# Two-Dimensional Ising Correlations: The SMJ Analysis* 

John Palmer<br>Department of Mathematics, University of Arizona, Tucson, Arizona 85721

AND
Craig Tracy
Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755

## Introduction

In this paper we make rigorous the analysis by Sato, Miwa, and Jimbo (henceforth, SMJ) [25-28] of the two-dimensional, zero external field Ising correlations. We also make use of this analysis to complete the verification of the Osterwalder-Schrader axioms (see [8]) for the correlations scaled from below $T_{c}$ by establishing rotational invariance in this case (Theorem 6.6).

In [3, 31, 32] Wu, McCoy, Tracy, and Barouch (WMTB) discovered that the scaled two point functions for the two-dimensional Ising model are expressible in terms of a Painlevé function of the third kind. This discovery was made by using results from the thesis of Myers [21] (which was in part based upon earlier work by Latta [16]) who established a connection between a linear integral equation arising in the problem of electromagnetic scattering from a strip and a differential equation of Painlevé type. A summary of the Latta-Myers method can be found in Appendix B of WMTB. Later in [19] McCoy, Tracy, and Wu improved the mathematical foundations of the two point function analysis, and developed connection formulas for a family of solutions to the Painlevé equation.

In an impressive series of papers, SMJ generalized the two point function result to the higher correlations, and these authors found what we believe is certainly the natural setting for these results. For an overview of this work, the reader will find [12] helpful. They were aware that all the Painlevé transcendents arise naturally in the integration of the simplest of the Schlesinger equations for monodromy preserving deformations of linear

[^0]differential equations (for a useful account of this connection see Flaschka and Newell [7]). There is no obvious candidate for an ordinary differential equation to associate with the Ising model. However, it is a relatively straightforward calculation that the Clifford algebra which plays a decisive role in the Onsager-Kaufman [14, 15, 22] analysis of the Ising model scales to the Clifford algebra associated with the free Dirac field in two space-time dimensions. This free field satisfies the Dirac equation; and by making inspired use of this observation, SMJ succeed in showing that the differential equation encountered in the two point function analysis arises as a particular case of a monodromy preserving deformation for a space of (multivalued) solutions to the Euclidean Dirac equation. More specifically, they introduce a family of single valued solutions to the Dirac equation in Minkowski space whose analytic continuation to the Euclidean domain is multivalued with branch points at $a_{1}, \ldots a_{r} \in \mathbb{R}^{2}$. The solutions they write down have monodromy which is formally determined by the commutation relations between the spin fields and the Dirac field. When the branch points $a_{1}, \ldots, a_{r}$ are moved, this monodromy remains unchanged. They develop a local operator product expansion which enables them to calculate the Fourier series expansion of the Minkowski wave functions about the points which "map" into $a_{1}, \ldots, a_{r}$ under analytic continuation. Under the analytic continuation to the Euclidean region, these Fourier coefficients suggest that the Euclidean solutions have restricted singularities at the branch points $a_{1}, \ldots, a_{r}$ and that the lowest order coefficients are computable in terms of scaled correlations and their derivatives (involving both "order" and "disorder" variables).
In [26] SMJ provide a penetrating analysis of monodromy preserving deformations of families of solutions to the Euclidean Dirac equation. One of the principal conclusions of their work is that the low order Fourier coefficients satisfy a total system of differential equations (the deformation equations). In the case of just two points $a_{1}$ and $a_{2}$, they reproduce the WMTB result in this manner.

In this paper our principal results concern the construction of the Euclidean wave functions. In [27] there do not seem to be any convergence results that would help make analytical sense of the construction for the Minkowski wave functions which is proposed. We believe the same can be said for the local operator product expansions in the Minkowski regime, and for the control of the analytic continuation to the Euclidean region. Here we circumvent these difficulties by first defining lattice analogues of the Euclidean wave functions. We then use results from [23] and [24] to control the convergence to continuum wave functions in the scaling limit. Because these functions are multivalued, the direct definition in the Euclidean region naturally leads to more than one analytical expression. In our approach, it is only the control of the convergence of the lattice wave
functions which guarantees that these many expressions for the continuum wave functions piece together consistently (Theorem 4.2). We avoid the local operator product formalism by scaling lattice formulas for normal ordered products and using results from [23] to establish an effective expression of continuity in the resulting limits (Theorem 4.1). The local Fourier expansion coefficients may then be computed directly, although at the cost of additional complications compared with the analogous SMJ calculation in the Minkowski regime (Theorem 5.1).

The first five sections of this paper are devoted to a proof of Theorem 5.1. Many of the details in the proofs are fussy rather than delicate but there are a few matters of independent interest we would like to mention. The first is that the lattice wave functions we define do satisfy a linear difference equation everywhere except on the branch cuts to the right of the points $a_{1}, \ldots, a_{r}$. This is a simple calculation everywhere except on the horizontal rays to the left of the points $a_{1}, \ldots, a_{r}$. At such points this result depends on the "local" character of the induced rotation for the transfer matrix. This property is ultimately responsible for the fact that the scaled wave functions satisfy the Euclidean Dirac equation on the horizontal rays to the left of the points $a_{1}, \ldots, a_{r}$ even though the lattice wave functions seem abruptly glued together along these rays in our definition.
The proper identification of the low order Fourier coefficients requires some identities for the derivatives of the scaled correlations (see Theorem 4.3). In SMJ such identities are deduced using the result that a particular integral operator kernel reduces to a product kernel in right coordinates in the continuum limit. Here we deduce these identities by scaling difference identities on the lattice. These difference identities (see Theorem 2.0) are of some interest since they are the sort of identities needed to make the connection between the Clifford algebra formalism developed here and the Montroll, Potts, Ward [20] formulas for the correlations in the Pfaffian formulation.

