on W_M) does not have 1 as an eigenvalue. Hence $\log T_M$ is invertible and we define Q_M to be the unitary part of the polar decomposition of $-\log T_M$. Since $\log T_M$ is self-adjoint it follows that Q_M is a self-adjoint idempotent. Furthermore, since T_M is P-orthogonal we have $T_M^* P T_M = P$ or using $T_M^* = T_M$ it follows that $PT = T^{-1}P$. Thus $P \log T_M = -(\log T_M)P$ and from this one may deduce that $PQ_M + Q_M P = 0$ (see, for example, p. 334 of Kato [33]). Note that $Q_M^+ = \frac{1}{2}(1 + Q_M)$ is the orthogonal projection on the eigenvectors for T_M with eigenvalues less than 1 and if we let $T_M^+ = Q_M^+ T_M$ then $\Gamma(T_M) = I \oplus T_M^+ \oplus (T_M^+ \otimes T_M^+) \oplus \cdots$ represents T_M in the Q_M-Fock representation of $\mathcal{C}(W_M, P)$ on $A(W_M^+)$. Since T_M^+ has eigenvalues strictly less than 1 the vacuum vector is clearly the unique eigenvector associated with the largest eigenvalue for $\Gamma(T_M)$ in this representation. If we define $Q'_M = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \oplus Q_M$ it is clear that $PQ'_M + Q'_M P = 0$ and that Q'_M is a self-adjoint idempotent.

Because the operators P, T'_M, s'_m, and Q'_M all respect the orthogonal decomposition $\mathbf{C}^2 \oplus W_M$ it is convenient to realize the Q'_M-Fock representation on the tensor product $A(\mathbf{C}) \otimes A(W_M^+)$ in the following manner:

$$\sqrt{2}\,F(e_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I,$$

$$\sqrt{2}\,F(e_2) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes I,$$

$$F(w) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes F(w), \qquad w \in W_M,$$

with vacuum vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes 1_{Q_M}$.

Evidently $\Gamma(T'_M) = I \otimes \Gamma(T_M)$ and $\Gamma(s'_m) = \Gamma\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \Gamma(s_m)$. Since $\Gamma\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ commutes with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and anticommutes with $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ we may take $\Gamma\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We then write $\Gamma(s'_M) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \Gamma(s_m)$ with the further normalization $\Gamma(s_m)^2 = I$ understood.

The vector $+$ is characterized as a unique common eigenvector for $\sigma_m (m = -M, \ldots, M)$ with eigenvalue 1. In the representation we are considering, this means $+ = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes (+)$ where $(+)$ on right side is the unique common eigenvector for $\Gamma(s_m)$ with eigenvalue 1.

If we substitute these results in (1.4) then this equation remains valid with V'_M replaced by $\Gamma(T_M)$, σ_m replaced by $\Gamma(s_m)$ and $+$ replaced by the common eigenvector for $\Gamma(s_m)$ on $A(W_M^+)$ with eigenvalue 1. The degeneracy in the spectrum of the transfer matrix has been removed and a standard spectral theory argument gives the $N \to \infty$ limit in (1.4) as a Fock expectation provided we know that the Q_M-Fock vacuum is not orthogonal to $(+)$. This is, however, a consequence of the lower bound in Appendix A of Abraham and Martin-Löf [6]. To simplify notation, we will write $\Gamma(T_M) = V_M$, and $\sigma_m = \Gamma(s_m)$ henceforth. We have sketched the proof of:

THEOREM 1.0. *Let $\langle \sigma_A \rangle^+_{M,\infty} = \lim_{N\to\infty} \langle \sigma_A \rangle^+_{M,N}$. Then*

$$\langle \sigma_A \rangle^+_{M,\infty} = \langle \sigma_{A_n} V_M \sigma_{A_{n-1}} \cdots V_M \sigma_{A_n} 1_{Q_M}, 1_{Q_M} \rangle, \tag{1.9}$$

where $A \subseteq [-M, M] \times [-n, n]$, $1_{Q_M} \in A(W_M^+)$ is the Q_M-Fock vacuum, $V_M = I \oplus T_M^+ \oplus (T_M^+ \otimes T_M^+) \oplus \cdots$ on $A(W_M^+)$ and $\sigma_m = \Gamma(s_m)$ with the further normalizations $\sigma_m^2 = I$ and $\langle \sigma_m \rangle^+_{M,\infty} > 0$ determining σ_m uniquely.

2.0. It is rather easy to guess the $M \to \infty$ limit in (1.9). Let $W = l^2(\mathbf{Z}_{1/2}, \mathbf{C}^2)$, where $\mathbf{Z}_{1/2} = \mathbf{Z} + 1/2$ and write $T = \lim_{M\to\infty} T_M$(informally). Then supposing that the boundary behavior does not play a role in the infinite-volume limit for T_M one has

$$\begin{aligned} T_M e_1(k) = &- (1/2)e_1(k-1) + c_1c_2e_1(k) - (1/2)e_1(k+1) \\ &- (i/2)(c_1 - s_1)e_2(k-1) + ic_2e_2(k) \qquad k \in \mathbf{Z}_{1/2}, \\ &- (i/2)(c_1 + s_1)e_2(k+1) \end{aligned} \tag{2.0}$$

$$\begin{aligned} T_m e_2(k) = &- (1/2)e_2(k-1) + c_1c_2e_2(k) - (1/2)e_2(k+1) \\ &+ (i/2)(c_1 + s_1)e_1(k-1) - ic_2e_1(k) \qquad k \in \mathbf{Z}_{1/2}, \\ &+ (i/2)(c_1 - s_1)e_1(k+1). \end{aligned}$$

In order to diagonalize T we introduce the Fourier transform $\hat{f}(\theta) = \Sigma_{k \in \mathbf{Z}_{1/2}} e^{ik\theta} f(k) (\theta \in (-\pi, \pi])$ for functions $f(k) = \begin{bmatrix} f_1(k) \\ f_2(k) \end{bmatrix}$ in $l^2(\mathbf{Z}_{1/2}, \mathbf{C}^2)$. In the Fourier transform variables T is the matrix-valued multiplication operator

$$\begin{aligned} T\hat{f}(\theta) &= \begin{bmatrix} c_1c_2 - \cos\theta, & s_1 \sin\theta - i(c_2 - c_1\cos\theta) \\ s_1\sin\theta + i(c_2 - c_1\cos\theta), & c_1c_2 - \cos\theta \end{bmatrix} \hat{f}(\theta) \\ &\equiv T(\theta)\hat{f}(\theta). \end{aligned} \tag{2.1}$$

We will take (2.1) for the definition of T and to simplify the description of the associated Q we follow Onsager [52] and introduce functions $\gamma(\theta) > 0$ and $\alpha(\theta)$ (called $\delta^*(w)$ by Onsager) defined by

$$\begin{aligned} ch\gamma(\theta) &= c_1c_2 - \cos\theta \\ sh\gamma(\theta)e^{i\alpha(\theta)} &= (c_2 - c_1\cos\theta) + is_1\sin\theta. \end{aligned} \tag{2.2}$$

The identity $ch^2\gamma - sh^2\gamma = 1$, which must be true for this definition to make sense is easily checked. Substituting (2.2) in (2.1) one finds

$$T(\theta) = \exp\left[\gamma(\theta)\begin{bmatrix} 0 & -ie^{i\alpha(\theta)} \\ ie^{-i\alpha(\theta)} & 0 \end{bmatrix}\right].$$

Thus Q is multiplication by $Q(\theta) = \begin{bmatrix} 0 & ie^{i\alpha(\theta)} \\ -ie^{-i\alpha(\theta)} & 0 \end{bmatrix}$ in the Fourier transform variables. Let $Q_\pm = (1/2)(1 \pm Q)$, $T_\pm = Q_\pm T$, and $W_\pm = Q_\pm W$. Then $\Gamma(T) = I \oplus T_+ \oplus (T_+ \otimes T_+) \oplus \cdots$ is a contraction on $A(W_+)$.

We define orthogonal maps s_m on W by

$$s_m e_j(k) = \operatorname{sgn}(m-k)e_j(k) \qquad (k \in \mathbf{Z}_{1/2}, m \in \mathbf{Z}, j = 1, 2). \tag{2.3}$$

We would like to define $\sigma_m = \Gamma(s_m)$ in the Q-Fock representation. Maps $\Gamma(s)$ implementing arbitrary P-orthogonals s do not exist in infinite-dimensional Fock representations. However, since each s_m is a real orthogonal map which commutes with P the necessary and sufficient condition for the existence of $\Gamma(s_m)$ is that the commutator $[s_m, Q]$ must be a Schmidt class operator on W [55, 66]. Since $[s_m, Q]$ is an integral operator in the Fourier transform variables with a bounded kernel this criterion is satisfied. Since $s(m)^2 = 1$ we may further normalize σ_m so that $\sigma_m^2 = I$. The representation for the infinite-volume correlations which suggests itself is

$$\langle \sigma_A \rangle^+ = \langle \sigma_{A_n} \Gamma(T) \sigma_{A_{n-1}} \cdots \Gamma(T) \sigma_{A_{-n}} 1_Q, 1_Q \rangle. \tag{2.4}$$

The rest of this section is devoted to a proof that this is correct for $T < T_c$ with the sign ambiguity resolved so that $\langle \sigma_m 1_Q, 1_Q \rangle > 0$.

For technical reasons our proof does not work for plus boundary conditions above T_c. This is probably a consequence of the fact that above T_c plus boundary conditions are not especially natural; the finite-volume disposition of spins to point up is "washed out" in the infinite-volume limit. In any case we use a variant of Kramers–Wannier duality to relate open boundary condition correlations above T_c to correlations for disorder variables with plus boundary conditions below T_c. The details of this analysis will be given in Section 3.

We now make preparations to state the results from [56] which we shall use. Suppose W is a Hilbert space with distinguished conjugation P and let Q be a self-adjoint idempotent on W which anticommutes with P. The real subspace of W relative to P is $\mathcal{X} \overset{\text{def}}{=} P_+ W$, where $P_\pm = (1/2)(1 \pm P)$. The complex structure i maps $P_- W$ onto $P_+ W$ and preserves the real orthogonal structure. If we identify W with $\mathcal{X} \oplus \mathcal{X}$ via the map $I \oplus (-i): P_+ W \oplus P_- W \to \mathcal{X} \oplus \mathcal{X}$, then in this representation P becomes $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the complex structure i on W becomes $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ on $\mathcal{X} \oplus \mathcal{X}$. Since iQ commutes with both P and i it follows that iQ has the matrix representation $\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}$ on $\mathcal{X} \oplus \mathcal{X}$. Since iQ is a complex structure it follows that Λ must be a complex structure on $\mathcal{X}$. The matrix representation of Q is thus $-i(iQ) = \begin{bmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{bmatrix}$. Now consider the real orthogonal map $D = (1/\sqrt{2}) \begin{bmatrix} 1 & \Lambda \\ 1 & -\Lambda \end{bmatrix}$. Then D is a unitary map from $\left(\mathcal{X} \oplus \mathcal{X}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$ to $(\mathcal{X} \oplus \mathcal{X}, \Lambda \oplus (-\Lambda))$, $D \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $D \begin{bmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{bmatrix} D^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ so that D diagonalizes Q. We will refer to $(\mathcal{X} \oplus \mathcal{X}, \Lambda \oplus (-\Lambda))$ as the Q representation of W since Q is diagonal in this representation.

One may verify without difficulty that $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ is the matrix of a P-orthogonal in the Q-representation of W if and only if

$$\begin{aligned} &T_{21}^{*}T_{11} + T_{11}^{*}T_{21} = 0, \qquad && T_{kk}\Lambda = \Lambda T_{kk},\ k = 1,2, \\ &T_{22}^{*}T_{12} + T_{12}^{*}T_{22} = 0, && T_{kj}\Lambda = -\Lambda T_{kj},\ k \neq j, \\ &T_{21}^{*}T_{12} + T_{11}^{*}T_{22} = 1. \end{aligned} \tag{2.5}$$

Let $w \to F(w)$ denote the Q-Fock representation of $\mathcal{C}(W, P)$ and define $G(W, Q)$ to be the collection of bounded operators on $A(W_+)$ such that $gF(w) = F(T(g)w)g$ for some bounded invertible P-orthogonal $T(g)$ on W. We will say $g \in G(W, Q)$ is factorable if the matrix element $T_{22}(g)$ of $T(g)$ in the Q-representation of W is invertible. If $g \in G(W, Q)$ is factorable we write

$$L(g) = \begin{bmatrix} T_{22}(g)^{*-1} & 0 \\ 0 & T_{22}(g) \end{bmatrix},$$

$$\Delta R(g) = \begin{bmatrix} 0 & T_{12}T_{22}^{-1}(g) \\ T_{22}^{-1}T_{21}(g) & 0 \end{bmatrix}. \tag{2.6}$$

Now suppose g_i is a factorable element of $G(W, Q)$ for $i = 1, \ldots, r$. Define ΔR to be the $r \times r$ block diagonal matrix with entries $(\Delta R)_{ij} = \delta_{ij}\Delta R(g_j)$ and define L to be the $r \times r$ block matrix with entries

$$\begin{aligned} L_{ij} &= -Q_+L(g_{i+1}) \cdots L(g_{j-1}), \qquad && j > i+1, \\ &= \quad -Q_+, && j = i+1, \\ &= \quad 0, && j = i, \\ &= \quad Q_-, && j = i-1, \\ &= Q_-L^{-1}(g_{i-1}) \cdots L^{-1}(g_{j+1}), && j < i-1. \end{aligned} \tag{2.7}$$

Writing $\langle g1_Q, 1_Q \rangle = \langle g \rangle_Q$ the theorem from [56] we wish to use is:

THEOREM (2.0). *Suppose g_i is a factorable element of $G(W, Q)$ for $i = 1, \ldots, r$. Then if $\langle g_1 \cdots g_r \rangle_Q \neq 0$ we have*

$$\langle g_1 \cdots g_r \rangle_Q^2 = \prod_{i=1}^{r} \langle g_i \rangle_Q^2 \det{}_2(I + L\Delta R).$$

The regularized determinant, $\det_2$, is defined in [67] and we refer the reader to this paper for the properties of $\det_2$ we will use.

A second result we require is

THEOREM (2.1). *Suppose g is a factorable element of $G(W,Q)$. Then* $|\langle g\rangle_Q|^2 = \|g1\|^2[\det(I+|T_{12}(g)T_{22}^{-1}(g)|^2)]^{-1/2}$.

This result is proved in Lemma 3.0 of [56].

We will apply these formulas to the semi-infinite correlations and then prove the convergence of the determinants explicitly. We begin, however, by identifying the various operators which appear in the final (infinite-volume) result. The reason for starting with these calculations is that the spin operators σ_i are factorable elements of $G(W,Q)$ only for $T<T_c$; this is apparent when one attempts to compute $T_{22}^{-1}(\sigma_i)$ in the infinite-volume limit.

If (m_i, n_i) are points in $\mathbf{Z}^2$ $(i=1,\ldots,r)$ with $n_1 \le n_2 \cdots \le n_r$ then the putative infinite-volume correlation $\langle\sigma(m_1,n_1)\cdots\sigma(m_r,n_r)\rangle_+$ is

$$\langle\sigma_{m_1}V^{n_2-n_1}\sigma_{m_2}\cdots V^{n_r-n_{r-1}}\sigma_{m_r}\rangle_Q.$$

Let n_{r+1} be an integer such that $n_r \le n_{r+1}$ and define $g_i = \sigma_{m_i}V^{(n_{i+1}-n_i)}$, $i=1,\ldots,r$. Then since $V1_Q = 1_Q$ it follows that:

$$\langle\sigma(m_1,n_1)\cdots\sigma(m_r,n_r)\rangle_+ = \langle g_1\cdots g_r\rangle_Q.$$

The P-real subspace of $l^2(\mathbf{Z}_{1/2},\mathbf{C}^2)$ is $\mathcal{X} = l^2(\mathbf{Z}_{1/2},\mathbf{R}^2)$ and it is clear that the Q representation of $T(V)$ is multiplication by $\begin{bmatrix} e^{-\gamma(\theta)} & 0 \\ 0 & e^{\gamma(\theta)}\end{bmatrix}$ in the Fourier transform variables. The action of $s(m) = T(\sigma_m)$ on W is given by (2.3). Since $s(m)$ commutes with i it follows that the Q representation of $s(m)$ is given by $D\begin{bmatrix} s(m) & 0 \\ 0 & s(m)\end{bmatrix}D^* = \begin{bmatrix} A(m) & B(m) \\ B(m) & A(m)\end{bmatrix}$, where $A(m) = (1/2)(s(m) - \Lambda s(m)\Lambda)$ is the Λ-linear part of $s(m)$ and $B(m) = (1/2)(s(m)+\Lambda s(m)\Lambda)$ is the Λ-skew linear part of $s(m)$. Evidently $s(m) = A(m)+B(m)$. The Q-representation of $T(g_i) = T(\sigma_{m_i})T^{n_{i+1}-n_i}$ is thus

$$T(g_i) = \begin{bmatrix} A(m_i) & B(m_i) \\ B(m_i) & A(m_i)\end{bmatrix}\begin{bmatrix} e^{-(n_{i+1}-n_i)\gamma} & 0 \\ 0 & e^{(n_{i+1}-n_i)\gamma}\end{bmatrix}.$$

If we suppose further that $A(m)$ is invertible then using $A(m_i)^* = A(m_i)$ one finds

$$L(g_i) = \begin{bmatrix} A(m_i)^{-1} & 0 \\ 0 & A(m_i)\end{bmatrix}\begin{bmatrix} e^{-(n_{i+1}-n_i)\gamma} & 0 \\ 0 & e^{(n_{i+1}-n_i)\gamma}\end{bmatrix},$$

$$\Delta R(g_i) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(n_{i+1}-n_i)\gamma}\end{bmatrix}\begin{bmatrix} 0 & BA^{-1}(m_i) \\ A^{-1}B(m_i) & 0\end{bmatrix}\begin{bmatrix} e^{-(n_{i+1}-n_i)\gamma} & 0 \\ 0 & 1\end{bmatrix}. \tag{2.8}$$

When $T < T_c$ it is true that $A(m)$ is invertible and we proceed next with the calculation of $A^{-1}(m)$. First observe that $A(m) = e^{im\theta}Ae^{-im\theta}$, where A is written for $A(0)$. Next factor $A = -(\Lambda/2)(s\Lambda + \Lambda s)$, where $s = s(0)$. To avoid introducing new notation we shall also use s to denote the map $sf(k) = -\text{sgn}(k)f(k)$ for real-valued f in $l^2(\mathbb{Z}_{1/2}, \mathbb{R})$. Let $s_\pm = (1/2)(1 \mp s)$, recall $\Lambda = \begin{bmatrix} 0 & -e^{i\alpha} \\ e^{-i\alpha} & 0 \end{bmatrix}$, and observe that

$$\Lambda\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\Lambda$$
$$= 2\begin{bmatrix} 0 & \left(s_+e^{i\alpha}s_+ - s_-e^{i\alpha}s_-\right) \\ \left(-s_+e^{-i\alpha}s_+ + s_-e^{-i\alpha}s_-\right) & 0 \end{bmatrix}.$$

Thus to invert $A(m)$ we need only invert $s_+\, e^{\pm i\alpha}s_+$ and $s_-\, e^{\pm i\alpha}s_-$. Since s_+ (s_-) is the projection on the space of functions on the unit circle all of whose negative (positive) half-integer Fourier coefficients vanish, we may use the standard Wiener–Hopf techniques to invert $s_+\, e^{i\alpha}s_+$ ($s_-\, e^{i\alpha}s_-$) when the index of $e^{i\alpha(\theta)}$ is zero [37].