In the final section of this paper, we provide a review of some of the results in SMJ [26, 27]. We make no claim to originality here since virtually every argument can be found somewhere in either SMJ [26] or SMJ [27]. We should perhaps mention here that we find the level of mathematical rigor in SMJ [26] and SMJ [27] rather different. We do not imagine that we have in any way improved on the mathematics in SMJ [26]. On the other hand, our confusion about the status of the work in SMJ [27] directly inspired the present effort. What we have done in the final section is present a simplified account of SMJ [26] leading as directly as possible to the deformation equations and the differential equation (6.26) for the logarithm of the correlation function. Recently Kadanoff and Kohmoto [13] have given a similar presentation of these results. However, partly because we did not understand their calculation of the diagonal elements $\Lambda_{k k}$ in the two point
function analysis, and partly because we wished to emphasize more mathematical aspects of the subject we feel justified in presenting our account. In particular, the connection between the deformation equations for $F$ and $G$ and the complete integrability of the extended holonomic system $d W=\Omega W$ is not mentioned in the Kadanoff-Kohmoto paper. We believe this connection provides an important clue for determining "boundary conditions" that might single out the particular solution of the deformation equations which is of interest. The problem of "boundary conditions" for the deformation equations is unsettled except for the two point case (see MTW [19]). The reader interested in this problem should be aware of the construction in SMJ [26] of solutions to the extended system $d W=\Omega W$ with generalized monodromy but the same deformation equations. For reasons of space and simplicity, this is not described in Section 6 of this paper. We recommend that the reader start with Section 6 of this paper or Kadanoff and Kohmoto [13] for orientation before proceeding with the developments in Sections 1-5.
In conclusion, we would like to mention some lines of research since the pioneering work of Myers, where Painlevé transcendents have been introduced to solve applied problems: (i) Bariev [2] has evaluated the scaled spontaneous magnetization of the half plane Ising model in terms of Painlevé transcendents of the third kind; (ii) Jimbo, Miwa, Mori, and Sato [11] express the density matrix of the one dimensional impenetrable boson model of Lenard [17] and Schultz [29] in terms of a certain Painlevé transcendent of the fifth kind; (iii) these authors also note that the solutions of the same Painlevé equation with different boundary conditions can be used to compute properties of random matrices which are of interest in the statistical theory of energy levels; (iv) Creamer, Thacker, and Wilkinson [6] employing the quantum inverse method express the density matrix of the delta function gas of Lieb and Liniger [18] as a perturbation expansion in powers of $1 / c$ ( $c$ is the coupling constant), and the explicit evaluation of the ( $1 / c$ ) term involves the fifth Painleve transcendent mentioned in (ii) as does the $(1 / c)^{2}$ term in the later work of Jimbo and Miwa [10] and of Creamer [5]; (v) starting with Ablowitz and Segur [1], several authors have made use of the second Painleve transcendent in the asymptotic analysis of the Korteweg-deVries (and related) equations; and finally, (vi) in the area where it all began, Jimbo, Kashiwara, and Miwa [9] employed these functions to study the electromagnetic scattering from a disk.

We begin with a description of the Fock representations in terms of which we will characterize the Ising correlations. Let $W$ denote a complex Hilbert
space with complex structure $i$ and distinguished conjugation $P$. Suppose $Q$ is a self-adjoint idempotent on $W$ such that $Q P+P Q=0$. Let $P_{ \pm}=$ $(1 / 2)(I \pm P), Q_{ \pm}=(1 / 2)(I \pm Q)$ and $W_{ \pm}=Q_{ \pm} W$. The $Q$-Fock representation of the Clifford relation on $(W, P)$ is $W \ni w \rightarrow F(w)=a^{*}\left(Q_{+} w\right)+$ $a\left(\overline{Q_{-}} \bar{W}\right)$, where $\bar{u}=P u$ and $a^{*}(\cdot)$ and $a(\cdot)$ are the standard creation and annihilation operators on $A\left(W_{+}\right)$, the complex alternating tensor algebra over $W_{+}[4]$. One may easily check that $F(u) F(w)+F(w) F(u)=\langle u, P w\rangle I$; these are the generator relations for the Clifford algebra $C(W, P)$.

The complex structure $i$ maps $P_{-} W$ onto $P_{+} W$. If we let $\mathcal{H}=P_{+} W$, then the $\operatorname{map} I \oplus(-i I): P_{+} W \oplus P_{-} W \rightarrow \mathscr{K} \oplus \mathscr{K}$ establishes a real orthogonal equivalence between $W$ and $\mathcal{H} \oplus \mathscr{H}$. The complex structure $i$ becomes $\left[\begin{array}{rr}0 & -1 \\ 1 & 0\end{array}\right]$ on $\mathscr{H} \oplus \mathscr{H}$ and $P$ becomes $\left[\begin{array}{rr}1 & 0 \\ 0 & -1\end{array}\right]$. Since $i Q$ commutes with both $P$ and $i$ it follows that $i Q$ has the matrix representation $\left[\begin{array}{cc}\Lambda & 0 \\ 0 & \Lambda\end{array}\right]$ on $\mathscr{H} \oplus \mathscr{H}$. Since $i Q$ is a complex structure it follows that $\Lambda$ must be a complex structure on $\mathscr{C}$. The matrix representation of $Q$ is thus $-i(i Q)=\left[\begin{array}{rr}0 & \Lambda \\ -\Lambda & 0\end{array}\right]$. Now consider the real orthogonal map $D=2^{-1 / 2}\left[\begin{array}{cc}I & \Lambda \\ I & -\Lambda\end{array}\right]$. Then $D$ is a unitary map from $(\mathcal{H} \oplus \mathcal{H}, i)$ to $(\mathscr{K} \oplus \mathscr{H}, \Lambda \oplus(-\Lambda))$, DPD* $=\left[\begin{array}{ll}0 & I \\ I & 0\end{array}\right]$ and $D Q D^{*}=\left[\begin{array}{rr}I & 0 \\ 0 & -1\end{array}\right]$. We will refer to $(\mathscr{H} \oplus \mathscr{H}, \Lambda \oplus(-\Lambda)$ ) as the $Q$ representation of $W$ since $Q$ is diagonal in this representation.