The following representation of $e^{i\alpha(\theta)}$ is well known [40]:

$$e^{i\alpha(\theta)} = \left[\frac{\left(1 - \alpha_1 e^{i\theta}\right)\left(1 - \alpha_2 e^{-i\theta}\right)}{\left(1 - \alpha_1 e^{-i\theta}\right)\left(1 - \alpha_2 e^{i\theta}\right)}\right]^{1/2},$$

where $\alpha_1 = thKthK^* = e^{-2K}thK$ and $\alpha_2 = thK^*/thK = e^{-2K}cthK$ and the branch of the square root is chosen so that $e^{i\alpha(\pi)} > 0$. The condition $T = T_c$ is characterized by $\alpha_2 = 1$, $T < T_c$ is characterized by $0 < \alpha_1 < \alpha_2 < 1$, and $T > T_c$ is characterized by $0 < \alpha_1 < 1 < \alpha_2$. For $T < T_c$ the index of $e^{i\alpha(\theta)}$ is zero and one has the factorization

$$e^{i\alpha(\theta)} = a_+(e^{i\theta})a_-(e^{i\theta}),$$

where

$$a_+(z) = \left[\frac{1 - \alpha_1 z}{1 - \alpha_2 z}\right]^{1/2} \quad \text{and} \quad a_-(z) = \left[\frac{1 - \alpha_2 z^{-1}}{1 - \alpha_1 z^{-1}}\right]^{1/2}.$$

The function $a_+(z)$ is analytic in the interior of the unit disk and $a_-(z)$ is analytic in the exterior of the unit disk.

For the convenience of the reader we present a simple argument to deal with half-integer Fourier transforms. Consider first $s_+e^{\pm i\alpha}s_+$. The map u_+

defined by

$$u_+ f(\theta) = \sum_{k \in \mathbf{Z}_{1/2}} f(k) e^{i(k-1/2)\theta}$$

is an isometry from $l^2(\mathbf{Z}_{1/2}, \mathbb{R})$ *into* $L^2([-\pi, \pi], \mathbf{C})$. Evidently s_+ projects $u_+ f$ onto the component which has an analytic extension into the interior of the unit disk. Furthermore $e^{i\alpha}$ is multiplication by $e^{i\alpha(\theta)}$ on $u_+ f(\theta)$ since u_+ is just the half-integer Fourier transform multiplied by $e^{-i\theta/2}$. The standard Wiener-Hopf method [37] applies and one finds

$$[s_+ e^{i\alpha} s_+]^{-1} = [s_+ a_+ a_- s_+]^{-1} = a_+^{-1} s_+ a_-^{-1},$$

$$[s_+ e^{-i\alpha} s_+]^{-1} = [s_+ a_+^{-1} a_-^{-1} s_+]^{-1} = a_+ s_+ a_-,$$

on $s_+ \, l^2(\mathbf{Z}_{1/2}, \mathbb{R})$.

To deal with $s_- \, e^{\pm i\alpha} s_-$ define $u_- f(\theta) = \sum_{k \in \mathbf{Z}_{1/2}} f(k) e^{i(k+1/2)\theta}$ and apply the Wiener-Hopf technique. The result is

$$[s_- e^{i\alpha} s_-]^{-1} = [s_- a_+ a_- s_-]^{-1} = a_-^{-1} s_- a_+^{-1},$$

$$[s_- e^{-i\alpha} s_-]^{-1} = [s_- a_+^{-1} a_-^{-1} s_-]^{-1} = a_- s_- a_+,$$

on $s_- l^2(\mathbf{Z}_{1/2}, \mathbb{R})$.

Turning to the calculation of A^{-1} and $A^{-1}B$ one finds

$$A^{-1} = \begin{bmatrix} (-a_+ s_+ a_+^{-1} + a_- s_- a_-^{-1}) & 0 \\ 0 & (-a_+^{-1} s_+ a_+ + a_-^{-1} s_- a_-) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} (a_+ s a_+^{-1} + a_- s a_-^{-1}) & 0 \\ 0 & (a_+^{-1} s a_+ + a_-^{-1} s a_-) \end{bmatrix}, \tag{2.9}$$

$$A^{-1}B = A^{-1}s - 1 = \frac{1}{2} \begin{bmatrix} (a_+ s a_+^{-1} - a_- s a_-^{-1}) & 0 \\ 0 & (a_+^{-1} s a_+ - a_-^{-1} s a_-) \end{bmatrix}.$$

These representations will be of further use when we consider the scaling limit. For present purposes we require only the result that A is invertible for $T < T_c$ and that B is a Schmidt class operator.

We will write P_M for the orthogonal projection on the real linear span of $\{e_j(k): |k| < M \text{ and } j = 1, 2\}$. To avoid cumbersome notation we shall also write P_M for $P_M \oplus P_M$ on $\mathcal{X} \oplus \mathcal{X}$. The following lemma will play an important role in the proof of the principal theorem of this section.

LEMMA 2.0. *Suppose A_n is a sequence of bounded operators on a Hilbert space which converges strongly to A. Suppose B_n is a sequence of Schmidt class operators which converges in Schmidt norm to B. Then A_nB_n converges in Schmidt norm to AB.*

Proof. Let $\{e_k\}$ denote an orthonormal basis for the Hilbert space. Then $\|AB - A_nB_n\|_2^2 = \Sigma_k \|(AB - A_nB_n)e_k\|^2 \leq 2\Sigma_k \|(A - A_n)Be_k\|^2 + 2\Sigma_k \|A_n(B - B_n)e_k\|^2$, where we have written $\|\cdot\|_2$ for the Schmidt norm. Since A_n converges strongly the operators norms $\|A_n\|$ are uniformly bounded by some constant C. Thus $\|(A - A_n)Be_k\|^2 \leq 4C^2\|Be_k\|^2$ and it follows from dominated convergence that $\lim_{n\to\infty}\Sigma_k\|(A - A_n)Be_k\|^2 = 0$. Since $\Sigma_k \|A_n(B - B_n)e_k\|^2 \leq C^2\|B - B_n\|_2^2$ we also have $\lim_{n\to\infty}\Sigma_k\|A_n(B - B_n)e_k\|^2 = 0$. □

In the following two lemmas we shall implicitly suppose that operators X_M on $P_M\mathcal{X}$(or P_MW) are extended to act as $X_M \oplus 0$ on $P_M\mathcal{X} \oplus (1 - P_M)\mathcal{X}$ (or $P_MW \oplus (1 - P_M)W$)).

LEMMA 2.1. *The operators $T_M^{\pm 1}$ converge strongly to $T^{\pm 1}$ on W as $M \to \infty$. The operator Q_M converges strongly to Q on W as $M \to \infty$.*

Proof. It is obvious from (1.8) that if ν is a finite linear combination of the basis vectors $\{e_\alpha(k)|\alpha = 1,2, k \in \mathbf{Z}_{1/2}\}$ then $T_M\nu = T\nu$ for all sufficiently large M. Since T_M is uniformly bounded in operator norm, strong convergence on the dense set of finite linear combinations of the vectors $e_\alpha(k)$ implies strong convergence on W. Observe that $P_MT^{-1}P_M$ converges strongly to T^{-1} as $M \to \infty$. However, $P_MT^{-1}P_M - T_M^{-1} = P_MT_M^{-1}(T_M - T)T^{-1}P_M$ and since $T_M - T$ goes strongly to zero and T_M^{-1} is uniformly bounded (the spectrum of T_M has a uniform gap in its spectrum about 0 [6]) it follows that $\text{s-lim}_{M\to\infty}(T^{-1} - T_M^{-1}) = 0$.

To prove the convergence of Q_M to Q observe first that if X_M is a sequence of self-adjoint operators which converges strongly to X and λ is a point of continuity for the spectral resolution $E(\lambda)$ associated with X then $\text{s-lim}_{M\to\infty}E_M(\lambda) = E(\lambda)$, where $E_M(\lambda)$ is the spectral resolution of X_M. This is a special case of Theorem 1.15 in Chapter VIII of Kato [33]. If we let $X_M = T_M$ then $Q_M = 2E_M(1) - I$ and $Q = 2E(1) - I$. Thus since T has a gap in its spectrum about 1 it follows that $\text{s-lim}_{M\to\infty}Q_M = Q$. □

One consequence of Lemma 2.1 is that $T_M^{\pm 1}$ in the Q_M representation converges strongly to T in the Q representation. To see this observe that the Q_M representation of $T_M^{\pm 1}$ is $D_MT_M^{\pm 1}D_M^*$ with $D_M = 2^{-1/2}\begin{bmatrix} 1 & \Lambda_M \\ 1 & -\Lambda_M \end{bmatrix}$. However, $\Lambda_M = iQ_MP_+$ so that $\text{s-lim}_{M\to\infty}\Lambda_M = \Lambda$ by Lemma 2.1 and it follows that $\text{s-lim}_{M\to\infty}D_MT_M^{\pm 1}D_M^* = DT^{\pm 1}D^*$.

The last preliminary before theorem 2.2 is the following lemma.

LEMMA 2.2. *Suppose that $T < T_c$. Then*

$$\lim_{M \to \infty} \| B(m) - B_M(m) \|_2 = 0,$$

$$\lim_{M \to \infty} \| A^{\pm 1}(m) - A_M^{\pm 1}(m) \|_2 = 0.$$

Proof. We begin with the proof that $\lim_{M \to \infty} \| B(m) - B_M(m) \|_2 = 0$. Since $B_M(m) = (1/2)(s_M(m) + \Lambda_M s_M(m) \Lambda_M) = (\Lambda_M/2)[s_M(m), \Lambda_M]$ and Λ_M converges strongly to Λ it suffices in view of Lemma 2.0 to prove that $[s_M(m), \Lambda_M]$ converges in Schmidt norm to $[s(m), \Lambda]$. However, since $[s(m), \Lambda]$ is a Schmidt class operator it follows that $P_M[s(m), \Lambda] P_M$ converges in Schmidt norm to $[s(m), \Lambda]$ as $M \to \infty$. Thus, since $s_M(m) = P_M s(m) P_M$, we are reduced to proving that $P_M[s(m), (\Lambda - \Lambda_M)] P_M$ tends to zero in Schmidt norm as $M \to \infty$. In order to deal with the difference $\Lambda - \Lambda_M$ we shall use the following contour integral representations for Λ and Λ_M (see, for example, p. 359 of Kato [33]):

$$\Lambda = -\frac{1}{\pi} \int_{\mathrm{Re}\, z = 1} (T - z)^{-1} \, dz,$$

$$\Lambda_M = -\frac{1}{\pi} \int_{\mathrm{Re}\, z = 1} (T_M - z)^{-1} \, dz.$$

Each of these integrals is understood as a symmetric strong limit $\int_{\mathrm{Re}\, z=1} \cdots dz = \lim_{N \to \infty} \int_{1-iN}^{1+iN} \cdots dz$. These formulas may be proved simply in the spectral representation for T (or T_M) using the fact that $+1$ is not in the spectrum of either T or T_M [6]. The simple estimates we use require additional information about the spectrum of T and T_M. The function $\gamma(\theta)$ has a positive lower bound for $\theta \in [-\pi, \pi]$ and $T < T_c$, so that the spectrum of $T = \{ e^{\pm \gamma(\theta)} | \theta \in [-\pi, \pi] \}$ has a gap around $+1$. For $T < T_c$, Abraham and Martin-Löf prove that the spectrum of T_M is a subset of $\{ e^{\pm \gamma(\theta)} | \theta \in [-\pi, \pi] \}$. Thus the gap in the spectrum of T_M is uniform in M. (As noted in [6] there is an exponentially small gap above T_c as $M \to \infty$ and this is the reason the proof in this section does not work for $T > T_c$.)

If we let "a" denote the gap in the spectrum of T about $+1$ then we have the elementary estimates

$$\left.\begin{aligned} \|(T - z)^{-1}\| &\le \frac{1}{(a^2 + y^2)^{1/2}} \\ \|(T_M - z)^{-1}\| &\le \frac{1}{(a^2 + y^2)^{1/2}} \end{aligned}\right\} \quad \mathrm{Re}\, z = 1,\ y = \mathrm{Im}\, z \tag{2.11}$$

Using the contour integral representations for Λ and Λ_M one computes

$$[s(m), \Lambda] = \frac{1}{\pi}\int_{\operatorname{Re} z=1} (T-z)^{-1}[s(m), T](T-z)^{-1}\,dz,$$

$$[s(m), \Lambda_M] = \frac{1}{\pi}\int_{\operatorname{Re} z=1} (T_M-z)^{-1}[s(m), T_M](T_M-z)^{-1}\,dz. \tag{2.13}$$

Since $[s(m), T_M] = [s(m), T]$ when $M > |m| + 1$ it follows that

$$\begin{aligned}[s(m), \Lambda_M - \Lambda] = \frac{1}{\pi}\int_{\operatorname{Re} z=1} &(T_M - z)^{-1}(T - T_M) \\ &\times (T-z)^{-1}[s(m), T](T-z)^{-1}\,dz + \frac{1}{\pi}\int_{\operatorname{Re} z=1}(T_M - z)^{-1} \\ &\times [s(m), T](T-z)^{-1}(T - T_M)(T_M - z)^{-1}\,dz \end{aligned} \tag{2.14}$$

for M larger than $|m| + 1$.

In the basis $\{e_\alpha(k)\}$ for $\mathcal{X}$ the matrix of $[s(m), T]$ has only 16 nonvanishing matrix elements connecting $\{e_j(m \pm 1/2), j = 1,2\}$. On the other hand $(T - T_M)P_M$ and $P_M(T - T_M)$ vanish except on the basis vectors $\{e_j(M - 1/2), e_j(-M + 1/2), j = 1,2\}$. If one multiplies (2.14) on both sides by P_M and observes that P_M and T_M commute then $P_M(T - T_M)(T - z)^{-1}[s(m), T]$ and $[s(m), T](T - z)^{-1}(T - T_M)P_M$ are each encountered as factors in the resulting integrands. One may estimate the Schmidt norm of each of these factors by an M independent constant times the square root of the sum of the squares of the 4×4 matrix elements of $(T - z)^{-1}$ connecting $\{e_j(m \pm 1/2), j = 1,2\}$ with $\{e_j(\pm M \mp 1/2), j = 1,2\}$. A typical such matrix element may be estimated using

$$\left\| \int_{-\pi}^{\pi} (T(\theta) - z)^{-1} e^{i(m \pm M)\theta}\,d\theta \right\| \leq \frac{C}{|m \pm M|}(a^2 + y^2)^{-1},$$

$$\operatorname{Re} z = 1,\ y = \operatorname{Im} z.$$

This result is easily obtained in one integration by parts using the fact that $T(\theta)$ is a differentiable 2π periodic function. It follows that the Schmidt norms of the finite-rank operators $P_M(T - T_M)(T - z)^{-1}[s(m), T]$ and $[s(m), T](T - z)^{-1}(T - T_M)P_M$ are dominated by (const.$/|m \pm M|$) $(a^2 + y^2)^{-1}$. This observation, the uniform bounds in (2.11), and (2.14) suffice to show that $P_M[s(m), \Lambda - \Lambda_M]P_M$ tends to zero in Schmidt norm as $M \to \infty$. This completes the demonstration that $\lim_{M\to\infty} \| B(m) - B_M(m)\|_2 = 0$.

Since $s_M(m) = P_M s(m) P_M$, $s(m) = A(m) + B(m)$, and $s_M(m) = A_M(m) + B_M(m)$ it follows that $P_M A(m) P_M - A_M(m) = -(P_M B(m) P_M - B_M(m))$ and hence that $\lim_{M\to\infty} \| P_M A(m) P_M - A_M(m) \|_2 = 0$.

It remains to show that $P_M A^{-1}(m) P_M$ and $A_M^{-1}(m)$ are close in Schmidt norm for large M. Observe first that since $P_M s(m)(1 - P_M) = (1 - P_M) s(m) P_M = 0$ we have

$$P_M A(m) P_M + (1 - P_M) A(m)(1 - P_M) = A(m) + E_M,$$

where $E_M = (1 - P_M) B(m) P_M + P_M B(m)(1 - P_M)$.

However, since $A(m)$ is invertible and E_M tends to zero in uniform norm as $M \to \infty$ it follows that $P_M A(m) P_M$ is invertible on $P_M \mathfrak{X}$ for all sufficiently large M and that $(P_M A(m) P_M)^{-1}$ is uniformly bounded in operator norm as $M \to \infty$. But $A_M(m)$ is arbitrarily close to $P_M A(m) P_M$ in operator norm for large M and so since $P_M A(m) P_M$ has a uniform gap in its spectrum about 0 it follows that $A_M(m)$ is invertible for all sufficiently large M and that $A_M^{-1}(m)$ (on $P_M \mathfrak{X}$) is uniformly bounded in operator norm as $M \to \infty$. We may also assert that $\lim_{M\to\infty} \| (P_M A(m) P_M)^{-1} - A_M^{-1}(m) \|_2 = 0$ (on $P_M \mathfrak{X}$) since $A_M(m)$ and $P_M A(m) P_M$ are close in Schmidt norm for large M.

To finish the proof we need only show that $P_M A^{-1}(m) P_M - (P_M A(m) P_M)^{-1}$ (on $P_M \mathfrak{X}$) converges to zero in Schmidt norm as $M \to \infty$. However,

$$P_M\left[A^{-1}(m) - (P_M A(m) P_M)^{-1}\right] P_M$$
$$= P_M A^{-1}(m)\left[P_M A(m) - A(m) P_M\right](P_M A(m) P_M)^{-1}.$$

But $[P_M, A(m)] = -[P_M, B(m)]$ and $\lim_{M\to\infty} \| [P_M, B(m)] \|_2 = 0$. This last limit is a consequence of the fact that the square of the Schmidt norm of $[P_M, B(m)] = (P_M - 1) B(m) P_M + P_M B(m)(1 - P_M)$ (the sum of the squares of the matrix elements in the basis $\{e_j(k)\}$) is the "tail end" of the convergent series for the square of Schmidt norm of $2B(m)$. Coupling this observation and the previously noted fact that $(P_M A(m) P_M)^{-1}$ is uniformly bounded as $M \to \infty$ it follows that $P_M A^{-1}(m) P_M - (P_M A(m) P_M)^{-1}$ does tend to zero in Schmidt norm as $M \to \infty$. □

THEOREM 2.2. *Suppose $T < T_c$ and let*

$$\langle \sigma_A \rangle_{M,\infty}^{+} = \langle \sigma_{m_1} V_M^{n_2 - n_1} \cdots V_M^{n_r - n_{r-1}} \sigma_{m_r} 1_{Q_M}, 1_{Q_M} \rangle,$$

where the integers n_i are ordered $n_1 \le n_2 \cdots \le n_r$. Then

$$\lim_{M\to\infty} [\langle \sigma_A \rangle_{M,\infty}^{+}]^2 = \langle \sigma \rangle_{+}^{2r} \det{}_2(1 + G).$$

The spontaneous magnetization $\langle\sigma\rangle_+$ is

$$\langle\sigma\rangle_+ = \left[\det\left(1 + |A^{-1}B|^2\right)\right]^{-1/4}.$$

The operator $G\colon W_r \to W_r$ is an $r \times r$ block matrix with entries

$$\begin{aligned} G_{ij} &= -T^{\Delta_{i+1}}L_{i+1}T^{\Delta_{i+2}}\cdots L_{j-1}T^{\Delta_j}Q_+\,\Delta R_j, && j > i+1,\\ &= -T^{\Delta_{i+1}}Q_+\,\Delta R_j, && j = i+1,\\ &= \quad 0, && j = 1,\\ &= \quad T^{-\Delta_i}Q_-\,\Delta R_j, && j = i-1,\\ &= T^{-\Delta_i}L_{i-1}^{-1}T^{\Delta_{i-1}}\cdots L_{j+1}^{-1}T^{-\Delta_{j+1}}Q_-\,\Delta R_j, && j < i-1, \end{aligned} \tag{2.18}$$

where $L_i = L(\sigma_{m_i})$, $\Delta R_i = \Delta R(\sigma_{m_i})$, and $\Delta_i = n_i - n_{i-1}$, and $W_r = W \oplus \cdots \oplus W$ (r times).