Definition. $G(W, Q)$ will denote the set of bounded operators on $A\left(W_{+}\right)$such that $g F(w)=F(T(g) w) g$ for some bounded invertible $P$ orthogonal $T(g)$ on $W$ (a $P$-orthogonal is a map which preserves the complex bilinear form $\langle\cdot, P \cdot\rangle$ ).

Because the $\{F(w) \mid w \in W\}$ generates an irreducible algebra on $A\left(W_{+}\right)$ [30], the map $g$ is determined up to multiplication by a constant by $T(g)$. We shall refer to $T(g)$ as the rotation induced by $g$. The semigroup $G(W, Q)$ is of interest to us because both the transfer matrix and the spin operator for the two-dimensional Ising model may be realized as elements of $G(W, Q)$ for suitable $(W, Q)$. We turn next to the characterization of the Ising correlations given in [24].

Let $\mathbf{Z}$ denote the set of all integers and write $\mathbf{Z}_{1 / 2}=\mathbf{Z}+(1 / 2)$. We let $\mathscr{H}$ be the real Hilbert space $l^{2}\left(\mathbb{Z}_{1 / 2}, \mathbb{R}^{2}\right)$ and $W$ the complexification of $\mathscr{K}$ which may naturally be identified with $l^{2}\left(\mathbf{Z}_{1 / 2}, \mathbb{C}^{2}\right)$. The distinguished conjugation on $W$ is ordinary complex conjugation. The interaction constant $K=J / k T$, where $T>0$ denotes the temperature, $k$ the Boltzmann constant, and $J$ a coupling constant. In this paper, $J$ will be considered fixed, and we will parametrize "temperature" dependent quantities by $T$ to conform with usage in physics. The critical point for the Ising model occurs at $K_{c}=(1 / 2) \ln (1+\sqrt{2})$ and the corresponding temperature will be denoted by $T_{c}$. Define the dual interaction constant $K^{*}$ by $\operatorname{sh} 2 K^{*} \operatorname{sh} 2 K=1$,
and let $s_{1}=\operatorname{sh} 2 K^{*}, c_{1}=\operatorname{ch} 2 K^{*}, s_{2}=\operatorname{sh} 2 K$, and $c_{2}=\operatorname{ch} 2 K$. We write $T^{*}=k K^{*} / J$. When $T>T_{c}$ it follows that $T^{*}<T_{c}$ and vice versa.

For $f \in W$ let

$$
\hat{f}(\theta)=(1 / 2 \pi) \sum_{k \in \mathbf{Z}_{1 / 2}} f(k) e^{i k \theta}
$$

The rotation, $T$, induced by the transfer matrix is defined by

$$
\widehat{T f}(\theta)=\left[\begin{array}{cc}
c_{1} c_{2}-\cos \theta & s_{1} \sin \theta-i\left(c_{2}-c_{1} \cos \theta\right) \\
s_{1} \sin \theta+i\left(c_{2}-c_{1} \cos \theta\right) & c_{1} c_{2}-\cos \theta \tag{1.1}
\end{array}\right],
$$

For future reference we include here the action of $T$ on the complex orthonormal basis

$$
\begin{align*}
\left\{e_{1}(k)=\right. & \left.\delta(k, \cdot) \oplus 0, e_{2}(k)=0 \oplus \delta(k, \cdot) \mid k \in \mathbb{Z}_{1 / 2}\right\} \quad \text { for } W: \\
T e_{1}(k)= & -(1 / 2) e_{1}(k-1)+c_{1} c_{2} e_{1}(k)-(1 / 2) e_{1}(k+1) \\
& -(i / 2)\left(c_{1}-s_{1}\right) e_{2}(k-1)+i c_{2} e_{2}(k) \\
& -(i / 2)\left(c_{1}+s_{1}\right) e_{2}(k+1)  \tag{1.2}\\
T e_{2}(k)= & -(1 / 2) e_{2}(k-1)+c_{1} c_{2} e_{2}(k)-(1 / 2) e_{2}(k+1) \\
& +(i / 2)\left(c_{1}+s_{1}\right) e_{1}(k-1)-i c_{2} e_{1}(k) \\
& +(i / 2)\left(c_{1}-s_{1}\right) e_{1}(k+1)
\end{align*}
$$

Following Onsager [22] we introduce functions $\gamma(\theta)>0$ and $\alpha(\theta)$ (called $\delta^{*}(\omega)$ by Onsager) defined by

$$
\begin{align*}
\operatorname{ch} \gamma(\theta) & =c_{1} c_{2}-\cos \theta \\
\operatorname{sh} \gamma(\theta) e^{i \alpha(\theta)} & =\left(c_{2}-c_{1} \cos \theta\right)+i s_{1} \sin \theta \tag{1.3}
\end{align*}
$$