Proof. We first apply Theorem 1.0 to the semi-infinite correlations $\langle\sigma_A\rangle_{M,\infty}^+$ with $g_M(i) = \sigma_{m_i}V_M^{n_{i+1}-n_i}$ $(i = 1,\ldots,r)$ and with n_{r+1} chosen so that $n_{r+1} \geq n_r$.

The representation independent analog of (2.8) is

$$\begin{aligned} L(g_M(i)) &= L_M(\sigma_{m_i})T_M^{\Delta_i},\\ \Delta R(g_M(i)) &= \left(Q_M^+ \oplus Q_M^- T_M^{-\Delta_i}\right)\Delta R_M(\sigma_{m_i})\left(Q_M^+ T_M^{\Delta_i} \oplus Q_M^-\right). \end{aligned}$$

Substituting this in (2.7) and making a similarity transformation in $\det_2(I + L_M\Delta R_M)$ to bring all the factors $Q_M^+T^{\Delta_i} \oplus Q_M^-$ on the right of $\Delta R_M(\sigma_{m_i})$ to act on the left of L_M one finds

$$\left(\langle\sigma_A\rangle_{M,\infty}^+\right)^2 = \prod_{j=1}^{r}\langle\sigma_{m_j}\rangle_{M,\infty}^2 \det{}_2(I + G_M),$$

where

$$\langle\sigma_{m_j}\rangle_{M,\infty}^+ = \left[\det\left(1 + |A_M^{-1}(m_j)B_M(m_j)|^2\right)\right]^{-1/4}$$

and G_M is the $r \times r$ matrix with entries

$$\begin{aligned} G_M(ij) &= -T_M^{\Delta_{i+1}}L_{M,i+1}\cdots L_{M,j-1}T_M^{\Delta_j}Q_M^+\,\Delta R_{M,j}, && j > i+1,\\ &= -T_M^{\Delta_{i+1}}Q_M^+\,\Delta R_{M,j}, && j = i+1,\\ &= \quad 0, && j = i,\\ &= \quad T_M^{-\Delta_i}Q_M^-\,\Delta R_{M,j}, && j = i-1,\\ &= \quad T_M^{-\Delta_i}L_{M,i-1}^{-1}\cdots L_{M,j+1}^{-1}T_M^{-\Delta_{j+1}}Q_M^-\,\Delta R_{M,j}, && j < i-1, \end{aligned}$$

where $L_{M,i} = L_M(\sigma_{m_i})$, $\Delta R_{M,i} = \Delta R_M(\sigma_{m_i})$.

Lemma 2.2 assures one that $A_M^{-1}B_M(m_i)$ converges in Schmidt norm to $A^{-1}B(m_i)$. Since $\det(1+K)$ is continuous in K in the trace norm it follows that $\langle\sigma_{m_j}\rangle_{M,\infty}^{+}$ converges as $M\to\infty$ to $[\det(1+|A^{-1}B(m_j)|^2)]^{-1/4}$. However, $|A^{-1}B(m)|^2=-(A^{-1}B(m))^2$ is similar to $-(A^{-1}B)^2=|A^{-1}B|^2$ so it follows that $\langle\sigma_A\rangle_{M,\infty}^{+}$ converges to $[\det(1+|A^{-1}B|^2)]^{-1/4}$ as $M\to\infty$.

To finish the proof we need only demonstrate that $\lim_{M\to\infty}\det_2(I+G_M)=\det_2(I+G)$. To avoid complicated notation we shall write P_M for the projection $P_M\oplus\cdots\oplus P_M$ of $W\oplus\cdots\oplus W$ on $W_M\oplus\cdots\oplus W_M$. It is not hard to see that since G is a Schmidt class operator, the operators $(1-P_M)GP_M$, $P_MG(1-P_M)$, and $(1-P_M)G(1-P_M)$ all converge to zero in Schmidt norm as $M\to\infty$. Since $\det_2(1+K)$ is continuous in K in the Schmidt norm, it will follow that $\det_2(I+G_M)$ converges to $\det_2(I+G)$ as $M\to\infty$ if we can show that $\lim_{M\to\infty}\|P_MGP_M-G_M\|_2=0$.

It is enough to show that the individual matrix elements differ by operators which go to zero in Schmidt norm as $M\to\infty$. We will write $X_1\cdots X_lY$ for a typical matrix element of G where each X_j represents one of the factors $L^{\pm1}(m)$ or $T^{\pm1}$, and Y is $\mp Q_\pm\,\Delta R_m$ $(m=1\cdots r)$. In a similar fashion we write $X_{M,1}\cdots X_{M,l}Y_M$ for a typical matrix element of G_M, where $X_{M,j}$ denotes one of the factors $L_{M,m}^{\pm1}$, or $T_M^{\pm1}$ and Y_M denotes $\mp Q_M^{\pm}$ $\Delta R_M(m)$ $(m=1,\ldots,r)$. The difference between the matrix elements of P_MGP_M and G_M may be expressed:

$$\begin{aligned} P_MX_1\cdots X_lYP_M-X_{M,1}\cdots X_{M,l}Y_M &\\ =P_M(X_1-X_{M,1})X_2\cdots X_lYP_M &\\ +P_MX_{M,1}(X_2-X_{M,2})X_3\cdots X_lYP_M &\\ \vdots\qquad\qquad &\\ +P_MX_{M,1}\cdots X_{M,l-1}(X_l-X_{M,l})YP_M &\\ +P_MX_{M,1}\cdots X_{M,l}(Y-Y_M)P_M. & \end{aligned}\tag{2.19}$$

Since each $X_{M,j}$ is uniformly bounded as $M\to\infty$ and since $(1-P_M)X_i\cdots X_lYP_M$ tends to zero in Schmidt norm as $M\to\infty$ we may insert a projection P_M to the right of each difference $X_i-X_{M,i}$ occurring in (2.19) making an error which tends to zero in Schmidt norm as $M\to\infty$. Since Lemma 2.2 implies that $P_ML_j^{\pm1}P_M-L_{M,j}^{\pm1}$ goes to zero in Schmidt norm we may confine our attention to those terms in (2.19) which contain $P_MTP_M-T_M$ or $P_MT^{-1}P_M-T_M^{-1}$. In the discussion which follows Lemma 2.1 it was shown that $P_MTP_M-T_M$ and $P_MT^{-1}P_M-T_M^{-1}$ tend strongly to zero as $M\to\infty$ (here it is understood that T_M occurs in the Q_M representation and T occurs in the Q representation). Together with Lemma 2.0 this controls those terms in (2.19) in which $P_MT^{\pm1}P_M-T_M^{\pm1}$ occurs and finishes the proof of the theorem. □

In the following corollary we show that the spontaneous magnetization $\langle\sigma\rangle_+$ of Theorem 2.1 is given by the famous Onsager–Yang formula [53, 74].

COROLLARY 2.0. *Suppose $T < T_c$, then*

$$\langle\sigma\rangle_+ = \left[\det\left(1 + |A^{-1}B|^2\right)\right]^{-1/4} = \left(1 - (sh2K)^{-4}\right)^{1/8}.$$

Proof. We first observe that since $A^{-1}B\colon \mathfrak{X} \to \mathfrak{X}$ is a Schmidt class operator, the determinant which appears above is the determinant of a trace class perturbation of I on $\mathfrak{X}$. For any trace class $T\colon \mathfrak{X} \to \mathfrak{X}$, let T_C denote the complexification of T: $T_C f = Tf_1 \oplus Tf_2$ with $f = f_1 \oplus f_2 \in \mathfrak{X} \oplus \mathfrak{X}$. We have

$$\det(1 + T_c) = \left(\det(1 + T)\right)^2.$$

We now regard $A^{-1}B$ as acting on $\mathfrak{X} \oplus \mathfrak{X}$ (we drop the subscript C), use $A^{-1}B = -BA^{-1}$ and $A^2 + B^2 = 1$ to conclude $1 + |A^{-1}B|^2 = A^{-2}$. Hence we have

$$(\langle\sigma_+\rangle)^8 = \det(A^2) = \det(1 - B^2),$$

where A and B are regarded as operators on W. Recalling the definition of B, we have $B^2 = \frac{1}{4}(s\Lambda - \Lambda s)^2$, and an elementary calculation gives

$$1 - B^2 = (s_-Qs_-)^2 + (s_+Qs_+)^2.$$

Since $W \simeq l^2(Z_{1/2}, \mathbf{C}^2)$ we may write $W = s_-W \oplus s_+W$ with the result that $s_-Qs_-(s_+Qs_+)$ acts on $s_-W(s_+W)$. Hence

$$\det(1 - B^2) = \det(s_+Qs_+)^2 \det(s_-Qs_-)^2.$$

We can now apply Szegö's theorem as formulated by Widom [71] to conclude (we use Widom's notation)

$$\begin{aligned} \det(s_+Qs_+)^2 &= \det T[Q]T[Q^{-1}] \\ &= E[Q] \\ &= \lim_{n\to\infty} \frac{\det T_n[Q]}{G[Q]^{n+1}}, \end{aligned}$$

where $T[Q]$ is the semi-infinite (block) Toeplitz operator s_+Qs_+ with symbol $Q(\theta)$, $T_n[Q]$ is the $(n+1) \times (n+1)$ finite section of $T[Q]$, and $G[Q] = \exp\{1/2\pi \int_0^{2\pi} \log \det Q(\theta)\, d\theta\}$. We can now apply the scalar version of Szegö's theorem to this last quantity (for details of this particular calculation see [40, 49]). Similar considerations apply to $\det(s_- Qs_-)^2$, and the result follows. □

One can avoid the use of Szegö's theorem in the evaluation of $\langle\sigma\rangle_+$ if one knows the explicit spectral theory of $A = \frac{1}{2}(s - \Lambda s \Lambda)$. One is led to the eigenvalue equation

$$\frac{P}{\pi i}\oint \frac{dz}{z}\frac{1}{1 - z/t}\left(1 + \frac{\Theta^*(t)}{\Theta^*(z)}\right)\varphi(z) = l\varphi(t),$$

where P denotes the Cauchy principal value, the contour of integration is the unit circle, $|t| = 1$, $l \in \mathbf{R}$, and $\Theta^*(e^{i\theta}) = \exp(i\alpha(\theta))$. This integral equation is essentially the equation studied by Yang [74] (see also Abraham [1]) in his derivation of the spontaneous magnetization.

Thus the abstract characterization of $\langle\sigma\rangle_+$ in Theorem 2.1 leads naturally to either the techniques employed by Montroll *et al.* [49] (Szegö's theorem) or those of Yang (spectral theory of A).

Combining Theorem 2.2 with Theorem 3.1 of [56] we have the following "abstract" characterization of the infinite-volume correlations.

THEOREM 2.3. *Suppose $T < T_c$. Then the infinite-volume correlation $\langle\sigma_A\rangle^+ = \langle\sigma(m_1, n_1)\cdots\sigma(m_r, n_r)\rangle$ is given by*

$$\langle\sigma_A\rangle^+ = \langle\sigma_{m_1}V^{n_2-n_1}\sigma_{m_2}\cdots V^{n_r-n_{r-1}}\sigma_{m_r}1_Q, 1_Q\rangle.$$

The vector 1_Q is the Q-Fock vacuum, where $Q = i/\pi\int_{\mathrm{Re}\,z=1}(T-z)^{-1}\,dz$, T is given by (2.1), *$V = \Gamma(T) = I \oplus T_+ \oplus (T_+ \otimes T_+) \oplus \cdots$, and $\sigma_m = \Gamma(s(m))$. The normalization of σ_m is determined by $\sigma_m^2 = I$ and $\langle\sigma_m 1_Q, 1_Q\rangle > 0$.*

This result can be used as a starting point for generating alternative "explicit" formulas. The factorization $\begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} 1 & BA \\ BA^{-1} & A^2 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}$ coupled with the idea of the proof of Theorem (3.0) of [56] leads to different formulas that have certain advantages over the ones presented here.

3.0. In this section we establish the duality relations that we shall use to deal with the spin correlations above the critical temperature. The idea for the duality presented below is taken from [23, 31].

The duality map d (defined below) is slightly different from the duality maps constructed in [23] and the explicit consideration of boundary conditions makes the treatment distinct from that in [31]. For the convenience of the reader we adopt the notation and ideas from [23] and refer the reader to [23] for a more complete treatment of duality than we present here.

Let $\Lambda = \{(m, n) \in \mathbf{Z}^2\colon |m| \le M, |n| \le N\}$ with $M, N \in \mathbf{Z}^+$, and identify each configuration of Λ with the subset of Λ at which sites the spin values of the configuration are -1. The set $\mathcal{P}(\Lambda)$ of subsets of Λ (now thought of as configurations) is a group under symmetric difference with the identity given by $\varnothing$. For $Y \in \mathcal{P}(\Lambda)$ define σ_Y: $\mathcal{P}(\Lambda) \to \{-1, 1\}$ by $\sigma_Y(X) = \prod_{i \in Y} \sigma_i$ (configuration X) $= (-1)^{|Y \cap X|}$, where $|Y \cap X|$ is the number of points in $Y \cap X$. The map $\mathcal{P}(\Lambda) \ni Y \to \sigma_Y(\cdot)$ is a group homomorphism from $\mathcal{P}(\Lambda)$ onto the characters of $\mathcal{P}(\Lambda)$. We now define a subset $\mathcal{B}$ of $\mathcal{P}(\Lambda)$ by

$$\mathcal{B} = \big\{ \{a, b\} | \, a \text{ and } b \text{ are nearest neighbors in } \Lambda \big\} \cup \partial\Lambda,$$

where

$$\partial\Lambda = \big\{ \{a\} | a \in \Lambda \text{ and either } |a_1| = M \text{ or } |a_2| = N \big\}.$$

This set will be the set of bonds in our model, and we will write $\overline{\mathcal{B}}$ for the subgroup of $\mathcal{P}(\Lambda)$ generated by $\mathcal{B}$.

Suppose now that K is a complex valued function on $\mathcal{B}$. The partition function $Z(\Lambda, K)$ we define by

$$Z(\Lambda, K) = \sum_{X \subseteq \Lambda} \exp \sum_{Y \subseteq \mathcal{B}} K(Y)\sigma_Y(X). \tag{3.1}$$

In order to construct the high- and low-temperature expansions for $Z(\Lambda, K)$, we introduce two subgroups of $\mathcal{P}(\mathcal{B})$, the power set of $\mathcal{B}$ with symmetric difference the group operation. The first subgroup is the kernel of the group homomorphism π: $\mathcal{P}(\mathcal{B}) \to \overline{\mathcal{B}}$ defined by $\pi\{B_1, \ldots, B_n\} = \prod_{i=1}^n B_i$. We write $\mathcal{K} = \ker \pi$ and refer to $\mathcal{K}$ as the set of closed graphs in $\mathcal{P}(\mathcal{B})$. The second subgroup is the image of the group homomorphism γ: $\mathcal{P}(\Lambda) \to \mathcal{P}(\mathcal{B})$ defined by $\gamma(X) = \{B \in \mathcal{B} | \sigma_X(B) = -1\}$. We write $\Gamma = \mathrm{Im}(\gamma)$.

The high-temperature expansion for $Z(\Lambda, K)$ is

$$Z(\Lambda, K) = 2^{|\Lambda|} \prod_{B \in \mathcal{B}} chK(B) \sum_{\beta \in \mathcal{K}} \prod_{B \in \beta} thK(B) \tag{3.2}$$

The low-temperature expansion for $Z(\Lambda, K)$ is

$$Z(\Lambda, K) = 2^{|\Lambda|} \prod_{B \in \mathcal{B}} e^{K(B)} \sum_{\beta \in \Gamma} \prod_{B \in \beta} e^{-2K(B)}. \tag{3.3}$$

The reader is referred to [23] for a proof and further discussion of these results.

We are now ready to introduce a dual system (Λ^*, K^*). Let Λ^* denote the subset of $\mathbf{Z}^2_{1/2}$ defined by

$$\Lambda^* = \big\{ (k, l) \in \mathbf{Z}^2_{1/2} | \, |k| < M, |l| < N \big\} \cup \partial\Lambda^*,$$

where

$$\partial\Lambda^* = \{(k,l) \mid |k| = M + 1/2, |l| < N\}$$

$$\cup \{(k,l) \mid |k| < M, |l| = N + 1/2\}$$

and $M, N \in \mathbf{Z}^+$ are the same as in the definition of Λ.

We define the set of dual bonds

$$\mathfrak{B}^* = \{\{a,b\} \mid a \text{ and } b \text{ are nearest neighbors in } \Lambda^*\} \cup C,$$

where $C = C_1 \cup C_2 \cup C_3 \cup C_4$ and the C_j are the corner bonds starting at the upper-left corner of Λ^* and moving clockwise to the lower-left corner of Λ^*. Thus, for example,

$$C_3 = \{(M + 1/2, -N + 1/2), (M - 1/2, -N - 1/2)\}.$$

We next describe a map d: $\mathfrak{B} \to \mathfrak{B}^*$. If B is a pair bond in $\mathfrak{B}$ then dB is the pair bond in $\mathfrak{B}^*$ which crosses B at right angles. If B is a point bond in $\mathfrak{B}$ then dB is the pair bond in $\mathfrak{B}^*$ with elements in $\partial\Lambda^*$ that lies "closest" to B. For example,

$$d\{(a,b),(a,b+1)\} = \{(a - 1/2, b + 1/2), (a + 1/2, b + 1/2)\},$$

$$d\{(M,N)\} = C_2,$$

$$d\{(M,l)\} = \{(M - 1/2, l + 1/2), (M + 1/2, l + 1/2)\},$$

$$|l| < N.$$

The dual interaction, K^*, we define by

$$e^{-2K(B)} = thK^*(B^*), \qquad \text{or more symmetrically}$$

$$sh2K(B)sh2K^*(B^*) = 1, \tag{3.4}$$

where $B^* = dB$.

Since $\mathcal{P}(\mathfrak{B})$ is generated by the elements $\{B_i\}(B_i \in \mathfrak{B})$ under symmetric difference it is clear how to extend d to a group homomorphism from $\mathcal{P}(\mathfrak{B})$ onto $\mathcal{P}(\mathfrak{B}^*)$ which we shall also denote by d. (Here we seem to part company with the formalism in [23].) The property of d: $\mathcal{P}(\mathfrak{B}) \to \mathcal{P}(\mathfrak{B}^*)$ which we wish to exploit is that the restriction of d to Γ establishes a bijection of Γ onto $\mathcal{K}^*$. We leave the proof of this to the reader.

Following Kadanoff and Ceva [31] we identify correlations as ratios of partition functions. For $\beta \in \mathcal{P}(\mathcal{B})$ define

$$\begin{aligned} K_\beta(B) &= K(B), && B \notin \beta, \\ &= -K(B), && B \in \beta. \end{aligned}$$

Let $\beta^* = d\beta$ and observe that as a consequence of (3.4)

$$\begin{aligned} K_\beta^*(B^*) &= K^*(B^*), && B^* \notin \beta^*, \\ &= K^*(B^*) + \frac{i\pi}{2}, && B^* \in \beta^*. \end{aligned}$$

Comparing the low-temperature expansion for the ratio $Z(\Lambda, K_\beta)/Z(\Lambda, K)$ with the high-temperature expansion for the ratio $Z(\Lambda^*, K_\beta^*)/Z(\Lambda^*, K^*)$ and making use of (3.4) and the bijective correspondence between Γ and K^* one finds

$$\begin{aligned} \frac{Z(\Lambda, K_\beta)}{Z(\Lambda, K)} &= \prod_{B\in\beta} e^{K_\beta(B)-K(B)} \prod_{B^*\in\beta^*} \frac{chK^*(B^*)}{chK_\beta^*(B^*)} \frac{Z(\Lambda^*, K_\beta^*)}{Z(\Lambda^*, K^*)} \\ &= (-i)^{|\beta|} \prod_{B\in\beta} e^{-2K(B)} \prod_{B^*\in\beta^*} cthK^*(B^*) \frac{Z(\Lambda^*, K_\beta^*)}{Z(\Lambda^*, K^*)} \\ &= (-i)^{|\beta|} \frac{Z(\Lambda^*, K_\beta^*)}{Z(\Lambda^*, K^*)}. \end{aligned} \tag{3.5}$$

Since $\sigma_{B^*} = (-i)e^{(i\pi/2)\sigma_{B^*}}$ it follows that

$$\left\langle \prod_{B^*\in\beta^*} \sigma_{B^*} \right\rangle_{(\Lambda^*, K^*)} = (-i)^{|\beta|} \frac{Z(\Lambda^*, K_\beta^*)}{Z(\Lambda^*, K^*)}, \beta^* \in \mathcal{P}(\mathcal{B}^*). \tag{3.6}$$

We now specialize the interaction to the Ising model which we wish to consider. Henceforth let

$$\begin{aligned} K(B) &= \frac{J}{kT} && \text{if } B \text{ is a two-point bond} \\ &= H > 0 && \text{if } B \text{ is a one-point boundary bond.} \end{aligned}$$

From (3.4) it follows that

$$\begin{aligned} K^*(B^*) &= J/kT^* && \text{if } B^* \text{ is a bond with at least one element in the interior of } \Lambda^*, \\ &= -\tfrac{1}{2}\ln(thH) && \text{if } B \text{ is a boundary bond.} \end{aligned}$$

Combining (3.5) and (3.6) for this interaction and letting $H \to +\infty$ one finds

$$\left\langle \prod_{B^* \in \beta^*} \sigma_B^* \right\rangle_{(\Lambda^*, K^*)}^{\mathrm{op}} = (-1)^{|\beta|} \frac{Z^+(\Lambda, K_\beta)}{Z^+(\Lambda, K)}. \tag{3.7}$$

The expectation on the left is for open boundary conditions; that is, the terms which couple boundary spins are omitted. The partition functions $Z^+(\Lambda, K_\beta)$ and $Z^+(\Lambda, K)$ are computed with + boundary conditions on Λ.

Suppose $A \subseteq \Lambda^*$, then $\langle \sigma_A \rangle_{(\Lambda^*, K^*)}^{\mathrm{op}} = 0$ unless A contains an even number of sites in Λ^*. In order to make use of (3.7) to compute $\langle \sigma_A \rangle_{(\Lambda^*, K^*)}^{\mathrm{op}}$ we shall describe a particular collection of bonds β^* such that $\sigma_A = \prod_{B^* \in \beta^*} \sigma_{B^*}$ when $|A|$ is even. If a row in Λ^* contains an even number of elements in A then include in β^* those bonds which connect the elements of A in a pairwise disjoint fashion. If a row in Λ^* contains an odd number of elements in A we include in β^* those bonds which connect the element in A with the smallest x-coordinate to the left end site of the row and those bonds which join the remaining elements of the row pairwise as before. Since $|A|$ is even there are an even number of rows which contain an odd number of sites in A, and we eliminate extraneous boundary spins by including in β^* those bonds along the left edge of Λ^* which connect the "odd spin" rows in a pairwise disjoint fashion. It is clear by construction that $\sigma_A = \prod_{B^* \in \beta^*} \sigma_{B^*}$.

The dual "path" β consists of vertical bonds in $\mathcal{P}(\mathfrak{B})$ which lie between pairs of points in A, vertical bonds between boundary points in Λ^* and points in A, and one point bonds on the boundary of Λ. Because of the plus boundary conditions, the point bonds in β do not affect the partition function $Z^+(\Lambda, K_\beta)$ and we may confine our attention to the effect of the vertical bonds in β. In particular we now show how to introduce "disorder" variables [31] in the transfer matrix formalism to reveal $Z^+(\Lambda, K_\beta)/Z^+(\Lambda, K)$ as a correlation. Suppose to begin that the jth row in Λ^* contains the points $(k_1, j), \ldots, (k_l, j)$ in A and that l is odd. Recalling the development of the transfer matrix in Section 1, one sees that the vertical bonds all occur in the factor $\prod_{m=-M}^{M} \otimes \left(e^K \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-K} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)_m$ (which becomes V_1 after factoring out $(2sh2K)^{M+1/2}$). To change the bond strength K to $-K$ at the mth site one need only multiply this factor in the transfer matrix by C_m. Now introduce $\mu'_k = p_k \sigma_{k+1/2}$, $k \in \mathbf{Z}_{1/2}$, and note that

$$C_{-M} \cdots C_{k_1 - 1/2} = \mu'_{k_1},$$

$$C_{k_i + 1/2} \cdots C_{k_{i+1} - 1/2} = \mu'_{k_i} \mu'_{k_{i+1}}.$$

Thus to incorporate the appropriate bond flips for the jth row it is sufficient to include the factor $\mu'_{k_1}\cdots\mu'_{k_l}$ next to V_1 in the transfer matrix at the jth row. Now define $\mu_k = V_2^{1/2}\mu'_k V_2^{-1/2}$. Then since $V_2^{1/2}p_k V_2^{-1/2} = (chK)p_k + i(shK)q_k$ and $V_2^{1/2}\sigma_{k+1/2}V_2^{-1/2} = \sigma_{k+1/2}$ we have $\mu_k = [(chK)p_k + i(shK)q_k]\sigma_{k+1/2}$. We have shown that

$$\langle\sigma_A\rangle^{\mathrm{op}}_{(\Lambda^*,K^*)} = \frac{\langle V_M \mu_{A_{N-1/2}} V_M \cdots \mu_{A_{-N+1/2}} V_M +, +\rangle}{\langle V_M^{2N+1}+, +\rangle}, \tag{3.8}$$

where μ_{A_j} denotes the product $\prod_{n=1}^{l}\mu_{k_n}$; that is, the product of disorder variables appearing in the jth row.

The analysis of the semi-infinite-volume limit given in Section 1 applies as well to the right-hand side of (3.8) and we have:

THEOREM 3.0. *Let* $\langle\sigma_A\rangle^{\mathrm{op}}_{M,K^*} = \lim_{N\to\infty}\langle\sigma_A\rangle^{\mathrm{op}}_{(\Lambda^*,K^*)}$. *Then if* $A \subseteq [-M, M]\times[-n, n]$, $n, M \in \mathbf{Z}^+$, *we have*

$$\langle\sigma_A\rangle^{\mathrm{op}}_{M,K^*} = \langle \mu_{A_{n-1/2}} V_M \cdots V_M \mu_{A_{-n+1/2}} 1_{Q_M}, 1_{Q_M}\rangle, \tag{3.9}$$

where $\mu_k = (chKp_k + ishKq_k)\sigma_{k+1/2}$, $k \in \mathbf{Z}_{1/2}$, *and the expectation on the right is evaluated at bond strength* K *determined by* $sh2Ksh2K^* = 1$. *The operators* V_M *and* Q_M *are the same as in Theorem* 2.1.

This theorem is of interest since when $T^* = J/kK^*$ is greater than T_c the dual temperature $T = J/kK$ is less than T_c. As before it is not hard to guess the infinite-volume limit of (3.9). The operators p_k and q_k have natural infinite-volume counterparts and since only even correlations occur the sign ambiguity in $\sigma_{k+1/2}$ which results from defining this operator in terms of its induced rotation is of no consequence.

Employing the same notation as in Theorem 2.0 we now state the result from [56] which we will use in conjunction with Theorem 3.0 to compute the infinite-volume correlations for $T > T_c$. The following result is a special case of Theorem 3.2 in [56].

THEOREM 3.1. *Suppose* g_i *is a factorable element of* $G(W, Q)$ *for* $i = 1, \ldots, r$. *Let* $w_i \in W$ $(i = 1, \ldots, r)$ *and define*

$$h_i =: w_i g_i : \overset{\mathrm{def}}{=} F(Q_+ w_i)g_i + g_i F(Q_- w_i);$$

then if $\langle g_1 \cdots g_r\rangle_Q \neq 0$ *we have*

$$\langle h_1 \cdots h_r\rangle_Q = \langle g_1 \cdots g_r\rangle_Q \,\mathrm{pf} H,$$

where $\mathrm{pf}H$ *is the Pfaffian of the* $r \times r$ *skew symmetric matrix* H *with entries* *above the diagonal given by*

$$H_{i,k} = -\langle (I + L\Delta R)^{-1} L I_k(w_k), I_i(\bar{w}_i)\rangle, \qquad i < k.$$

The map I_k is the injection of W into the k'th slot in $(W \oplus \cdots \oplus W)_r$ and the inner product in the definition of $H_{i,k}$ is $\langle x_1 \oplus \cdots \oplus x_r, y_1 \oplus \cdots \oplus y_r\rangle = \sum_{i=1}^r \langle x_i, y_i\rangle$.

In order to use this theorem we need to write μ_k in "normal ordered form." Suppose $w \in W$. Then by definition $:w\sigma_m: = F(Q_+ w)\sigma_m + \sigma_m F(Q_- w) = F((s_m Q_- + Q_+)w)\sigma_m$. Since $chKp_k + ishKq_k = \sqrt{2}\, F(chKe_1(k) + ishKe_2(k))$ it follows that $\mu_k = :w(k)\sigma_{k+1/2}:$ provided that $(s_{k+1/2}Q_- + Q_+)w(k) = \sqrt{2}\,(chKe_1(k) + ishKe_2(k))$. To solve for $w(k)$ we will invert $s_m Q_- + Q_+$. For $T < T_c$ this is possible and it is easy to see how to do it in the Q-representation. In this representation

$$s_m Q_- + Q_+ = \begin{bmatrix} A(m) & B(m) \\ B(m) & A(m) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & B(m) \\ 0 & A(m) \end{bmatrix}.$$

Below T_c the operator $A(m)$ is invertible and we have

$$(s_m Q_- + Q_+)^{-1} = \begin{bmatrix} 1 & -BA^{-1}(m) \\ 0 & A^{-1}(m) \end{bmatrix}.$$

Applying this operator ($m = k + 1/2$) to the vector $D\begin{bmatrix} (chK)e_1(k) \\ (shK)e_2(k) \end{bmatrix}$ and making use of $I - BA^{-1}(m) = I + A^{-1}B(m) = A^{-1}(m)s_m$ and $s_m e_j(k) = e_j(k)$ for $m = k + 1/2$ one finds

$$w(k) = \sqrt{2}\, D \begin{bmatrix} (chK)A^{-1}(k + 1/2)e_1(k) \\ (shK)A^{-1}(k + 1/2)e_2(k) \end{bmatrix}. \tag{3.10}$$

The same calculation is meaningful for $(s_m Q_M^- + Q_M^+)^{-1}$ and one finds

$$w_M(k) = \sqrt{2}\, D_M \begin{bmatrix} (chK)A_M^{-1}(k + 1/2)e_1(k) \\ (shK)A_M^{-1}(k + 1/2)e_2(k) \end{bmatrix}.$$

We are now prepared to state the principal result of this section.

THEOREM 3.2. *Suppose $T^* > T_c$ and let*

$$\langle \sigma_A \rangle^{\mathrm{op}}_{M,K^*} = \langle \mu_{k_1} V_M^{l_2 - l_1} \cdots V_M^{l_r - l_{r-1}} \mu_{k_r} 1_{Q_M}, 1_{Q_M}\rangle,$$

where $k_i, l_i \in \mathbf{Z}_{1/2}$ and the half integers l_i are ordered $l_1 \le l_2 \le \cdots \le l_r$. Then

$$\lim_{M \to \infty} [\langle \sigma_A \rangle^{\mathrm{op}}_{M,K^*}]^2 = \langle \sigma \rangle_+^{2r} \det{}_2(I + G)\det H. \tag{3.11}$$

In this formula $\langle \sigma \rangle_+$ is the spontaneous magnetization at temperature $T < T_c$.

The operator G is the same as the operator G in Theorem 2.1 *evaluated at* $m_i = k_i + 1/2$, $\Delta_i = l_i - l_{i-1}$ *and temperature* $T < T_c$. *Finally H is the* $r \times r$ *skew symmetric matrix with entries above the diagonal given by*

$$H(ij) = -\langle (1+G)^{-1} K w_j, \bar{w}_i \rangle, \qquad i, j \in \mathbf{Z}, \tag{3.12}$$

where $w_i = I_i(w(k_i))$ *and K is the* $r \times r$ *block matrix with entries*

$$\begin{aligned} K(ij) &= -\, T^{\Delta_{i+1}} L_{i+1} T^{\Delta_{i+2}} \cdots L_{j-1} T^{\Delta_j} Q_+, && j > i+1, \\ &= \quad -\, T^{\Delta_{i+1}} Q_+ && j = i+1, \\ &= \qquad 0, && j = 1, \\ &= \quad T^{-\Delta_i} Q_-, && j = i-1, \\ &= T^{-\Delta_i} L_{i-1}^{-1} T^{-\Delta_{i-1}} \cdots L_{j+1}^{-1} T^{-\Delta_{j+1}} Q_-, && j < i-1. \end{aligned} \tag{3.13}$$

The notation is the same as in Theorem 2.1.

Proof. Let $g_M(i) = \sigma_{m_i} T_M^{\Delta_{i+1}}$ and observe that $((chK)p_{m_i - 1/2} + i(shK)q_{m_i - 1/2}) g_M(i) = \,:w_M(k_i)\sigma_{m_i}: T^{\Delta_{i+1}} = \,:(Q_M^- T^{-\Delta_{i+1}} \oplus Q_M^+) w_M(k_i) \sigma_{m_i} T^{\Delta_{i+1}}:$. If we write $h_i = \,:(Q_M^- T^{-\Delta_{i+1}} + Q_M^+) w_M(k_i) \sigma_{m_i} T^{\Delta_{i+1}}:$ and successively apply Theorems 3.1 and 2.0 to $\langle h_1 \cdots h_r \rangle_{Q_M}$ we obtain the semi-infinite analog of (3.11) except for a minor discrepancy due to a similarity transformation which we now explain. In the following discussion we will use the subscript M to identify obvious semi-infinite analogs of infinite-volume operators. Let E denote the block diagonal matrix with entries $E(ij) = \delta_{ij}(Q_M^- T_M^{-\Delta_{i+1}} + Q_M^+)$. Then as already noted in Theorem 2.1 $\bar{E}^* L_M \Delta R_M \bar{E}^{*-1} = G_M$ and the factor $\det_2(I + G_M)$ is obtained from $\det_2(I + L_M \Delta R_M)$ by a similarity transformation. For the matrix element $H(ij)$ we have the formula

$$H_M(ij) = -\left\langle (I + L_M \Delta R_M)^{-1} L_M E w_{M,j}, \overline{E w_{M,i}} \right\rangle.$$

Observing that $\bar{E}^* L_M E = K_M$ we have

$$H_M(ij) = -\langle (I + G_M)^{-1} K_M w_{M,j}, \bar{w}_{M,i} \rangle.$$

In order to prove the convergence of the infinite-volume limit to the product in (3.11) we may ignore the first two factors since their convergence has already been proved in Theorem 2.1. Griffith's inequalities imply that $\langle \sigma_A \rangle_M^+ \geq \Pi_{a \in A} \langle \sigma_a \rangle^+$. Thus $\det_2(I + G_M) \geq 1 (T < T_c)$. It follows that $\det_2(I + G) \geq 1$ and $I + G$ is invertible. Indeed since G_M was shown to converge in Schmidt norm to G (in the appropriate representation) it follows that $(I + G_M)^{-1}$ converges uniformly to $(I + G)^{-1}$. Thus to show that $H_M(ij)$ converges to $H(ij)$ (so that $\lim_{M \to \infty} \det H_M = \det H$) we need only show that $K_M w_{M,j}$ and $w_{M,j}$ converge in W to $K w_j$ and w_j, respectively. This is an elementary consequence of the strong convergence of $T_M^{\pm 1}$, Q_M, and $A_M^{\pm 1}$ established in Lemmas 2.1 and 2.2. □

The following characterization of the infinite-volume correlations above T_c is a direct consequence of Theorem 3.2 in this paper and Theorem 3.2 in [56].

THEOREM 3.3. *If* $A = \{(k_i, l_i) | i = 1, \ldots, r\} \subseteq \mathbf{Z}^2_{1/2}$ *and* $\langle \sigma_A \rangle_{K^*}$ *is the infinite-volume limit above* T_c *then*

$$\langle \sigma_A \rangle_{K^*} = \langle \mu_{k_1} V^{l_2 - l_1} \cdots V^{l_r - l_{r-1}} \mu_{k_r} 1_Q, 1_Q \rangle,$$

where $\mu_k = \sqrt{2}\,[(chK)F(e_1(k)) + i(shK)F(e_2(k))]\sigma_{k+1/2}$, $V = \Gamma(T)$, *and* 1_Q *is the Fock state at temperature* T *dual to* T^*.

4.0. Before we turn to the consideration of the scaling limit we shall restate Theorems 2.2 and 3.2 in a more convenient form; in particular we shall introduce a similarity transformation to simplify the kernels for A^{-1} and $A^{-1}B$. We will also take the trouble to give translation invariant formulas for the correlations; this would not have been useful in the original statements since we wanted formulas with semi-infinite analogs.

We first introduce a Hilbert space $\mathcal{X}(\delta, T)$. Here $\delta > 0$ and T refers to temperature. $\mathcal{X}(\delta, T)$ will denote the Hilbert space of functions $f{:}[-\pi/\delta, \pi/\delta] \to \mathbf{C}^2$ satisfying the reality condition $f(-p) = \overline{f(p)}$ (ordinary complex conjugation) with the real inner product:

$$s(f, g) = \mathrm{Re}\int_{-\pi/\delta}^{\pi/\delta} \left[f_1(p)\overline{g_1(p)} + f_2(p)\overline{g_2(p)} \right] [2\pi sh\gamma(\delta p)]^{-1} \delta\, dp$$

and complex structure $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In the formula for the inner product the function γ is evaluated at temperature T. The parameter δ is introduced for convenience and will play a role in the later discussion of scaling.