Substituting (1.3) in (1.1) one finds

$$
T(\theta)=\exp \left[\gamma(\theta)\left[\begin{array}{cc}
0 & -i e^{i \alpha(\theta)} \\
i e^{-i \alpha(\theta)} & 0
\end{array}\right]\right]
$$

The map $Q$ on $W$ is multiplication by

$$
Q(\theta)=\left[\begin{array}{cc}
0 & i e^{i \alpha(\theta)} \\
-i e^{-i \alpha(\theta)} & 0
\end{array}\right]
$$

in the Fourier transform variables. If we let $T_{ \pm}=Q_{ \pm} T$, then $V=I \oplus T_{+}$
$\oplus\left(T_{+} \otimes T_{+}\right) \oplus \cdots$ on $A\left(W_{+}\right)$is an element of $G(W, Q)$ such that $T(V)=$ $T$. The map $V$ will be referred to as the transfer matrix.

We define a family of real orthogonal maps $s_{m}$ on $W$ by

$$
s_{m} e_{j}(k)=\operatorname{sgn}(m-k) e_{j}(k), \quad m \in \mathbb{Z}, k \in \mathbb{Z}_{1 / 2}
$$

In [24] it was shown that for $T<T_{c}$ there exists an element $\sigma_{m} \in G(W, Q)$ such that $T\left(\sigma_{m}\right)=s_{m}$. The map $\sigma_{m}$ is uniquely determined by this condition and the further normalizations $\sigma_{m}^{2}=I$ and $\left\langle\sigma_{m} 1,1\right\rangle>0$. Here 1 denotes the vacuum vector $1 \oplus 0 \oplus 0 \cdots \in A\left(W_{+}\right)$. We refer to $\sigma_{m}$ as a spin operator on $A\left(W_{+}\right)$.

Suppose now that $a \in \mathbb{Z}^{2}$, we define $\sigma(a)=V^{a_{2}} \sigma_{a_{1}} V{ }^{a_{2}}$. Since $V^{b}$ is an unbounded operator when $b<0$, it requires some care to define $\sigma(a)$ as an operator. In this paper we will be interested in "time ordered" products of the operators $\sigma(a)$. In such products the power of the transfer matrix $V$ sandwiched between two spin operators is always nonnegative, and the transfer matrices on the ends of such products encounter the vacuum state with the result $V^{b} 1=1$. For our purposes then the symbol $\sigma(a)$ may be understood as a convenient notational device. In Section 2 we will show that, in any case, $\sigma(a)$ does extend naturally to an element of $G(W, Q)$.

At this point it is convenient to introduce the time ordering symbol $\mathfrak{T}$. Suppose $\varphi_{l}\left(a_{l}\right)(l=1, \ldots, r)$ is a collection of operators on $A\left(W_{+}\right)$depending on the parameters $a_{l} \in \mathbb{R}^{2}$, and write $\pi_{j}\left(a_{l}\right)$ for the $j$ 'th component of $a_{l}$. If all the second components are distinct (that is $\pi_{2}\left(a_{i}\right) \neq \pi_{2}\left(a_{m}\right)$ unless $l=m$ ) then we write

$$
\operatorname{T}_{\varphi_{1}}\left(a_{1}\right) \cdots \varphi_{r}\left(a_{r}\right)=\varphi_{s(1)}\left(a_{s(1)}\right) \cdots \varphi_{s(r)}\left(a_{s(r)}\right)
$$

where the permutation, $s$, of $(1, \ldots, r)$ is chosen so that $\pi_{2}\left(a_{s(1)}\right)<\pi_{2}\left(a_{s(2)}\right)$ $\cdots<\pi_{2}\left(a_{s(r)}\right)$.

In [24] it was proved that the "plus" state infinite volume correlations below the critical temperature are given by

$$
\begin{equation*}
\left\langle\sigma_{a_{1}} \cdots \sigma_{a_{r}}\right\rangle_{T<T_{c}}^{+}=\left\langle\mathscr{T} \sigma\left(a_{1}\right) \cdots \sigma\left(a_{r}\right) 1,1\right\rangle_{T} \tag{1.4}
\end{equation*}
$$

with 1 the vacuum in $A\left(W_{+}\right)$. To simplify notation we will commonly omit the vacuum vector 1 , writing $\langle A 1,1\rangle=\langle A\rangle$.

The reader should note that the right-hand side of (1.4) depends on the temperature, since both the transfer matrix and $Q$ depend on $T$ through (1.1) and (1.3). Since the spin operators $\sigma_{m}$ and $\sigma_{n}$ commute, it follows that the time ordered product in (1.4) is unambiguous even when there are coincidences among the second coordinates $\pi_{2}\left(a_{l}\right)(l=1, \ldots, r)$. In such circumstances (1.4) still represents the spin correlations.

Now let $p_{k}=\sqrt{2} F\left(e_{1}(k)\right)$ and $q_{k}=\sqrt{2} F\left(e_{2}(k)\right)$, and define $\mu_{k}=$ $\left((\operatorname{ch} K) p_{k}+i(\operatorname{sh} K) q_{k}\right) \sigma_{k+1 / 2}\left(k \in \mathbb{Z}_{1 / 2}\right)$. If $a \in \mathbb{Z}_{1 / 2} \times \mathbf{Z}$ then, as above,
we introduce the notation $\mu(a)=V^{a_{2}} \mu_{a_{1}} V^{-a_{2}}$. The infinite volume correlations for the Ising model above the critical temperature are given by [24]

$$
\begin{equation*}
\left\langle\sigma_{a_{1}} \cdots \sigma_{a_{r}}\right\rangle_{T>T_{c}}=\left\langle\mathscr{J} \mu\left(a_{1}\right) \cdots \mu\left(a_{r}\right) 1,1\right\rangle_{T^{*}} \tag{1.5}
\end{equation*}
$$

The vacuum expectation on the right is evaluated at temperature $T^{*}<T_{c}$. As above, because $\mu_{m} \mu_{n}=\mu_{n} \mu_{m}$ (this needs to be checked!) the time ordered product is unambiguous even when there are coincidences among the coordinates $\pi_{2}\left(a_{l}\right)$. The spin correlations continue to be represented by (1.5) in such circumstances.