The Fourier series $f(\theta) = \Sigma_{k \in \mathbf{Z}_{1/2}} f_k e^{ik\theta}$ identifies $l^2(\mathbf{Z}_{1/2}, \mathbf{R}^2)$ with the functions in $L^2([-\pi, \pi], \mathbf{C}^2)$ which satisfy $f(-\theta) = \overline{f(\theta)}$. By the Plancherel theorem the real inner product on $\mathcal{X} = l^2(Z_{1/2}, \mathbf{R}^2)$ becomes $\mathrm{Re}\int_{-\pi}^{\pi} [f_1(\theta)\overline{g_1(\theta)} + f_2(\theta)\overline{g_2(\theta)}](2\pi)^{-1} d\theta$. We introduce a real orthogonal map U_T mapping $\mathcal{X}$ onto $\mathcal{X}(1, T)$ defined by

$$U_T f(\theta) = \begin{bmatrix} e^{-i\alpha(\theta)/2} & 0 \\ 0 & e^{i\alpha(\theta)/2} \end{bmatrix} (sh\gamma(\theta))^{1/2} f(\theta)$$

This definition is partly inspired by some calculations in S.M.J. [63] and some identities in [73] but might also be motivated by observing that it is a natural orthogonal conjugation transforming $\Lambda = \begin{bmatrix} 0 & -e^{i\alpha} \\ e^{-i\alpha} & 0 \end{bmatrix}$ into the canonical complex structure $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. That is, $U_T \Lambda U_T^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Thus U_T

is a unitary map from $(\mathcal{X}, \Lambda)$ to $\mathcal{X}(1, T)$. If we now transform A^{-1} and $A^{-1}B$ given in (2.9) by this similarity we find

$$
\begin{aligned}
U_T A^{-1} U_T^* &= 2^{-1}(sh\gamma)^{1/2}(asa^{-1} + a^{-1}sa)(sh\gamma)^{-1/2}J,\\
U_T A^{-1} B U_T^* &= 2^{-1}(sh\gamma)^{1/2}(asa^{-1} - a^{-1}sa)(sh\gamma)^{-1/2}I,
\end{aligned}
\tag{4.1}
$$

where a is multiplication by $(a_+(e^{i\theta})/a_-(e^{i\theta}))^{1/2}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Since $a_-(e^{i\theta}) = \overline{a_+(e^{i\theta})^{-1}}$ it follows that $a_+(e^{i\theta})/a_-(e^{i\theta}) = |a_+(e^{i\theta})|^2$. From this one may deduce that

$$
a = |a_+(e^{i\theta})| = (2thK/sh\gamma(\theta))^{1/2} ch(\gamma(\theta)/2)
$$

and $a^{-1} = (2/(thK)sh\gamma(\theta))^{1/2} sh(\gamma(\theta)/2)$. If we substitute these results in (4.1) along with the principal value kernel, $-i\sin((\theta - \theta')/2)^{-1}$, for s (relative to $d\theta/2\pi$) we obtain the following kernels for A^{-1} and $A^{-1}B$ (relative to the measure $(2\pi sh\gamma(\theta))^{-1}d\theta$):

$$
\begin{aligned}
A^{-1}(\theta, \theta') &= -i\frac{sh((\gamma(\theta) + \gamma(\theta'))/2)}{\sin((\theta - \theta')/2)} I,\\
A^{-1}B(\theta, \theta') &= i\frac{sh((\gamma(\theta) - \gamma(\theta'))/2)}{\sin((\theta - \theta')/2)} J,
\end{aligned}
\tag{4.2}
$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The first kernel is to be understood in the principal value sense. In fact, although (4.2) is a convenient shorthand, when we come to estimate A^{-1} we shall understand (4.2) in the spirit of (4.1) as a sum of operators each of which is s bracketed by multiplication operators. Since $\gamma(\theta)$ is a differentiable 2π periodic function the kernel $A^{-1}B(\theta, \theta')$ is not singular.

We now restate Theorems 2.2 and 3.2 in a form which will make it easy for us to prove the scaling limit result in Theorem 4.2. The reader might find it useful to note that A^{-1} will scale to an unbounded operator and the scaling limit of $A^{-1}B$ will no longer be Schmidt class. The formulas we present below will (partially) overcome these problems by an elementary factorization which expresses everything in terms of $e^{-t\gamma}A^{-1}e^{-s\gamma}$ and $e^{-t\gamma}A^{-1}Be^{-t\gamma}(s, t > 0)$ the first of which scales to a bounded operator, the second of which scales to a Schmidt class operator. We write $\langle\sigma_A\rangle_{T<T_c}$ for the infinite-volume '+' state correlations for $T < T_c$ and $\langle\sigma_{A'}\rangle_{T>T_c}$ for the infinite-volume correlations above T_c with $A' = \{a - (1/2, 1/2) | a \in A\}$.

THEOREM 4.1.

$$
\begin{aligned}
\langle\sigma_A\rangle^2_{T<T_c} &= \langle\sigma\rangle_T^{2r}\det{}_2(I + G(T)),\\
\langle\sigma_{A'}\rangle^2_{T>T_c} &= \langle\sigma\rangle_{T^*}^{2r}\det{}_2(I + G(T^*))\det H(T^*),
\end{aligned}
$$

where $sh(2J/kT)sh(2J/kT^*) = 1$ *and* $\langle\sigma\rangle_T$ *is the spontaneous magnetization at temperature* T.

$G(T)$ *is the* $r \times r$ *block matrix with entries*

$$\begin{aligned} G_{ij}(T) &= -K_{ij}(T)D_1(\Delta n_j)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, && j > i+1, \\ &= z^{\Delta m_{i+1}} D_1(\Delta n_j)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, && j = i+1, \\ &= 0, && j = i, \\ &= z^{-\Delta m_i} D_1(\Delta n_{j+1})\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, && j = i-1, \\ &= K_{ij}(T)D_1(\Delta n_{j+1})\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, && j < i-1, \end{aligned}$$

where $z = e^{i\theta}$, $D_1(s) = e^{-s\gamma/2}A^{-1}Be^{-s\gamma/2}$, $\Delta m_i = m_i - m_{i-1}$, $\Delta n_i = n_i - n_{i-1}$ *and* $K(T)$ *is the* $r \times r$ *matrix with entries*

$$\begin{aligned} K_{ij}(T) &= -z^{\Delta m_{i+1}}C_1(\Delta n_{i+1}, \Delta n_{i+2}) \cdots z^{\Delta m_{j-1}}C_1(\Delta n_{j-1}, \Delta n_j)z^{\Delta m_j}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ & \qquad\qquad j > i+1, \\ &= -z^{\Delta m_{i+1}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad j = i+1, \\ &= 0 \qquad j = i, \\ &= z^{-\Delta m_i}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad j = i-1, \\ &= z^{-\Delta m_i}C_1(\Delta n_i, \Delta n_{i-1}) \cdots z^{-\Delta m_{j+2}}C_1(\Delta n_{j+2}, \Delta n_{j+1})z^{-\Delta m_{j+1}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ & \qquad\qquad j < i-1, \end{aligned}$$

where $C_1(s, t) = e^{-s\gamma/2}A^{-1}e^{-t\gamma/2}$.

$H(T)$ *is the* $r \times r$ *skew symmetric matrix with entries above the diagonal given by*

$$H_{ij}(T) = -\left\langle (I + G(T))^{-1}K(T)u_j, \bar{u}_i \right\rangle, \tag{4.3}$$

where the vector $u_i \in (W \oplus \cdots \oplus W)_r$ *is given by*

$$u_i = I_i\left\{e^{-i\theta/2}(sh2K)^{1/2}\begin{bmatrix} e^{-(\Delta n_i + 1)\gamma(\theta)/2} \\ 0 \end{bmatrix} \oplus \begin{bmatrix} e^{-(\Delta n_{i+1} - 1)\gamma(\theta)/2} \\ 0 \end{bmatrix}\right\}.$$

With u_k given by this formula the operators A^{-1} and $A^{-1}B$ in the definition of $C_1(t, s)$ and $D_1(s)$ are to be understood as integral operators on $\mathcal{X}(1, T)$ with the kernels given in (4.2).

The complex structure implicit in (4.3) is the direct sum of r copies of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on $\mathcal{X}(1, T) \oplus \mathcal{X}(1, T)$; the conjugation $u \to \bar{u}$ is the direct sum of r copies of $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ on $\mathcal{X}(1, T) \oplus \mathcal{X}(1, T)$ with I the identity on $\mathcal{X}(1, T)$.

Proof. This result is a similarity transformation away from the conclusions of theorems 2.2 and 3.2. Let M denote the $r \times r$ block diagonal matrix with entries $M_{ij} = \delta_{ij} e^{-im_j\theta}(Q_+ e^{-\Delta n_{j+1}\gamma(\theta)/2} + Q_- e^{\Delta n_j \gamma(\theta)/2})$. Let G and K denote the operators defined in Theorems 2.2 and 3.2. The operator $G(T)$ is MGM^{-1} in the Q-representation transformed further by the similarity in (4.1). The operator $K(T)$ is $MK\overline{M}^*$ in the Q-representation transformed by the similarity in (4.1). Finally, to make the formula for $H(T)$ correct the vector u_k must be related to $w(m_k)$ in Theorem 3.2 as follows:

$$u_k = I_k(U_T \oplus U_T) e^{-im_k\theta}\left(Q_- e^{-\Delta n_{k+1}\gamma/2} + Q_+ e^{-\Delta n_k \gamma/2}\right) w(m_k).$$

The result quoted for u_k in the theorem is a relatively simple calculation using (3.10) and (2.9). □

We now describe the scaling limit for the correlations. First we introduce a lattice spacing δ for the two-dimensional Ising model. At fixed temperature this has the obvious effect of multiplying the correlation length by δ. As we wish to let $\delta \to 0$ we choose the temperature in a δ dependent fashion so that the correlation length approaches a finite value as $\delta \to 0$. Since the correlation length diverges to $+\infty$ as one approaches the critical temperature T_c [40] this is possible provided $T(\delta) \to T_c$ as $\delta \to 0$. However, since $\lim_{T \to T_c^-} \langle \sigma \rangle_T = \lim_{T \to T_c^+} \langle \sigma \rangle_{T^*} = 0$ it turns out to be essential for a nontrivial result to divide the r point function by $\langle \sigma \rangle^r$ before passing to the limit. In the language of quantum field theory, $\langle \sigma \rangle$ is the wave-function renormalization constant $Z_3(\delta)$. For more discussion of this point see McCoy and Wu [48] and Glimm and Jaffe [21, 22]. Two distinct limits are possible depending on whether $T(\delta)$ approaches T_c from above or below.

We define a unitary scale transformation from $\mathcal{X}(1, T)$ to $\mathcal{X}(\delta, T)$ by $f(\theta) \to f(\delta p)$ ($p \in [-\pi/\delta, \pi/\delta]$). Under similarity transformation by this scaling operator the kernels of A^{-1} and $A^{-1}B$ become (relative to the measure $(2\pi sh\gamma(\delta p))^{-1}\delta dp$)

$$A^{-1}(p, p') = \chi_\delta(p) \frac{sh((\gamma(\delta p) + \gamma(\delta p'))/2)}{i \sin(\delta(p - p')/2)} \chi_\delta(p') I,$$

$$A^{-1}B(p, p') = -\chi_\delta(p) \frac{sh((\gamma(\delta p) - \gamma(\delta p'))/2)}{i \sin(\delta(p - p')/2)} \chi_\delta(p') J,$$

where

$$\chi_\delta(p) = 1, \qquad |p| \le \pi/\delta,$$
$$= 0, \qquad |p| > \pi/\delta.$$

Next we define δ so that $\delta\cdot$(correlation length) is 1 in the limit $T \to T_c$. The correlation length is known [13, 40] to be asymptotically given by $(z(1-z^2))|z^2+2z-1|^{-1}$ ($z = thK$) for T near T_c. If we define $\delta = |z^2 + 2z - 1|(z(1-z^2))^{-1}$, $a = (1+z^2)^2$, and $b = 2z(1-z^2)$ then

$$ch\gamma(\theta) = \frac{a}{b} - \cos\theta$$

$$= 1 + \left(\frac{a}{b} - 2\right) + (1 - \cos\theta)$$

$$= 1 + \frac{\delta^2}{2} + (1 - \cos\theta).$$

Introducing the variable $p = \theta/\delta$ as before, we define $\gamma(p, \delta)$ by

$$ch\gamma(p, \delta) = 1 + (1/2)\delta^2 + (1 - \cos\delta p).$$

To obtain the appropriate temperature dependence in the kernels for A^{-1} and $A^{-1}B$ we define $A_\delta^{-1}(p, p')$ to be the kernel $A^{-1}(p, p')$ with $\gamma(\delta q)$ replaced by $\gamma(q, \delta)$ and $A_\delta^{-1}B_\delta(p, p')$ to be the kernel $A^{-1}B(p, p')$ with $\gamma(\delta q)$ replaced by $\gamma(q, \delta)$. The operators A_δ^{-1} and $A_\delta^{-1}B_\delta$ act on $\mathcal{X}(\delta) = \mathcal{X}(\delta, T(\delta))$ where $T(\delta)$ is implicitly defined by $\gamma_{T(\delta)}(\delta p) = \gamma(p, \delta)$.

We introduce the obvious analogs of the operators C_1 and D_1 in Theorem 4.1:

$$C_\delta(p, p') = e^{-s\gamma(p,\delta)/\delta} A_\delta^{-1}(p, p') e^{-t\gamma(p',\delta)/\delta},$$

$$D_\delta(p, p') = e^{-s\gamma(p,\delta)/\delta} A_\delta^{-1} B_\delta(p, p') e^{-s\gamma(p',\delta)/\delta}.$$

We have dropped the explicit t and s dependence to unburden the notation. The following three lemmas provide the technical core of the convergence proof for the scaling limit. Throughout the discussion preceeding Theorem 4.2 we suppose t and s are fixed reals *strictly* greater than zero.

LEMMA 4.1. *The function $\gamma(p, \delta)$ has the following properties*:

(1) *There are constants $m > 0$, and $M > 0$ independent of p and δ such that $m\omega(p) \le \delta^{-1}\gamma(p, \delta) \le M\omega(p)$ for $p \in [-\pi/\delta, \pi/\delta]$.*

(2) $\lim_{\delta\to 0}\delta^{-1}\gamma(p, \delta) = \omega(p)$ *uniformly for p in any compact subset of* $\mathbf{R}$.

Proof. Let $\omega_\delta^2(p) = 1 + \delta^{-2}(1 - \cos\delta p)$ so that $ch\gamma(p,\delta) = 1 + (1/2)\delta^2\omega_\delta^2(p)$. The formula for the inverse hyperbolic cosine yields $\gamma(p,\delta) = \log(1+\delta x)$, where $x = \omega_\delta(1 + (1/4)\delta^2\omega_\delta^2)^{1/2} + (1/2)\delta\omega_\delta^2$. Since $\log(1+\delta x)^{1/\delta} = \delta^{-1}\int_1^{1+\delta x} t^{-1}\,dt$ we have the elementary estimates: $\log(1+\delta x)^{1/\delta} \le x$ and $\log(1+\delta x)^{1/\delta} \ge x/(1+\delta x)$ $(x > 0)$. Thus $\delta^{-1}\gamma(p,\delta) = \log(1+\delta x)^{1/\delta} \le x \le \omega_\delta(1+\delta\omega_\delta)$ and $\delta^{-1}\gamma(p,\delta) = \log(1+\delta x)^{1/\delta} \ge \log(1+\delta\omega_\delta)^{1/\delta} \ge \omega_\delta/(1+\delta\omega_\delta)$. We have demonstrated

$$\omega_\delta(p)(1+\delta\omega_\delta(p))^{-1} \le \delta^{-1}\gamma(p,\delta) \le \omega_\delta(p)(1+\delta\omega_\delta(p)). \quad (4.3)$$

Since $\omega_\delta(p)$ converges uniformly to $\omega(p)$ for p in a bounded subset of $\mathbb{R}$ part 2 of the lemma follows from (4.3). To obtain a uniform upper bound from (4.3) observe that $1 - \cos x \le x^2/2$ so that $\omega_\delta^2(p) \le 1 + p^2 = \omega^2(p)$ and $\delta^{-1}\gamma(p,\delta) \le \omega(p)(1+\delta\omega(p)) \le (1+\sqrt{1+\pi^2})\omega(p)$ for $p \in [-\pi/\delta, \pi/\delta]$ and $\delta < 1$. Finally since $(1-\cos\theta)/\theta^2$ is bounded away from zero for $\theta \in [-\pi,\pi]$ it follows that $\omega_\delta(p) \ge \text{const.}\,\omega(p)$ for $p \in [-\pi/\delta, \pi/\delta]$. From this observation and the bound $\delta\omega_\delta(p) \le \text{const.}$, $p \in [-\pi/\delta, \pi/\delta]$, it follows from (4.3) that $\delta^{-1}\gamma(p,\delta) \ge m\omega(p)$ for some constant $m > 0$ and all $p \in [-\pi/\delta, \pi/\delta]$. $\square$

Let $\mathcal{X}(0)$ denote the Hilbert space of functions $f:\mathbb{R} \to \mathbb{C}^2$ subject to the reality condition $f(-p) = \overline{f(p)}$ with real inner product

$$\mathrm{Re}\int_{-\infty}^{\infty} \left(f_1(p)\overline{g_1(p)} + f_2(p)\overline{g_2(p)}\right)(2\pi\omega(p))^{-1}dp$$

and complex structure $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Define an isometric injection $i_\delta:\mathcal{X}(\delta) \to \mathcal{X}(0)$ by

$$\begin{aligned} i_\delta f(p) &= \left(\delta\omega(p)/sh\gamma(p,\delta)\right)^{1/2} f(p), \qquad && p \in [-\pi/\delta, \pi/\delta] \\ &= 0, && p \notin [-\pi/\delta, \pi/\delta]. \end{aligned}$$

We write $i_\delta(p) = (\delta\omega(p)/sh\gamma(p,\delta))^{1/2}$ and note that $i_\delta^*:\mathcal{X}(0) \to \mathcal{X}(\delta)$ is multiplication by $\mathcal{X}_\delta(p)i_\delta(p)^{-1}$. Let

$$D(p,p') = ie^{-s\omega(p)}\frac{\omega(p)-\omega(p')}{p-p'}e^{-s\omega(p')J}$$

and define a bounded operator on $\mathcal{X}(0)$ by

$$Df(p) = \int_{-\infty}^{\infty} D(p,p')f(p')(2\pi\omega(p'))^{-1}\,dp'.$$

Let

$$C(p, p') = -ie^{-s\omega(p)} \frac{\omega(p) + \omega(p')}{p - p'} e^{-t\omega(p')}I$$

and define a bounded operator on $\mathcal{X}(0)$ by

$$Cf(p) = \int_{-\infty}^{\infty} C(p, p') f(p') (2\pi\omega(p'))^{-1} \, dp',$$

where the integral is understood in the principal value sense.

LEMMA 4.2. *Suppose $s > 0$. The difference $\chi_\delta D \chi_\delta - i_\delta D_\delta i_\delta^*$ tends to zero in Schmidt norm as $\delta \to 0$.*

Proof. We must show that

$$\lim_{\delta \to 0} \int \frac{dp}{\omega(p)} \int \frac{dp'}{\omega(p')} |\chi_\delta(p) D(p, p') \chi_\delta(p') - i_\delta D_\delta i_\delta^*(p, p')|^2 = 0.$$

By Lemma 4.1, $i_\delta(p)$ and $i_\delta(p)^{-1}$ are δ-uniformly bounded for p in $[-\pi/\delta, \pi/\delta]$ and converge pointwise to 1 as $\delta \to 0$. It is a further consequence of Lemma 4.1 that $D_\delta(p, p')$ converges pointwise to $D(p, p')$. We may thus employ the dominated convergence theorem provided we establish an adequate bound for $D_\delta(p, p')$. For brevity in the following discussion we write $\gamma(p) = \gamma(p, \delta)$. To obtain a bound for $D_\delta(p, p')$ we proceed as follows:

$$\left| \frac{sh(1/2)(\gamma(p) - \gamma(p'))}{\sin(1/2)\delta(p - p')} \right| \leq (1/2) \left| \frac{sh(\gamma(p) - \gamma(p'))}{\sin(1/2)\delta(p - p')} \right|$$

$$= (1/2) \left| sh\gamma(p) \frac{ch\gamma(p') - ch\gamma(p)}{\sin(1/2)\delta(p - p')} + ch\gamma(p) \frac{sh\gamma(p) - sh\gamma(p')}{\sin(1/2)\delta(p - p')} \right|.$$

Thus (recall the identity $(shx - shy)/(chx - chy) = (chx + chy)/(shx + shy)$)

$$\left| \frac{sh(1/2)(\gamma(p) - \gamma(p'))}{\sin(1/2)\delta(p - p')} \right| \leq (1/2) \left[sh\gamma(p) + ch\gamma(p) \frac{ch\gamma(p') + ch\gamma(p)}{sh\gamma(p') + sh\gamma(p)} \right]$$

$$\times \left| \frac{ch\gamma(p) - ch\gamma(p')}{\sin(1/2)\delta(p - p')} \right|.$$

But $ch\gamma(p) - ch\gamma(p') = \cos\delta p' - \cos\delta p$, $(\cos\delta p' - \cos\delta p)/(\sin(1/2)\delta(p - p')) = 2\sin(1/2)\delta(p + p')$, and $sh\gamma(p) + sh\gamma(p') \geq \gamma(p)$

$+\gamma(p')$ so that

$$\left|\frac{sh(1/2)(\gamma(p)-\gamma(p'))}{\sin(1/2)\delta(p-p')}\right| \le \left[sh\gamma(p)+ch\gamma(p)\frac{ch\gamma(p')+ch\gamma(p)}{\gamma(p')+\gamma(p)}\right] \times |\sin(1/2)\delta(p+p')|.$$

Finally observe that $|\delta^{-1}\sin(1/2)\delta(p+p')/\delta^{-1}(\gamma(p)+\gamma(p'))| \le |(1/2)(p+p')/m(\omega(p)+\omega(p'))| \le 1/2m$ and that $|\gamma(p,\delta)|$ is uniformly bounded in p and δ. Consequently

$$|D_\delta(p,p')| \le (\text{const.})e^{-mt\omega(p)}e^{-ms\omega(p')},\ p,p' \in [-\pi/\delta, \pi/\delta].$$

This bound and the analog for $D(p,p')$ are evidently sufficient for dominated convergence. □

LEMMA 4.3. *Suppose $s>0$ and $t>0$. The difference $\chi_\delta C\chi_\delta - i_\delta C_\delta i_\delta^*$ tends to zero in operator norm as $\delta \to 0$.*

Proof. Observe that $i_\delta C_\delta i_\delta^*$ is the sum of two operators each of which is s bracketed by multiplication operators. Now s is uniformly bounded and the presence of exponential factors in each of the multiplication operators makes it possible to conclude from Lemma 4.1 that each of these multiplication operators converges uniformly to its pointwise scaling limit. Thus if we replace the multiplication operators in $i_\delta C_\delta i_\delta^*$ by their pointwise scaling limits we make an error which tends to zero in uniform norm as $\delta \to 0$. To prove the lemma we are thus reduced to showing that the operator with kernel

$$e^{-t\omega(p)}\frac{(\omega(p)+\omega(p'))\chi_\delta(p)\chi_\delta(p')(\delta/2)}{\sin(\delta/2)(p-p')}e^{-s\omega(p')}$$

and the operator with kernel

$$e^{-t\omega(p)}\frac{(\omega(p)+\omega(p'))\chi_\delta(p)\chi_\delta(p')}{p-p'}e^{-s\omega(p')}$$

differ by an operator which tends to zero in uniform norm as $\delta \to 0$. Let

$$\Delta_\delta(p,p') = \chi_\delta(p)\left[\frac{\delta/2}{\sin(\delta/2)(p-p')} - \frac{1}{p-p'}\right]\chi_\delta(p')$$

and

$$g(p,p') = e^{-t\omega(p)}e^{-s\omega(p')}(\omega(p)+\omega(p')).$$

We wish to estimate the operator norm of the operator with kernel $g(p,p')\Delta_\delta(p,p')$. To deal with this kernel it is convenient to make an additional subtraction which eliminates "spurious" singularities at $(p, p') = (\pm\pi/\delta, \mp\pi/\delta)$. Consider the kernel $g(\pi/\delta, \pi/\delta)\Delta_\delta(p, p')$. Since $g(\pi/\delta, \pi/\delta)$ tends to 0 as $\delta \to 0$ and $\Delta_\delta(p, p')$ is the kernel of a uniformly bounded operator it follows that we may work with the kernel $[g(p, p') - g(\pi/\delta, \pi/\delta)]\Delta_\delta(p, p')$ rather than $g(p, p')\Delta_\delta(p, p')$ making an error which tends to zero in uniform norm as $\delta' \to 0$.

If $K(p, p')$ is the kernel of an integral operator K on $\mathfrak{X}(0)$ then the following estimate gives a well-known bound for the operator norm of K:

$$2\pi \| K \| \leq \left[\sup_p \int_{-\infty}^{\infty} |K(p, p')| \frac{dp'}{\omega(p')} \right]^{1/2} \left[\sup_{p'} \int_{-\infty}^{\infty} | K(p, p')| \frac{dp}{\omega(p)} \right]^{1/2}.$$

We shall use this to estimate the norm of the operator with kernel $K(p, p') = [g(p, p') - g(\pi/\delta, \pi/\delta)]\Delta_\delta(p, p')$. Observe first that

$$\begin{aligned} \int_{-\infty}^{\infty} |K(p, p')| \frac{dp'}{\omega(p')} \leq & \int_p^{\pi/\delta} [g(p, p') - g(\pi/\delta, \pi/\delta)] |\Delta_\delta(p, p')| \frac{dp'}{\omega(p')} \\ & + \int_{-\pi/\delta}^{p} [g(p, p') - g(\pi/\delta, \pi/\delta)] \\ & \times |\Delta_\delta(p, p')| \frac{dp'}{\omega(p')}. \end{aligned} \tag{4.4}$$

Now since, $\sin x \geq x - \pi^{-1}x^2$ ($x \in [0, \pi]$) we have

$$0 \leq \frac{x}{\sin x} - 1 \leq \frac{x}{x - \pi^{-1}x^2} - 1 = \frac{x}{\pi - x} \quad (x \in [0, \pi]).$$

Thus

$$\begin{aligned} &\left| \frac{\delta/2}{\sin(\delta/2)(p - p')} - \frac{1}{p - p'} \right| = \left| \frac{(\delta/2)(p - p')}{\sin(\delta/2)(p - p')} - 1 \right| \frac{1}{p - p'} \\ &\leq \frac{(\delta/2)(p - p')}{\pi - (\delta/2)(p - p')} \frac{1}{(p - p')} = \frac{\delta/2}{\pi - (\delta/2)(p - p')}, \\ &\qquad\qquad\qquad\qquad 0 \leq p - p' \leq \frac{2\pi}{\delta}. \end{aligned}$$

In a similar fashion

$$\begin{aligned} &\left| \frac{\delta/2}{\sin(\delta/2)(p - p')} - \frac{1}{p - p'} \right| \leq \frac{\delta/2}{\pi + (\delta/2)(p - p')}, \\ &\qquad\qquad\qquad\qquad 0 \geq p - p' \geq -\frac{2\pi}{\delta}. \end{aligned}$$

Making use of these estimates in (4.4) one finds

$$\int_{-\infty}^{\infty}|K(p,p')|\frac{dp'}{\omega(p')}\le\delta\int_{p}^{\pi/\delta}\frac{g(p,p')-g(\pi/\delta,\pi/\delta)}{[\pi-\delta p]+[\pi+\delta p']}\frac{dp'}{\omega(p')}$$
$$+\delta\int_{-\pi/\delta}^{p}\frac{g(p,p')-g(\pi/\delta,\pi/\delta)}{[\pi+\delta p]+[\pi-\delta p']}\frac{dp'}{\omega(p')}$$
$$\le\delta\int_{p}^{\pi/\delta}\frac{g(p,p')-g(\pi/\delta,p')}{\pi-\delta p}\frac{dp'}{\omega(p')}$$
$$+\delta\int_{p}^{\pi/\delta}\frac{g(\pi/\delta,p')-g(\pi/\delta,\pi/\delta)}{\pi+\delta p'}\frac{dp'}{\omega(p')}$$
$$+\delta\int_{-\pi/\delta}^{p}\frac{g(p,p')-g(\pi/\delta,p')}{\pi+\delta p}\frac{dp'}{\omega(p')}$$
$$+\delta\int_{-\pi/\delta}^{p}\frac{g(\pi/\delta,p')-g(\pi/\delta,\pi/\delta)}{\pi-\delta p'}\frac{dp'}{\omega(p')}.$$

It is now straightforward to estimate the terms on the righthand side of this last inequality to show that they go to zero uniformly in p as $\delta\to 0$. The same estimates also control $\sup_{p'}\int_{-\infty}^{\infty}|K(p,p')|dp'/\omega(p')$ and we have finished the proof of the lemma. □

Before we state Theorem 4.2 we introduce some notation. First observe that the formulas in Theorem 4.1 for the Ising correlations on the lattice make sense for nonintegral "lattice sites." If $x\in\mathbb{R}^2$ we write $\pi_j(x)$ for the j'th component of x. Suppose $x_1,\dots,x_r\in\mathbb{R}^2$ and $\pi_2(x_1)<\pi_2(x_2)\cdots<\pi_2(x_r)$. We define

$$\begin{aligned}\sigma_T^2(x_1,\dots,x_r)&=\det{}_2(I+G(T)), && T<T_c,\\ &=\det{}_2(I+G(T^*))\det H(T^*), && T>T_c.\end{aligned}\tag{4.5}$$

The parameters in the definitions of $G(T)$ and $H(T)$ are $\Delta m_i=\pi_1(x_i)-\pi_1(x_{i-1})$ and $\Delta n_i=\pi_2(x_i)-\pi_2(x_{i-1})$. If $x_1\cdots x_r\in\mathbb{R}^2$ and the x_i all have distinct second coordinates, let α denote the permutation of $1,\cdots,r$ such that $\pi_2(x_{\alpha(1)})<\cdots<\pi_2(x_{\alpha(r)})$. We define $\sigma_T(x_1\cdots x_r)=\sigma_T(x_{\alpha(1)}\cdots x_{\alpha(r)})$. We make no attempt to define $\sigma_T(x_1\cdots x_r)$ at points where there are coincidences among $\pi_2(x_1),\dots,\pi_2(x_r)$.

We now define

$$\begin{aligned}S_\delta^-(x_1,\dots,x_r)&=\sigma_{T(\delta)}(\delta^{-1}x_1,\dots,\delta^{-1}x_r),\\ S_\delta^+(x_1,\dots,x_r)&=\sigma_{T(\delta)^*}(\delta^{-1}x_1,\dots,\delta^{-1}x_r),\end{aligned}\tag{4.6}$$

where $T(\delta)$ is the temperature below T_c implicitly defined by $ch\gamma(\theta) = 2 + (1/2)\delta^2 - \cos\theta$. The reader may note that there is some ambiguity in σ_T caused by taking a square root. For points on the integer lattice it is easy to remove the ambiguity by taking a positive square root. We will discuss the elimination of the ambiguity in the scaling limits, $\lim_{\delta\to 0} S_\delta^{\pm}$, in the course of the proof of Theorem 4.2.

In the proof of Theorem 4.2 we shall write i_δ for $i_\delta \oplus \cdots \oplus i_\delta$ to avoid clumsy notation. We will also make use of the fact that if $G_\delta : \mathfrak{X}(\delta) \oplus \cdots \oplus \mathfrak{X}(\delta) \to \mathfrak{X}(\delta) \oplus \cdots \oplus \mathfrak{X}(\delta)$ is a Schmidt class operator, then $\det_2(I + G_\delta) = \det_2(I + i_\delta G_\delta i_\delta^*)$. This follows from the isometric character of i_δ.

THEOREM 4.2. *Let $x_1, \ldots, x_r \in \mathbb{R}^2$ and suppose $\pi_2(x_1) < \pi_2(x_2) \cdots < \pi_2(x_r)$. Then*

$$\lim_{\delta\to 0} \left[S_\delta^-(x_1, \ldots, x_r)\right]^2 = \det{}_2(I + G),$$

$$\lim_{\delta\to 0} \left[S_\delta^+(x_1, \ldots, x_r)\right]^2 = \det{}_2(I + G)\det H.$$

G is the $r \times r$ block matrix with entries

$$
\begin{aligned}
G_{ij} &= -K_{ij} D(\Delta n_j)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & j &> i+1, \\
&= z^{\Delta m_{i+1}} D(\Delta n_j)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & j &= i+1, \\
&= 0 & j &= i, \\
&= z^{-\Delta m_i} D(\Delta n_{j+1})\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, & j &= i-1, \\
&= K_{ij} D(\Delta n_{j+1})\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, & j &< i-1,
\end{aligned}
$$

where $z = e^{ip}$, $\Delta m_i = \pi_1(x_i) - \pi_1(x_{i-1})$, $\Delta n_i = \pi_2(x_i) - \pi_2(x_{i-1})$, $D(s)$ is the integral operator on $\mathfrak{X}(0)$ with kernel

$$ie^{-s\omega(p)/2} \frac{\omega(p) - \omega(p')}{p - p'} e^{-s\omega(p')/2} J,$$

and K is the $r \times r$ block matrix with entries

$$K_{ij} = -z^{\Delta m_{i+1}} C(\Delta n_{i+1}, \Delta n_{i+2}) \cdots \qquad z^{\Delta m_{j-1}} C(\Delta n_{j-1}, \Delta n_j) z^{\Delta m_j} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad j > i+1,$$

$$= \qquad -z^{\Delta m_{i+1}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad j = i+1,$$

$$= \qquad 0 \qquad j = i,$$

$$= \qquad z^{-\Delta m_i} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad j = i-1,$$

$$= \qquad z^{\Delta m_i} C(\Delta n_i, \Delta n_{i-1}) \cdots z^{-\Delta m_{j+2}} C(\Delta n_{j+2}, \Delta n_{j+1}) z^{-\Delta m_{j+1}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad j < i-1,$$

where $C(s, t)$ is the bounded operator on $\mathcal{X}(0)$ with principal value kernel

$$e^{-s\omega(p)/2} \frac{\omega(p) + \omega(p')}{i(p - p')} e^{-t\omega(p')/2}.$$

H is the $r \times r$ skew symmetric matrix with entries above the diagonal given by

$$H_{ij} = -\left\langle (I + G)^{-1} K u_j, \bar{u}_i \right\rangle.$$

The inner product in this definition is the hermitian symmetric inner product on the direct sum of r copies of $\mathcal{X}(0) \oplus \mathcal{X}(0)$ with complex structure $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and conjugation $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. The vector u_i is given by

$$u_i = I_i \left\{ e^{-\Delta n_i \omega(p)/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus e^{-\Delta n_{i+1} \omega(p)/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Proof. First introduce the unitary scale transformation into the formulas in Theorem 4.1. This has the effect of changing $\mathcal{X}(1, T)$ to $\mathcal{X}(\delta, T)$. Next introduce the scaled parameters $\Delta m_j/\delta$ and $\Delta n_j/\delta$ for Δm_j and Δn_j and evaluate the result at $T = T(\delta)$. One finds $S_\delta^-(x_1 \cdots x_r) = \det_2(I + G_\delta)$ and $S_\delta^+(x_1 \cdots x_r) = \det_2(I + G_\delta)\det H_\delta$, where $\Delta m_j/\delta = \pi_1(x_j) - \pi_1(x_{j-1})$, $\Delta n_j/\delta = \pi_2(x_j) - \pi_2(x_{j-1})$, G_δ is obtained from $G(T)$ by replacing $C(t, s)$ and $D(s)$ by $C_\delta(t, s)$ and $D_\delta(s)$, and H_δ is obtained from $H(T)$

by the same substitutions and the further replacement of u_i by its scaled counterpart.

Let χ_δ denote the projection on $\mathcal{X}(0)$ given by multiplication by $\chi_\delta(p)$. As above we also use χ_δ to denote direct sums of copies of χ_δ. To establish the convergence of S_δ^- we may follow the argument in the proof of Theorem 2.2 to reduce the problem to showing that $\chi_\delta G \chi_\delta - i_\delta G_\delta i_\delta^*$ converges to zero in Schmidt norm as $\delta \to 0$. Inserting additional projections χ_δ in the matrix elements for $\chi_\delta G \chi_\delta$ and using the convergence results in Lemmas 4.2 and 4.3 we have $\lim_{\delta \to 0} \|\chi_\delta G \chi_\delta - i_\delta G_\delta i_\delta^*\|_2 = 0$.

As in the proof of Theorem 3.2 we wish to show that $\det_2(I + G) \neq 0$ from which we may conclude that $(I + G)$ is invertible and that $(I + G_\delta)^{-1}$ converges to $(I + G)^{-1}$ uniformly. Suppose $(x_1, \ldots, x_r)$ is an r-tuple of points in $\mathbb{R}^2$ with dyadic rational components and distinct second components. Let $\delta_n = 2^{-n}$. Then since for sufficiently large n we have $(x_1, \ldots, x_r) \in (\delta_n \mathbb{Z}^2)^r$ it follows from Griffith's inequalities that $\det_2(I + G) = \lim_{n \to \infty} \det_2 (I + G_{\delta_n}) \geq 1$, where G and G_δ are evaluated at the lattice points $(x_1, \ldots, x_r)$. However, G depends continuously (in Schmidt norm) on the parameters $(x_1, \ldots, x_r)$ in regions which stay away from coincident second coordinates. This is easily proved using the observation that the Schmidt class continuity of an operator valued function $x \to U(x)V$ follows from the *strong* continuity of $x \to U(x)$ and the Schmidt class character V. The continuity of G and the density of the dyadic rationals imply $\det_2(I + G) \geq 1$ at all r-tuples in $(\mathbb{R}^2)^r$ without coincident second components. Thus $(1 + G)$ is invertible at such points and Lemmas 4.2 and 4.3 supply all the additional information needed to show that H_δ converges to H. The reader should note that the ambiguity in S_δ^- is naturally resolved by taking a positive square root. The remaining ambiguity in S_δ^+ is settled using $\det H = (\mathrm{pf} H)^2$, (see Theorem 3.1). $\square$

5.0. In this section we will establish some of the probabilistic and field theoretic properties which are expected for the scaled n-point functions. Since the classic work of Dobrushin [14, 15] and of Lanford and Ruelle [38] it is customary to consider the infinite volume Ising model as a particular example of a Gibbs random field. We briefly review this probabilistic framework. Let $\sigma \in \Omega = \{-1, 1\}^{\mathbb{Z}^2}$ and take Σ to be the σ-algebra of subsets of Ω generated by the cylinder subsets of the form $\{\sigma(\cdot) | \sigma(x_i) = \sigma_i, i = 1, \ldots, n\}$, where $\sigma_i \in \{-1, 1\}$, $i = 1, \ldots, n$. A Gibbs state μ_T at temperature T for the two-dimensional Ising model is a probability measure on (Ω, Σ) whose conditional expectations satisfy the appropriate D.L.R. equations. The set of Gibbs states at temperature T is a Choquet simplex. For $T > T_c$ this set has only one extremal point [39] and for $T < T_c$ there are exactly two extremal Gibbs states μ_+ and μ_- [7, 8]. The measures μ_+ and μ_- are obtained by passing to the thermodynamic limit with + and −

boundary conditions [15]. Since the random variable σ at sites x assumes only the values ± 1, the finite-dimensional distributions of the random field μ are directly computable from the moments $\int_\Omega \sigma(x_1)\cdots\sigma(X_n)d\mu_T$, where $x_i \in \mathbf{Z}^2$. Thus for fixed $T \neq T_c$, Theorems 2.2, 2.3, 3.2, and 3.3 give a concrete characterization of the finite dimensional distributions of the two dimensional Ising field.