We now introduce the lattice analogues of free Fermi fields at imaginary times. The reason for introducing linear combinations of $p_{k}$ and $q_{k}$ rather than working directly with $\left(p_{k}, q_{k}\right)$ has to do with a simplification in the analysis of the monodromy for the associated Euclidean Dirac equation that will appear later. Suppose $x \in \mathbb{Z}_{1 / 2} \times \mathbb{Z}$, we define

$$
\begin{equation*}
\psi_{j}(x)=F\left(T^{x_{2}} f_{j}\left(x_{1}\right)\right), \quad j= \pm 1 \tag{1.6}
\end{equation*}
$$

where $f_{j}(k)=2^{-1 / 2}\left[-j e_{1}(k)+e_{2}(k)\right]$.
Let $\nabla_{1} f(x)=(1 / 2)\left(f\left(x_{1}+1, x_{2}\right)-f\left(x_{1}-1, x_{2}\right)\right)$ and $\nabla_{2} f(x)=$ $(1 / 2)\left(f\left(x_{1}, x_{2}+1\right)-f\left(x_{1}, x_{2}-1\right)\right)$. Then $\left[\begin{array}{c}\psi_{1}(x) \\ \psi_{-1}(x)\end{array}\right]$ satisfies the following lattice version of the Euclidean Dirac equation:

$$
\left[\begin{array}{rr}
0 & -i \\
i & 0
\end{array}\right] \nabla_{2}\left[\begin{array}{c}
\psi_{1}(x) \\
\psi_{-1}(x)
\end{array}\right]+\left[\begin{array}{cc}
0 & s_{1} \\
s_{1} & 0
\end{array}\right] \nabla_{1}\left[\begin{array}{c}
\psi_{1}(x) \\
\psi_{-1}(x)
\end{array}\right]-M\left[\begin{array}{c}
\psi_{1}(x) \\
\psi_{-1}(x)
\end{array}\right]=0
$$

where
$M\left[\begin{array}{c}\psi_{1}(x) \\ \psi_{-1}(x)\end{array}\right]=c_{2}\left[\begin{array}{c}\psi_{1}(x) \\ \psi_{-1}(x)\end{array}\right]-\frac{1}{2} c_{1}\left[\begin{array}{c}\psi_{1}\left(x_{1}+1, x_{2}\right)+\psi_{1}\left(x_{1}-1, x_{2}\right) \\ \psi_{-1}\left(x_{1}+1, x_{2}\right)+\psi_{-1}\left(x_{1}-1, x_{2}\right)\end{array}\right]$.
This is most easily checked by computing $\left[\begin{array}{cc}0 & -i \\ i & 0\end{array}\right] \nabla_{2}\left[\begin{array}{c}\psi_{1}(x) \\ \psi_{-1}(x)\end{array}\right]$ in the $Q$ representation of $W$ making use of $i F(e)=F(\Lambda \oplus(-\Lambda) e)$ and $\operatorname{sh} \gamma(\theta) e^{i \alpha(\theta)}=\left(c_{2}-c_{1} \cos \theta\right)+i s_{1} \sin \theta$. As may be easily verified, the complex structure $\Lambda$ is multiplication by $\left[\begin{array}{cc}0 & -e^{i \alpha(\theta)} \\ e^{-i \alpha(\theta)} & 0\end{array}\right]$ in the Fourier transform variables for $\mathcal{H}=l^{2}\left(\mathbb{Z}_{1 / 2}, \mathbb{R}^{2}\right)$, and the rotation $T$, induced by the transfer matrix, becomes multiplication by $\left[\begin{array}{cc}e^{-\gamma(\theta)} & 0 \\ 0 & e^{\gamma(\theta)}\end{array}\right]$ on $\mathscr{H} \oplus \mathscr{H}$ in the $Q$ representation of $W$. One should also note that $f_{j}(x)$ goes to $D f_{j}(x)=$ $2^{-1 / 2}\left[f_{j}(x) \oplus f_{j}(x)\right]$ in the $Q$ representation.

We are now prepared to define the lattice order-disorder correlations whose scaled versions play a decisive role in the SMJ analysis of the Ising correlations.