Theorem 4.2 gives the scaling limit of these expectations when the one-point functions are divided out. It is natural to ask if the resulting functions are the moments of a generalized random field on $\mathcal{S}'(\mathbb{R}^2)$ [16, 20]. In order to answer this question we introduce random fields $\sigma_\delta^{\pm}(\cdot)$ as follows. Let $f \in \mathcal{S}(\mathbb{R}^2)$, $\delta > 0$, and define $\sigma_\delta^-(f) = \langle \sigma \rangle_{T(\delta)}^{-1} \Sigma_{n \in \mathbf{Z}^2} \sigma(n) f(\delta n)\delta^2$ as a random variable on $(\Omega, \Sigma, d\mu_{T(\delta)})$ and $\sigma_\delta^+(f) = \langle \sigma \rangle_{T(\delta)}^{-1} \Sigma_{n \in \mathbf{Z}^2} \sigma(n) f(\delta n)\delta^2$ as a random variable on $(\Omega, \Sigma, d\mu_{T^*(\delta)})$. The characteristic functions of these random fields are

$$\chi_\delta^-(f) = \int_\Omega \exp\left[i\sigma_\delta^-(f)\right] d\mu_{T(\delta)},$$

$$\chi_\delta^+(f) = \int_\Omega \exp\left[i\sigma_\delta^+(f)\right] d\mu_{T^*(\delta)}. \tag{5.1}$$

Since $\chi_\delta^{\pm}(\cdot)$ are the characteristic functions of random fields they are necessarily positive definite in the following sense [20]. Suppose $f_j \in \mathcal{S}(\mathbb{R}^2)$ and $\alpha_j \in \mathbf{C}$ $(j = 1, \ldots, n)$ then

$$\sum_{j,k} \alpha_j \bar{\alpha}_k \chi_\delta^{\pm}(f_j - f_k) \geq 0. \tag{5.2}$$

Since $\sigma_\delta^{\pm}(f)$ is a bounded random variable the power series for $\exp[i\sigma_\delta^{\pm}(f)]$ converges uniformly and it is permissible to integrate this series term by term to obtain

$$\chi_\delta^{\pm}(f) = 1 + \sum_{l=1}^{\infty} \frac{(i)^l}{l!} \sum_{n_1 \cdots n_l \in \mathbf{Z}^2} f(\delta n_1) \cdots f(\delta n_l) S_\delta^{\pm}(\delta n_1, \ldots, \delta n_l)\delta^{2l}. \tag{5.3}$$

Given the pointwise convergence (almost everywhere) in Theorem 4.2 it is natural to conjecture that $\lim_{\delta \to 0^+} \chi_\delta^{\pm}(f) = \chi^{\pm}(f)$, where

$$\chi^{\pm}(f) = 1 + \sum_{l=1}^{\infty} \frac{(i)^l}{l!} \int dx_1 \cdots dx_l f(x_1) \cdots f(x_l) S^{\pm}(x_1 \cdots x_l). \tag{5.4}$$

Much of the rest of this section will be devoted to a proof of this convergence. As we wish to use the Bochner–Minlos theorem (see, for instance, [20, 24]) to show that $\chi^{\pm}(f)$ are the characteristic functions of generalized random fields on $\mathcal{S}'(\mathbb{R}^2)$ the convergence of $\chi_\delta^{\pm}(f)$ to $\chi^{\pm}(f)$ and (5.2) will establish positive definiteness for $\chi^{\pm}(f)$. The reader should note that the formulas for $S^{\pm}(x_1,\ldots,x_l)$ given in Theorem 4.2 do not manifest this positivity in an obvious fashion. The continuity of $\chi^{\pm}(f)$ as a function of f in $\mathcal{S}(\mathbb{R}^2)$ and $\chi^{\pm}(0)=1$ are the other hypothesis of the Bochner–Minlos theorem. The second property is evident from the definition of $\chi^{\pm}(f)$ and the continuity will be apparent in the course of the proof of the convergence of $\chi_\delta^{\pm}(f)$ to $\chi^{\pm}(f)$.

The scheme of our proof is to establish integrable bounds for $S_\delta^{\pm}(x_1,\ldots,x_l)$ uniform in δ so that we may employ dominated convergence to pass from (5.3) to (5.4). Gaussian domination [51] reduces this problem to finding bounds for the two-point function. The formula in the following theorem is of interest in its own right and will yield the estimate we desire directly.

THEOREM 5.0. *Let $T<T_c$. Then if $\pi_2(x_2)\geq\pi_2(x_1)$*

$$S_\delta^-(x_1,x_2)=\sum_{k=0}^{\infty}F_{2k}(\Delta m,\Delta n),\qquad \Delta m=\pi_1(x_2)-\pi_1(x_1),$$

$$\Delta n=\pi_2(x_2)-\pi_2(x_1),$$

where $F_0(\Delta m,\Delta n)=1$ and

$$F_{2k}(s,t)=(2k!)^{-1}\int_{-\pi/\delta}^{\pi/\delta}\frac{dp_1}{2\pi}\cdots\frac{dp_{2k}}{2\pi}$$

$$\times\prod_{j=1}^{2k}\frac{e^{-isp_j-t\gamma(p_j)/\delta}}{\delta^{-1}sh\gamma(p_j)}\prod_{1\leq i<j\leq 2k}(h_{ij})^2$$

with

$$h_{ij}=\frac{sh[(\gamma(p_i)-\gamma(p_j))/2]}{\sin[\delta(p_i+p_j)/2]}$$

and $\gamma(p)=\gamma(p,\delta)$.

Proof. We will give the proof for $\delta=1$ and leave it to the reader to introduce the obvious change of scale. By Theorem 4.1 we have $S_1^-(x_1,x_2)^2$

$= \det_2(I + G(T))$, where

$$G(T) = \begin{bmatrix} 0 & 0 & 0 & z^{\Delta m} D(\Delta n) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z^{-\Delta m} D(\Delta n) & 0 & 0 & 0 \end{bmatrix}$$

It we define a kernel $g(\theta_1, \theta_2)$ by

$$g(\theta_1, \theta_2) = e^{(i\theta_1 \Delta m - \gamma(\theta_1)\Delta n)/2} h(\theta_1, \theta_2) e^{(i\theta_2 \Delta m - \gamma(\theta_2)\Delta n)/2},$$

where

$$h(\theta_1, \theta_2) = \frac{sh[(\gamma(\theta_1) - \gamma(\theta_2))/2]}{\sin[(\theta_1 + \theta_2)/2]}.$$

then it is an elementary calculation that

$$\mathrm{Tr}[G(T)^n] = 0 \quad n \text{ odd}, \qquad n \geq 3,$$

$$\mathrm{Tr}[G(T)^n] = 2\int_{-\pi}^{\pi} \mathbf{d}\boldsymbol{\theta}_1 \cdots \int_{-\pi}^{\pi} \mathbf{d}\boldsymbol{\theta}_n g(\theta_1, \theta_2) g(\theta_2, \theta_3) \cdots g(\theta_n, \theta_1), \qquad n \text{ even},$$

where $\mathbf{d}\boldsymbol{\theta} = (2\pi sh\gamma(\theta))^{-1} d\theta$.

To verify this result one should make the changes of variables $\theta_j \to -\theta_j$ for even integers j (making use of $\gamma(\theta) = \gamma(-\theta)$) and when computing the trace of an integral operator on $\mathcal{X}(1, T)$ one should keep in mind that this space is $l^2(\mathbf{Z}_{1/2}, \mathbf{C})$ to avoid introducing an extra factor of 2.

Since $g(\theta_1, \theta_2) = -g(\theta_2, \theta_1)$ it follows that $\mathrm{Tr}[G(T)^n] = 2\mathrm{Tr}(g^n)$ for all $n \geq 2$. The Plemelj–Smithies formula [67] expresses $\det_2(I + G(T))$ in terms of the traces $\mathrm{Tr}[G(T)^n]$ and one finds

$$\det_2(I + G(T)) = (\det_2(1 + g))^2.$$

We now use the Hilbert formula for $\det_2(1 + g)$ to obtain

$$\det_2(1 + g) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\pi}^{\pi} \mathbf{d}\boldsymbol{\theta}_1 \cdots \int_{-\pi}^{\pi} \mathbf{d}\boldsymbol{\theta}_k \, g\begin{pmatrix} \theta_1 \cdots \theta_k \\ \theta_1 \cdots \theta_k \end{pmatrix},$$

where $g\begin{pmatrix} \theta_1 \cdots \theta_k \\ \theta_1 \cdots \theta_k \end{pmatrix} = \det[g_{ij}]_{k \times k}$ has entries $g_{ij} = g(\theta_i, \theta_j)$. Zeros along the diagonal are required for the Hilbert formula but because of the antisymmetry $g(\theta_i, \theta_i) = 0$ in any case. Observe also that since the matrix $[g_{ij}]_{k \times k}$ is antisymmetric, $\det[g_{ij}]_{k \times k} = 0$ for odd k.

If we now pull the factors $e^{(i\theta_k \Delta m - \gamma(\theta_k)\Delta n)/2}$ out of $\det[g_{ij}]_{k\times k}$ we find

$$S_1^-(x_1, x_2) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi}$$

$$\cdots \int_{-\pi}^{\pi} \frac{d\theta_{2k}}{2\pi} \prod_{j=1}^{2k} \frac{e^{-i\theta_j \Delta m - \gamma(\theta_j)\Delta n}}{sh\gamma(\theta j)} h\begin{pmatrix} \theta_1 \cdots \theta_{2k} \\ \theta_1 \cdots \theta_{2k} \end{pmatrix},$$

where $h\begin{pmatrix} \theta_1 \cdots \theta_{2k} \\ \theta_1 \cdots \theta_{2k} \end{pmatrix} = \det[h_{ij}]_{2k\times 2k}$, $h_{ij} = h(\theta_i, \theta_j)$.

To finish this proof it remains to establish that

$$\det[h_{ij}]_{2k\times 2k} = \prod_{1\le i<j\le 2k} (h_{ij})^2 \tag{5.5}$$

This identity is crucial for the estimate we wish to make.

To prove (5.5) we find it necessary to first express h_{ij} in terms of Jacobi elliptic functions. That such a parametrization is possible is suggested by the work of Onsager [52] and by work of Johnson *et al.* (in particular, see their Eq. (3.5)) [27].

We proceed to uniformize $a_+(z)$ and $a_-(z)$. The fractional linear transformation

$$z = \frac{\alpha_2 - kz'}{1 - \alpha_2 kz'}, \qquad k = (sh2K)^{-2},$$

puts the branch points into canonical positions

$$\begin{aligned} a_+\ (z(z')) &= \left[\frac{1-\alpha_1\alpha_2}{1-\alpha_2^2}\right]^{1/2} (1 - k^2 z')^{1/2}, \\ a_-\ (z(z')) &= \left[-\frac{1-\alpha_2^2}{1-\alpha_1\alpha_2}\right]^{1/2} \left[\frac{z'}{1-z'}\right]^{1/2}, \end{aligned} \tag{5.6}$$

and then the substitution $z' = sn^2(u, k)$ gives the desired uniformization once one recalls the identities

$$sn^2(u,k) + cn^2(u,k) = 1,$$

$$dn^2(u,k) + k^2 sn^2(u,k) = 1,$$

for the Jacobi elliptic functions $sn(u, k)$, $cn(u, k)$, and $dn(u, k)$ [17].

The identity

$$\tfrac{1}{2}\left[a_+(z_1)a_-(z_2) - a_-(z_1)a_+(z_2)\right]\left[\sin\tfrac{1}{2}(\theta_1+\theta_2)\right]^{-1}$$
$$= e^{i\alpha(\theta_1)/2}\left[sh\gamma(\theta_1)\right]^{-1/2}h(\theta_1,\theta_2)e^{i\alpha(\theta_2)/2}\left[sh\gamma(\theta_2)\right]^{-1/2} \tag{5.7}$$

follows from (4.1), where $z_j = e^{i\theta_j}$, $j = 1,2$. In this identity we make use of two further identities

$$\left[\sin\tfrac{1}{2}(\theta_1+\theta_2)\right]^{-1} = -2R(z_1')\left(1 - k^2 z_1' z_2'\right)^{-1}R(z_2')$$

and

$$e^{-i\alpha(\theta)}sh\gamma(\theta) = k^{1/2}(1 - z')\left[R(z')\right]^{-2},$$

where $R(z') = [(1-\alpha_2^2)^{-1}(1-\alpha_2 kz')(\alpha_2 - kz')]^{1/2}$ and z_j' is the image of z_j under the fractional linear transformation above. To prove the last identity first recall (2.2), multiply this by $e^{-i\alpha}$, use $e^{-i\alpha} = [a_+(z)a_-(z)]^{-1}$ and (5.6), and simplify the resulting expression.

The result is an expression for $h(\theta_1,\theta_2)$. Now make the substitutions $z_j' = sn^2(u_j,k)$, $j = 1,2$ and use the addition formula

$$sn(u_1-u_2,k) = \frac{s_1c_2d_2 - c_1d_1s_2}{1 - k^2s_1^2s_2^2}$$

(we employ the notation $s_j = sn(u_j,k)$, etc.) to obtain

$$h_{12} = -k^{1/2}sn(u_1-u_2,k),$$

where $h_{12} = h(\theta_1,\theta_2)$. When $-\pi < \theta < \pi$ the u variable is on the line segment $x + \frac{1}{2}i\mathbf{K}'$, $|x| < \mathbf{K}$ with $\mathbf{K}$ ($\mathbf{K}'$) the real (imaginary) quarter period (see, for example, p. 377 of [17]).

To prove (5.5) we show that

$$\mathrm{pf}(h_{ij}) = \prod_{1\le i<j\le 2k} h_{ij}, \tag{5.8}$$

where $h_{ij} = -k^{1/2}sn(u_i-u_j,k)$. The case $k=1$ is obvious. We can use the expansion formula of the pfaffian to write the left-hand side as $\sum_{j=2}^{2k}(-1)^j h_{1j}\mathrm{pf}(h[1j])$, where $h[1j]$ denotes the matrix obtained from h by striking out the 1st row and column and the jth row and column. Hence $\mathrm{pf}(h[1j])$ does not depend upon the variable u_1 for all $j = 2,3,\ldots,2k$. We take the variables $u_2, u_3, \ldots, u_{2k}$ as distinct, otherwise (5.8) is trivial. A simple induction is now possible. We use the inductive hypothesis on the expression $\mathrm{pf}(h[1j])$, $j = 2, 3, \ldots, 2k$. It is then clear that as a function of the complex variable u_1 the product $\prod_{1 \le i < j \le 2k} h_{ij}$ and $\sum_{j=2}^{2k}(-1)^j h_{1j}\mathrm{pf}(h[1j])$ are both elliptic functions, have the same periods, and both have simple poles (we restrict our attention to the fundamental parallelogram) at $u_j + i\mathbf{K}'$, $j = 2, 3, \ldots, 2k$ and at $u_j + 2\mathbf{K} + i\mathbf{K}'$, $j = 2, 3, \ldots, 2k$. (Recall $sn(z, k)$ has in the fundamental parallelogram two simple poles at $i\mathbf{K}'$ and $2\mathbf{K} + i\mathbf{K}'$ with residues k^{-1} and $-k^{-1}$, respectively.) If we show that the residues at these poles are equal, then as a simple consequence of Liouville's theorem it follows they differ only by an additive constant. The proof is now reduced to evaluating the residues at the poles $u_j + i\mathbf{K}'$ and $u_j + 2\mathbf{K} + i\mathbf{K}'$, $j = 2, 3, \ldots, 2k$. It is straightforward to see that these residues are equal once one makes use of the identities [17] $sn(z - iK')sn(z) = k^{-1}$ and $sn(z - 2K - iK')sn(z) = -k^{-1}$. The additive constant can be shown to be zero by comparing the two functions at a zero.

For real z, $|sn(z, k)| \le 1$. Hence if $\theta_j, \theta_k \in (-\pi, \pi)$ it follows that $u_j - u_k$ is real. Hence $|h_{ij}| \le k^{1/2} < 1$ for $T < T_c$, and we obtain from (5.5) the inequality

$$\left|\det\left[h_{ij}\right]\right| \le 1. \quad \square \tag{5.9}$$

As mentioned in the introduction, the infinite-volume correlations which result from passage to the thermodynamic limit in the two-stage process natural for the transfer matrix formalism, and the correlations which result from letting the sides of a square box tend simultaneously to infinity are identical for plus boundary conditions [39]. Since the correlations in a square box are evidently invariant under simultaneous rotation of all sites by $\pi/2$ (about the origin of the box) the same can be said for the plus state infinite-volume correlations (rotated about any lattice origin). The correlations we computed above T_c are also invariant under rotations by $\pi/2$ since they are correlations of $e^{2K\sigma} = (ch 2K) + (sh 2K)\sigma$ calculated with plus boundary conditions below T_c. Invariance under reflections across lattice axes follows in the same manner. This dihedral group invariance extends the convergence domain in theorem 4.2. In particular the two-point functions $S_\delta^{\pm}(x_1, x_2)$ converge as $\delta \to 0$ except when $x_1 = x_2$. The invariance under rotation by $\pi/2$ will be of some use in the following preliminary to Theorem 5.1.

LEMMA 5.1. *There exists a constant $a > 0$ such that*

$$S_\delta^{\pm}(x_1, x_2) \le ch\left[\tfrac{1}{2}K_0(ar)\right],$$

where $r^2 = \|x_2 - x_1\|^2$, $x_1, x_2 \in \delta\mathbf{Z}^2$, and $K_0(\cdot)$ is the modified Bessel function.

Proof. We first consider $S_\delta^-(x_1, x_2)$. Since $S_\delta^-(x_1, x_2)$ is actually a function of the difference $x_2 - x_1$ and is furthermore invariant under rotation of $x_2 - x_1$ by $\pi/2$ we may confine our attention to the sector of the upper plane defined by $\pi_2(x_2 - x_1) = \Delta n \geq \Delta m = \pi_1(x_2 - x_1)$. In this sector we have $\Delta n \geq 2^{-1/2} r$. If we estimate $S_\delta^-(x_1, x_2)$ using the formula in Theorem 5.0 by bringing absolute values inside the sums and integrals and using the bound

$$\left| \frac{sh[(\gamma(\theta_1) - \gamma(\theta_2))/2]}{\sin[(\theta_1 + \theta_2)/2]} \right| \leq 1$$

established in the proof of Theorem 5.0 then we find

$$S_\delta^-(x_1, x_2) \leq 1 + \sum_{k=1}^{\infty} \frac{1}{(2k)!} \left[\frac{1}{2\pi} \int_{-\pi/\delta}^{\pi/\delta} \frac{e^{-r\gamma(p)/\sqrt{2}\,\delta}}{\delta^{-1} sh\gamma(p)} dp \right]^{2k}. \tag{5.10}$$

In Lemma 4.1 we proved the δ-uniform lower bound $\delta^{-1}\gamma(p, \delta) \geq m\omega(p)$ for $p \in [-\pi/\delta, \pi/\delta]$. Let $a = 2^{-1/2} m$ then

$$e^{-r\gamma(p)/\sqrt{2}\,\delta} \leq e^{-ar\omega(p)}, \qquad p \in [-\pi/\delta, \pi/\delta]. \tag{5.11}$$

Employing the notation of Lemma 4.1 we have $ch\gamma(p) = 1 + (1/2)\delta^2\omega_\delta^2(p)$ so that $sh\gamma = (ch^2\gamma - 1)^{1/2} = (ch\gamma + 1)^{1/2}(ch\gamma - 1)^{1/2} = ((ch\gamma + 1)/2)^{1/2}\delta\omega_\delta(p)$. Thus $\delta^{-1} sh\gamma(p) \geq \omega_\delta(p)$. However,

$$\frac{\omega_\delta^2(p)}{\omega^2(p)} = \frac{1 + 2\dfrac{[1 - \cos\delta p]}{(\delta p)^2} p^2}{1 + p^2} \geq \frac{4}{\pi^2}$$

since $\theta^{-2}(1 - \cos\theta)$ attains its minimum on $[-\pi, \pi]$ at the end points $\theta = \pm\pi$. We have then

$$\delta^{-1} sh\gamma(p) \geq \omega_\delta(p) \geq \frac{2}{\pi}\omega(p), \qquad p \in [-\pi/\delta, \pi/\delta] \tag{5.12}$$

Substituting (5.11) and (5.12) in (5.10) and then pushing the integration limits to $\pm\infty$ one finds

$$S_\delta^-(x_1, x_2) \leq ch\left[\frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{-ar\omega(p)}}{\omega(p)} dp \right].$$

However,

$$K_0(s) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-s\omega(p)}}{\omega(p)} dp, s > 0,$$

so that we have finished the proof for $S_\delta^-(x_1, x_2)$.