Let $s$ denote an integer from 1 to $r$. Suppose that $x, a_{s} \in \mathbb{Z}_{1 / 2} \times \mathbf{Z}$ and $a_{l} \in \mathbf{Z}^{2}$ for $l=1, \ldots, r, l \neq s$. Suppose that the second coordinates $\pi_{2}\left(a_{l}\right)$ ( $l=1, \ldots, r$ ) are all distinct, and for convenience suppose $\pi_{2}\left(a_{1}\right)<\pi_{2}\left(a_{2}\right)$ $\cdots<\pi_{2}\left(a_{r}\right)$. We define

$$
\begin{array}{rlrl}
w_{j}^{s}(x, a) & =\left\langle\mathscr{T} \psi_{j}(x) \mu\left(a_{s}\right) \prod_{l \neq s} \sigma\left(a_{l}\right)\right\rangle /\left\langle\mathscr{T} \prod_{l} \sigma\left(a_{l}\right)\right\rangle, & & \pi_{2}(x)<\pi_{2}\left(a_{s}\right) \\
& =-\left\langle\mathscr{T} \psi_{j}(x) \mu\left(a_{s}\right) \prod_{l \neq s} \sigma\left(a_{l}\right)\right\rangle /\left\langle\mathscr{T} \prod_{l} \sigma\left(a_{l}\right)\right\rangle, & \pi_{2}\left(a_{s}\right)<\pi_{2}(x) \tag{1.7}
\end{array}
$$

There is no need to specify the order in the product $\prod_{l=s} \sigma\left(a_{l}\right)$, since this is controlled by the time ordering symbol. We will be interested in $w_{j}^{s}(x, a)$ as a function of $(x, j)$ with $s=1, \ldots, r$ and $a=\left(a_{1}, \ldots, a_{r}\right)$ held fixed. In particular, it is interesting to see what happens when $\pi_{2}(x)$ agrees with $\pi_{2}\left(a_{l}\right)$ for some $a_{l}(l=1, \ldots, r)$. Since $\psi_{j}(k, 0) \sigma_{l}=\operatorname{sgn}(l-k) \sigma_{l} \psi_{j}(k, 0)$ and $\psi_{j}(k, 0) \mu_{s}=-\operatorname{sgn}(s-k) \mu_{s} \psi_{j}(k, 0)(k \neq s)$ it follows that the natural extension of $w_{j}^{s}(x, a)$ to points where $\pi_{2}(x)$ agrees with $\pi_{2}\left(a_{l}\right)$, for some $a_{l}$, is a two-valued function. The definition of $w_{j}^{s}(x, a)$ was split into two parts so that the "branch cuts" for $w_{j}^{s}(x, a)$ lie on the horizontal rays to the right of each of the points $a_{l}$. If $x$ makes a local circuit of $a_{l}$ starting on the horizontal ray to the right of $a_{l}$ and moving counterclockwise, then when it returns to its starting point $w_{j}^{s}(x, a)$ has changed by a sign. Since we do not have a notion of continuity on the lattice this monodromy property may not seem significant. However, the reader should note that as a consequence of (1.6) the function $w_{j}^{s}(x, a)$ satisfies an elliptic difference equation as a function of $(x, j)$, at least away from the points where $\pi_{2}(x)$ agrees with some $\pi_{2}\left(a_{l}\right)$. Actually, one may show that $w_{j}^{s}(x, a)$ satisfies the same difference equation even when $\pi_{2}(x)=\pi_{2}\left(a_{l}\right)$ as long as $\pi_{1}(x)<\pi_{1}\left(a_{l}\right)$. In the notation of Lemma 2.2 of [24] this follows from the fact that the only nonzero matrix elements of the commutator $[s(l), T]$ connect the vectors $\left\{e_{j}(l \pm 1 / 2) \mid j=1,2\right\}$. This implies that $w_{j}^{s}(x, a)$ jumps in a highly "regular" fashion from one site to the next as $x$ makes the local circuit of $a_{l}$ described above. One should also observe that as the points $a_{1}, \ldots, a_{r}$ are varied the monodromy of the solutions $w_{j}^{s}(x, a)$ to the difference equation (1.6) remains fixed. In the continuum limit it is this monodromy preserving property that SMJ exploit [26,27] to demonstrate that the scaled $n$-point functions satisfy nonlinear Pfaffian systems of differential equations. One might pursue a similar analysis on the lattice, but this will not be attempted here (a slightly different choice for $w_{j}^{s}(x, a)$ seems appropriate for such an analysis). Instead we will scale the functions $w_{j}^{s}(x, a)$ to their continuum limits using results from [23] and [24] to control the convergence. To avoid introducing extra notation we will use $w_{j}^{s}(x, a)$ to denote the scaled counterpart of the lattice wave functions $w_{j}^{s}(x, a)$ when there is no cause for
confusion. Our principal results concern the following properties of the scaled functions $w_{j}^{s}(x, a)$ :
(1) The functions $w_{j}^{s}(x, a)$ satisfy the Euclidean Dirac equation in the variables $(x, j)$ except at the points $a_{l}$, where they have branch cuts and fixed monodromy as $a_{l}$ and $s$ vary.
(2) The order of growth of $w_{j}^{s}(x, a)$ is restricted at each singularity $a_{l}$ and the functions $w_{j}^{s}(x, a)$ are exponentially small for $x$ near $\infty$.
(3) Let $\left(r_{l}, \theta_{l}\right)$ denote polar coordinates in $\mathbb{R}^{2}$ centered at $a_{l}$. The "Fourier coefficients" of $w_{j}^{s}\left(r_{l}, \theta_{l}, a\right)$ in the $\theta_{l}$ variables are Bessel functions of $r_{l}$ multiplied by functions of $a=\left(a_{1}, \ldots, a_{r}\right)$. The functions of $a \cdot \in \mathbb{R}^{2 r}$ which multiply the most singular (at $r_{l}=0$ ) Bessel functions are expressible in terms of scaled correlations for $\sigma\left(a_{m}\right), \mu\left(a_{m}\right)(m=1, \ldots, r)$ and their $a_{l}$ derivatives.

The precise statement of results will be given later. Conditions (1) and (2) will single out an $r$-dimensional family of solutions to the Euclidean Dirac equation [26] and (3) will permit us to show that $\left\{w_{j}^{s}(x, a) \mid s=1, \ldots, r\right\}$ is a basis for this finite-dimensional vector space. The general analysis of monodromy preserving deformations of the Euclidean Dirac equation in [26] coupled with the identification of $n$-point functions in (3) above determines Pfaffian systems of differential equations for the scaled $n$-point correlations.

Since the analysis we have just sketched is identical to that presented by SMJ [26, 27], it might be useful for the reader to understand what contributions we believe we have made to this analysis. In large measure what is novel in this paper concerns the definition of the functions $w_{j}^{s}(x, a)$ and the rigorous calculation of the local expansions in terms of scaled correlations. In [27] the wavefunction $w_{j}^{s}(x, a)$ is introduced as a multivalued analytic continuation of a single valued entity defined in the Minkowski region for $x$ and $a$. A local operator product expansion is developed to compute the Fourier expansion coefficients in the Minkowski regime, and the results are analytically continued to the Euclidean region. While the intuition behind the two-valued nature of the analytic continuation is instructive, and the calculation of the expansion coefficients is simpler than the calculation presented here, we believe that a complete definition of the single valued Minkowski entity (valid almost everywhere in space-time) together with a rigorous account of the operator product expansion, the analytic continuation of the wave functions, and the validity of analytically continuing the Fourier coefficients would be a considerable undertaking.

By working directly with the "Euclidean" wave functions $w_{j}^{s}(x, a)$ we avoid problems associated with analytic continuation and we are able to exploit formulas from [23] and [24] which are valid almost everywhere.

In this section we develop identities for the Ising correlation functions that we shall use later to obtain formulas for the derivatives of the scaled correlations. These identities will play an important role in properly identifying the low-order expansion coefficients of the wavefunctions $w_{j}^{s}(x, a)$. To state our main result we introduce some notation. Let $P(k)=(\operatorname{ch} K) p_{k}+$ $i(\operatorname{sh} K) q_{k}$ and $Q(k)=(\operatorname{ch} K) q_{k}-i(\operatorname{sh} K) p_{k}$. The difference identities we require are all consequences of the following result.

Theorem 2.0. The map $V \sigma_{0} V^{-1}$ extends from an operator densely defined on the span of the product vectors $\Pi_{j, k} F\left(e_{j}(k)\right) 1$ to a bounded operator on $A\left(W_{+}\right)$. The operator $V^{-1} \sigma_{0} V=\left(V \sigma_{0} V^{-1}\right)^{*}$ is a bounded operator on $A\left(W_{+}\right)$. Furthermore

$$
\begin{align*}
& \text { (1) } \sigma_{1}=i q(1 / 2) p(1 / 2) \sigma_{0}, \\
& \text { (2) } \mu_{1 / 2}=-i Q(1 / 2) \sigma_{0}, \\
& \text { (3) } V \sigma_{0} V^{-1}=\exp \left[-2 K^{*} i Q(1 / 2) P(-1 / 2)\right] \sigma_{0}, \\
& \text { (4) } V^{-1} \mu_{-1 / 2} V=P^{*}(-1 / 2) \sigma_{0} . \tag{2}
\end{align*}
$$

Proof. The first identity may be easily verified by consulting the finite dimensional representation for $p_{k}$ and $q_{k}$ in the discussion following (1.4) in [24]. The argument needed to extend this identity to the infinite-dimensional situation is easily supplied (see below).
To prove (2) observe first that $\mu_{1 / 2}=P(1 / 2) \sigma_{1}$. It is a simple calculation to check that $q(1 / 2) p(1 / 2)=Q(1 / 2) P(1 / 2)$. Thus $\mu_{1 / 2}=P(1 / 2)(i Q(1 / 2)$ $\left.P(1 / 2) \sigma_{0}\right)=-i Q(1 / 2) \sigma_{0}$.

Let $T=T(V)$ be the rotation induced by the transfer matrix. Then $V^{-1} \Pi_{j, k} F\left(e_{j}(k)\right) 1=\Pi_{j, k} F\left(T^{-1} e_{j}(k)\right) 1$ (the order of the terms in the product is irrelevant except that this order should be the same on both sides of the equality). We shall find a useful expression for $V \sigma_{0} V^{-1}$ by computing the induced rotation of $\sigma_{0} V \sigma_{0} V^{-1}$. This induced rotation is $s T s T^{-1}$ and it has a simple form because the commutator $[s, T]$ has only a small number of non zero matrix elements in the basis $\left\{e_{j}(k)\right\}$. We take advantage of this by writing $s T s T^{-1}=I+K$, from which we deduce that $K=[s, T] s T^{-1}$.

From (1.2) one easily computes

$$
\begin{array}{rlrl}
{[s, T] e_{1}(k)} & =-e_{1}(-1 / 2)-i\left(c_{1}-s_{1}\right) e_{2}(-1 / 2), \\
& =e_{1}(1 / 2)+i\left(c_{1}+s_{1}\right) e_{2}(1 / 2), & & k=1 / 2, \\
& =0, & & k=-1 / 2, \\
{[s, T] e_{2}(k)} & =-e_{2}(-1 / 2)+i\left(c_{1}+s_{1}\right) e_{1}(-1 / 2), & & k=1 / 2, \\
& =e_{2}(1 / 2)-i\left(c_{1}-s_{1}\right) e_{1}(1 / 2), & & k=-1 / 2, \\
& =0, & & k=1 / 2,-1 / 2,
\end{array}
$$