Now suppose $T < T_c$. In a finite box for $+$ boundary conditions, Griffith's inequalities [36] imply that the correlations are decreasing functions of the temperature so that $\langle\sigma(x_1)\sigma(x_2)\rangle_{T^*}^+ \leq \langle\sigma(x_1)\sigma(x_2)\rangle_T^+$. Above T_c it is known that boundary conditions do not influence the correlations in the infinite-volume limit [39]. It follows that $S_\delta^+(x_1, x_2) \leq S_\delta^-(x_1, x_2)$. The estimate we proved for S_δ^- is consequently true for S_δ^+ as well. □

THEOREM 5.2. *Let $f_j \in \mathcal{S}(\mathbb{R}^2)$ $(j = 1, \ldots, n)$. The scaled correlations $S^\pm(f_1, \ldots, f_n) = \int S^\pm(x_1, \ldots, x_n) f_1(x_1) \cdots f_n(x_n) dx_1 \cdots dx_n$ are the moments $\langle\sigma^\pm(f_1) \cdots \sigma^\pm(f_n)\rangle$ of random fields $\sigma^\pm(f)$ on $\mathcal{S}'(\mathbb{R}^2)$. The functions $S^\pm(x_1, \ldots, x_n)$ are locally integrable.*

Proof. A partition $\{\Lambda_1, \ldots, \Lambda_m\}$ of a set $\{x_1, \ldots, x_n\}$ into disjoint subsets is said to be a pair partition if each Λ_j has two elements when n is even and if exactly one Λ_j has a single element and the rest have two elements when n is odd. We write $S_\delta^\pm(\Lambda) = S_\delta^\pm(x_i, x_j)$ if $\Lambda = \{x_i, x_j\}$, $S_\delta^+(\Lambda) = 0$ if $\Lambda = \{x_i\}$, and $S_\delta^-(\Lambda) = 1$ if $\Lambda = \{x_i\}$. Theorem 3 in [51] implies the following correlation inequalities

$$S_\delta^\pm(x_1, \ldots, x_n) \leq \Sigma S_\delta^\pm(\Lambda_1) \cdots S_\delta^\pm(\Lambda_m), \tag{5.13}$$

where the sum on the right is over all pair partitions of $\{x_1, \ldots, x_n\}$. We will use this inequality to reduce all estimates to bounds on the two-point functions.

Suppose $f_1, f_2 \in \mathcal{S}(\mathbb{R}^2)$. We shall first demonstrate that

$$\lim_{\delta\to 0} \sum_{x_1, x_2 \in \delta\mathbf{Z}^2} f_1(x_1) f_2(x_2) S_\delta^\pm(x_1, x_2)\delta^4 = \int f_1(x_1) f_2(x_2) S^\pm(x_1, x_2)\, dx_1\, dx_2. \tag{5.14}$$

Observe that $S_\delta^\pm(x, x) = \langle\sigma\rangle_{T(\delta)}^{-2} = 0(\delta^{-1/4})$ (see Corollary 2.0 and the definition of $T(\delta)$). Thus the terms in the sum on the left-hand side of (5.14) on the diagonal $x_1 = x_2$ do not make a contribution in the limit $\delta \to 0$.

Next define functions $G_\delta^\pm(x_1, x_2)$ on $\mathbb{R}^4$ in the following manner. If (x_1, x_2) is an element of the open cube of side δ and center $(y_1, y_2) \in \delta\mathbf{Z}^4$ define $G_\delta^\pm(x_1, x_2) = f_1(y_1) f_2(y_2) S_\delta^\pm(y_1, y_2)$ if $y_1 \neq y_2$, and $G_\delta^\pm(x_1, x_2) = 0$ if $y_1 = y_2$. Set $G_\delta^\pm(x_1, x_2) = 0$ on the remaining set of measure zero in $\mathbb{R}^4$.

We have

$$\sum_{\substack{x_1, x_2 \in \delta \mathbf{Z}^2 \\ x_1 \neq x_2}} f_1(x_1) f_2(x_2) S_\delta^{\pm}(x_1, x_2) \delta^4 = \int G_\delta^{\pm}(x_1, x_2)\, dx_1\, dx_2 .$$

Theorem 4.2 implies $G_\delta^{\pm}(x_1, x_2)$ converges almost everywhere to $f_1(x_1) f_2(x_2) S^{\pm}(x_1, x_2)$. Thus to prove (5.14) we need only show that $G_\delta^{\pm}(x_1, x_2)$ is dominated by a "sequence" of positive functions which converges in L^1. Since f_1 and f_2 are in $\mathcal{S}(\mathbf{R}^2)$ there exists a constant $C > 0$ such that $|f_1(x)| \le C(1 + |x|^2)^{-2}$ and $|f_2(x)| \le C(1 + |x|^2)^{-2}$. Let $D(x_1, x_2) = C^2(1 + |x_1|^2)^{-2}(1 + |x_2|^2)^{-2} ch(1/2) K_0(a|x_2 - x_1|)$ and define $D_\delta(x_1, x_2)$ as follows: if (x_1, x_2) is in the open cube with edge δ and center $(y_1, y_2) \in \delta \mathbf{Z}^4$ let $D_\delta(x_1, x_2) = D(y_1, y_2)$ if $y_1 \neq y_2$ and $D_\delta(x_1, x_2) = 0$ if $y_1 = y_2$. Set $D_\delta(x_1, x_2) = 0$ on the remaining set of measure zero in $\mathbf{R}^4$. It is an obvious consequence of Lemma 5.1 that $|G_\delta^{\pm}(x_1, x_2)| \le D_\delta(x_1, x_2)$. But $\int D_\delta(x_1, x_2)\, dx_1\, dx_2$ is a Riemann sum approximation to the integral $\int D(x_1, x_2)\, dx_1\, dx_2$. Since $ch(1/2) K_0(ar)$ is $O(r^{-1/2})$ as $r \to 0$ and is asymptotic to 1 as $r \to \infty$ it follows that $D(x_1, x_2)$ is absolutely integrable. Furthermore $\lim_{\delta \to 0} \int D_\delta(x_1, x_2)\, dx_1\, dx_2 = \int D(x_1, x_2)\, dx_1\, dx_2$. This is a consequence of the fact that it is possible to make the contributions to the Riemann sums for $D(x_1, x_2)$ coming from a neighborhood of the diagonal $x_1 = x_2$ and near ∞ uniformly small in δ by choosing "small" enough neighborhoods of $x_1 = x_2$ and ∞. The monotone behavior of $ch(1/2) K_0(a|x_2 - x_1|)$ near $x_1 = x_2$ and of $(1 + |x_1|^2)^{-2}$ near ∞ make it straightforward to give the appropriate estimates. We have finished the proof of (5.14).

Now suppose $f_l(x) \in \mathcal{S}(\mathbf{R}^2)$ $(l = 1, \ldots, n)$; the correlation inequality (5.13) and the behavior of the two-point function $S_\delta^{\pm}(x, x) = \langle \sigma \rangle_{T(\delta)}^{-2}$ at coincidence are enough to show that the sum over pair coincidences $x_i = x_j$ makes no contribution to the sum $\sum_{x \in \delta \mathbf{Z}^{2n}} f_1(x_1) \cdots f_n(x_n) S_\delta^{\pm}(x_1, \ldots, x_n) \delta^{2n}$ in the limit $\delta \to 0$. As above we define functions $G_\delta^{\pm}(x_1, \ldots, x_n)$. Let $G_\delta^{\pm}(x_1, \ldots, x_n) = f_1(y_1) \cdots f(y_n) S_\delta^{\pm}(y_1, \ldots, y_n)$ if $(x_1, \ldots, x_n)$ is in the open cube with edge δ and center $(y_1 \cdots y_n) \in \delta \mathbf{Z}^{2n}$ and all the points $y_1, \ldots, y_n$ are distinct. Let $G_\delta^{\pm}(x_1, \ldots, x_n) = 0$ otherwise. Then

$$\sum_{\substack{x \in \delta \mathbf{Z}^{2n} \\ x_i \neq x_j}} f_1(x_1) \cdots f_n(x_n) S_\delta^{\pm}(x_1, \ldots, x_n) \delta^{2n} = \int G_\delta^{\pm}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n .$$

Choose $C > 0$ large enough so that $|f_l(x)| \le C(1 + |x|^2)^{-2}$ for $l = 1, \ldots, n$. If n is even we may use (5.13) to give the estimate:

$$|G_\delta^{\pm}(x_1, \ldots, x_n)| \le \sum D_\delta(\Lambda_1) \cdots D_\delta(\Lambda_{n/2}), \tag{5.15}$$

where the sum is over all pair partitions. Since each $D_\delta(\Lambda_l)$ converges in L^1 it follows that the right-hand side of (5.15) converges in L^1. Theorem 4.2 implies that $G_\delta^\pm(x_1,\ldots,x_n)$ converges almost everywhere so that by dominated convergence

$$\lim_{\delta\to 0}\int G_\delta^\pm(x_1\cdots x_n)\,dx_1\cdots dx_n$$
$$=\int f_1(x_1)\cdots f_n(x_n)S^\pm(x_1\cdots x_n)\,dx_1\cdots dx_n.$$

When n is odd only trivial changes are necessary to obtain the same result.

We have shown that

$$\lim_{\delta\to 0}\sum_{x\in\delta\mathbf{Z}^{2n}} f_1(x_1)\cdots f_n(x_n)S_\delta^\pm(x_1,\ldots,x_n)\delta^{2n}$$
$$=\int f_1(x_1)\cdots f_n(x_n)S^\pm(x_1,\ldots,x_n)\,dx_1\cdots dx_n. \qquad (5.16)$$

When n is even we have

$$\sum_{x\in\delta\mathbf{Z}^{2n}} f(x_1)\cdots f(x_n)\sum D_\delta(\Lambda_1)\cdots D_\delta(\Lambda_{n/2})\delta^{2n}$$
$$=\frac{n!}{2^{n/2}(n/2)!}(D_\delta(f,f))^{n/2}, \qquad (5.17)$$

where $D_\delta(f,f)=\sum_{x\in\delta\mathbf{Z}^4} f(x_1)f(x_2)D_\delta(x_1,x_2)\delta^4$. When n is odd

$$\left|\sum_{x\in\delta\mathbf{Z}^{2n}} f(x_1)\cdots f(x_n)\sum D_\delta(\Lambda_1)\cdots D_\delta(\Lambda_{(n-1)/2})\delta^{2n}\right|$$
$$\le\frac{Cn!}{2^{(n-1)/2}[(n-1)/2!]}(D_\delta(f,f))^{(n-1)/2}, \qquad (5.18)$$

where the constant C is chosen to dominate $|\sum_{x\in\delta\mathbf{Z}^2} f(x)\delta^2|$. Since $D_\delta(f,f)$ converges as $\delta\to 0$ the estimates in (5.17) and (5.18) can be made uniform in δ. This observation coupled with (5.16) is sufficient to conclude that $\chi_\delta^\pm(f)\to\chi^\pm(f)$ as $\delta\to 0$. As remarked earlier this convergence suffices to demonstrate the positivity necessary for the characteristic function of a random field on $\mathcal{S}'(\mathbb{R}^2)$. The continuity of $\chi^\pm(f)$ as a function of $f\in\mathcal{S}(\mathbb{R}^2)$ is evident from estimates given already. The Bochner–Minlos theorem applies and we have completed the proof of the theorem. $\square$

THEOREM 5.4. *The two-point Schwinger functions* $S^{\pm}(x_1 - x_2)$, $x_1, x_2 \in \mathbb{R}^2$ *have the representation*

$$S^-(x_1 - x_2) = 1 + \sum_{n=1}^{\infty} F_{2n}(r) \qquad (r > 0) \tag{5.19}$$

and

$$S^+(x_1 - x_2) = \sum_{n=0}^{\infty} F_{2n+1}(r) \qquad (r > 0), \tag{5.20}$$

where $r = \|x_1 - x_2\|$ *and*

$$F_n = \frac{1}{n!}\int_0^{\infty} \frac{du_1}{2\pi u_1} \cdots \frac{du_n}{2\pi u_n} \prod_{j=1}^{n} \exp\left[-\frac{1}{2}r\left(u_j + u_j^{-1}\right)\right] \prod_{1 \le i < j \le n} \left(\frac{u_i - u_j}{u_i + u_j}\right)^2$$

for $n = 1, 2, \ldots$.

Proof. To derive (5.19) for $S^-(x_1 - x_2)$ first take the scaling limit $\delta \to 0^+$ of $S_\delta^-(x_1, x_2)$ in Theorem 5.0. Now make the change of variables $p_j = \frac{1}{2}(u_j + u_j^{-1})$, $j = 1, 2, \ldots, 2n$. To see that the result is rotationally invariant rotate the u_j-contours (see [73] for details).

The derivation of (5.20) for $S^+(x_1 - x_2)$ is somewhat more involved since we have not yet derived the analog of Theorem 5.0. To do this we first make a Neumann expansion of the inverse appearing in the expression for pf H and obtain for the two-point function

$$\mathrm{pf} H = \sum_{n=0}^{\infty} g_{2n+1,\delta}(\Delta m_2, \Delta n_2),$$

where

$$g_{2n+1,\delta}(\Delta m_2, \Delta n_2) = \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_{2n+1}}{2\pi} \prod_{j=1}^{2n+1} \frac{e^{-i\Delta m_2 \theta_j - \Delta n_2 \gamma(\theta_j)}}{sh\gamma(\theta_j)}$$
$$\times \prod_{j=1}^{2n} \frac{sh\frac{1}{2}\left[\gamma(\theta_j) - \gamma(\theta_{j+1})\right]}{\sin\frac{1}{2}\left[\theta_j + \theta_{j+1}\right]} \exp\left[(i/2)(\theta_1 - \theta_{2n+1}) + (1/2)(\gamma(\theta_1) - \gamma(\theta_{2n+1}))\right].$$

This agrees with earlier results of other authors [73]. For the higher n-point functions our expression for pf$H$ is quite different from the corresponding formulas in [4, 10, 43, 57]. We could derive an expression for $S_\delta^+(x_1 - x_2)$; however, since we are mainly interested in the scaling limit $\delta \to 0^+$ we proceed as follows. Take the scaling limit of pf$H$ and multiply the result by

$\det_2(1+G)$; one obtains

$$S^+(x_1-x_2)=\sum_{n=0}^{\infty} g_{2n+1}(r)\sum_{n=1}^{\infty} F_{2n}(r),$$

where $g_{2n+1}(r)$ is the scaling limit of $g_{2n+1,\delta}(\Delta m_2, \Delta n_2)$. Define $F_{2n+1}(r)$ for $n=0,1,2\ldots$ by $F_{2n+1}(r)=\Sigma' g_{2k+1}(r)F_{2l}(r)$, where the sum Σ' is over all $k\geq 0$ and $l\geq 1$ subject to the restriction $k+l=n$. It is now a combinatoric problem to rewrite this sum in a more tractable form. For details of this see Nappi [50]. The result is

$$S^+(x_1-x_2)=\sum_{n=0}^{\infty} F_{2n+1}(r),$$

where

$$F_{2n+1}(r)=\frac{1}{(2n+1)!}\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\cdots\frac{dp_{2n+1}}{2\pi}$$
$$\times\prod_{j=1}^{2n+1}\frac{e^{-isp_j-t\omega(p_j)}}{\omega(p_j)}\det(A_{\alpha\beta})\Big|_{\alpha,\beta=1}^{2n+2},$$

where $(A_{\alpha\beta})$ is a $(2n+2)\times(2n+2)$ matrix with elements $A_{\alpha\alpha}=0$, $A_{\alpha\beta}=(\omega(p_\alpha)-\omega(p_\beta))/(p_\alpha+p_\beta)$, $\alpha,\beta=1,2,\ldots,2n+1$, and $A_{\alpha,2n+2}=-A_{2n+2,\alpha}=1$, $\alpha=1,2,\ldots,2n+1$. As before one changes to the u_j variables, $j=1,2,\ldots,2n+1$, rotates the u_j-contours to obtain explicit rotational invariance for $S^+(x_1-x_2)$, and finally evaluates the resulting determinant using the same method that is used to evaluate Cauchy determinants. One obtains (5.20). □

We have only sketched the proof of this corollary since many of the calculations are already in the literature [73]. The final "simple" expression for F_n is new.

It follows from the local integrability of $S^{\pm}(x_1-x_2)$ and simple estimates from (5.19) that $S^+(x_1-x_2)$ and $S^-(x_1-x_2)-1$ are in $L^1(\mathbf{R}^2)$. Thus the Fourier transforms of S^+ and $S^- - 1$ exist; making use of the rotational invariance of $S^{\pm}$, these transforms are easily seen to be $2\pi\int_0^\infty r\,dr\,J_0(rp)S^+(r)$ and $2\pi\int_0^\infty r\,dr\,J_0(rp)[S^-(r)-1]$, respectively. Here p is the magnitude of the Fourier transform variable $\vec{p}\in\mathbf{R}^2$ and $J_0(x)$ is the zeroth-order Bessel function. Using the corollary above, series expansions for the spectral densities are easily derived (the identity $\int_0^\infty r\,dr\,J_0(rp)e^{-\alpha r}=\alpha(\alpha^2+p^2)^{-3/2}$ is useful). These Fourier transforms are of considerable interest in physics, and have been studied in [69]. The interpretation as the spectral density of a generalized stationary process is possible in light of Theorem 5.2. For a indication of the rate of convergence of these series expansions for the spectral densities at small p see [47, 70, 73]. This work is numerical, but it suggests that at $p = 0$ the first three or four terms in the series give ten decimal place accuracy!

We conclude with some remarks concerning the Osterwalder–Schrader axioms for the scaled correlations. Symmetry is enforced as a matter of definition, and the distribution property is a consequence of the estimates given in the proof of the preceeding theorem. O.S. positivity on the lattice is a consequence of the positive self-adjoint nature of the transfer matrix. Since we can replace sums by integrals, as in (5.16), the positivity property carries over to the scaled correlations. The cluster property is the one nontrivial axiom which follows from our formulas directly. We will sketch how this goes below T_c. Suppose A and B are finite collections of sites in $\mathbb{R}^2$, and $a \in \mathbb{R}^2$ with $a_2 > 0$ (this can be arranged by making one or two rotations by $\pi/2$ if it is not true to begin with). We wish to show that $\lim_{\lambda \to +\infty} \langle \sigma_A \sigma_{B+\lambda a} \rangle^+ = \langle \sigma_A \rangle^+ \langle \sigma_B \rangle^+$. Making use of the determinant formulas one finds $\langle \sigma_A \sigma_{B+\lambda a} \rangle^+ = \det_2 \left(I + \begin{bmatrix} G(A) & G_{12}(\lambda a) \\ G_{21}(\lambda a) & G(B) \end{bmatrix} \right)$. The off-diagonal piece $G_{12}(\lambda a)$ contains the exponentials $e^{-\lambda a_2 \omega}$ and goes to zero in Schmidt norm. Thus $\lim_{\lambda \to \infty} \langle \sigma_A \sigma_{B+\lambda a} \rangle^+ = \det_2 \left(I + \begin{bmatrix} G(A) & 0 \\ 0 & G(B) \end{bmatrix} \right)$.

The function $\det_2$ is not in general multiplicative; however, $G(A)$ and $G(B)$ are manifestly limits (in Schmidt norm) of finite rank trace 0 operators. We have then

$$\det \left(I + \begin{bmatrix} G(A) & 0 \\ 0 & G(B) \end{bmatrix} \right) = \det_2(I + G(A))$$

$$\times \det_2(I + G(B)) = \langle \sigma_A \rangle^+ \langle \sigma_B \rangle^+ .$$

Similar considerations apply above T_c. Rotational invariance is the one axiom we have not proved. We remark that this property is closely related to the continuity of the scaled correlations at collections of sites in which two sites have coincident second coordinates. Our formulas are not well adapted to prove such continuity results.

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