Since $T$ is a $P$-orthogonal map we may determine its inverse by computing its transpose. One finds that $K$ is zero except on the subspace of vectors spanned by $\left\{e_{j}( \pm 1 / 2) \mid j=1,2\right\}$. The induced rotation $I+K$ is thus the identity on the complement of this subspace. The element of the Clifford group with induced rotation $I+K$ may thus be expressed in terms of $p( \pm 1 / 2)$, and $q( \pm 1 / 2)$, and we proceed next with the calculation of this element along the lines of Theorem 1.1 of [23]. If $U$ denotes the restriction of $s T s T^{-1}$ to the span of $\left\{e_{j}( \pm 1 / 2) \mid j=1,2\right\}$ then the matrix of $U$ in the (ordered) basis $\left\{e_{1}(1 / 2), e_{2}(1 / 2), e_{1}(-1 / 2), e_{2}(-1 / 2)\right\}$ is

$$
\left[\begin{array}{cccc}
c_{1}\left(c_{1}-s_{1}\right) & -i s_{1} & c_{1} & i c_{1}\left(c_{1}-s_{1}\right) \\
-i s_{1} & c_{1}\left(c_{1}+s_{1}\right) & i c_{1}\left(c_{1}+s_{1}\right) & -c_{1} \\
-c_{1} & -i c_{1}\left(c_{1}+s_{1}\right) & c_{1}\left(c_{1}+s_{1}\right) & i s_{1} \\
-i c_{1}\left(c_{1}-s_{1}\right) & c_{1} & i s_{1} & c_{1}\left(c_{1}-s_{1}\right)
\end{array}\right]
$$

Let

$$
\begin{aligned}
& G=\left[\begin{array}{cccc}
c_{1}\left(c_{1}-s_{1}\right) & 0 & c_{1} & 0 \\
0 & c_{1}\left(c_{1}+s_{1}\right) & 0 & -c_{1} \\
-c_{1} & 0 & c_{1}\left(c_{1}+s_{1}\right) & 0 \\
0 & c_{1} & 0 & c_{1}\left(c_{1}-s_{1}\right)
\end{array}\right], \\
& H=\left[\begin{array}{cccc}
0 & -s_{1} & 0 & c_{1}\left(c_{1}-s_{1}\right) \\
-s_{1} & 0 & c_{1}\left(c_{1}+s_{1}\right) & 0 \\
0 & -c_{1}\left(c_{1}+s_{1}\right) & 0 & s_{1} \\
-c_{1}\left(c_{1}-s_{1}\right) & 0 & s_{1} & 0
\end{array}\right] .
\end{aligned}
$$

Then the matrix $U$ relative to the real orthogonal decomposition, span $\left\{e_{j}( \pm 1 / 2)\right\} \oplus \operatorname{span}\left\{i e_{j}( \pm 1 / 2)\right\}$, is $\left[\begin{array}{rr}G & -H \\ H\end{array}\right]$. Let $\Lambda_{0}$ denote the complex structure on the span of $\left\{e_{j}(+1 / 2)\right\}$ determined by $\Lambda_{0} e_{1}(+1 / 2)=$ $e_{2}( \pm 1 / 2)$ and $\Lambda_{0} e_{2}( \pm 1 / 2)=-e_{1}( \pm 1 / 2)$. Let $Q_{0}=\left[\begin{array}{cc}0 & -\Lambda_{0} \\ \Lambda_{0} & 0\end{array}\right]$ be the associated idempotent. We write $D_{0}=2^{-1 / 2}\left[\begin{array}{cc}1 & \Lambda_{0} \\ 1 & -\Lambda_{0}\end{array}\right]$ and make the transformation to the $Q_{0}$ representation

$$
D_{0}\left[\begin{array}{cc}
G & -H \\
H & G
\end{array}\right] D_{0}^{*}=\left[\begin{array}{cc}
U_{11} & U_{12} \\
U_{21} & U_{22}
\end{array}\right] .
$$

The $R$ matrix associated with $U$ and $\Lambda_{0}$ is then

$$
R=\left[\begin{array}{cc}
U_{22}^{*-1}-1 & U_{12} U_{22}^{-1} \\
U_{22}^{-1} U_{21} & 1-U_{22}^{-1}
\end{array}\right] .
$$

Let $I=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right], J=\left[\begin{array}{rr}1 & 0 \\ 0 & -1\end{array}\right]$ and define $R_{1}=\left[\begin{array}{ll}0 & I \\ I & 0\end{array}\right]$ and $R_{2}=\left[\begin{array}{cc}0 & -J \\ J & 0\end{array}\right]$. One may verify without difficulty that

$$
R=\left[\begin{array}{cc}
-\left(s_{1} / c_{1}\right) R_{1} & {\left[\left(c_{1}-1\right) / c_{1}\right] R_{2}} \\
-\left[\left(c_{1}+1\right) / c_{1}\right] R_{2} & \left(s_{1} / c_{1}\right) R_{1}
\end{array}\right] .
$$

According to Theorem 1.1 of [23], an element of the Clifford algebra with induced rotation $U$ is given by $\theta \exp (1 / 2) \Sigma_{k} R \bar{f}_{k} \wedge f_{k}$, where the sum is over any $i$-complex orthonormal basis of $W, \bar{f}=P f$, and $\theta$ is the $Q_{0}$ normal ordering map from the Grassmann algebra over $W$ onto the Clifford algebra, $C(W, P)$ (see [23]). In the $Q_{0}$ representation $i=\Lambda_{0} \oplus\left(-\Lambda_{0}\right)$. Thus if $u_{j}(j=1, \ldots, 8)$ is the standard basis for $\mathbb{R}^{8}$, we may choose $f_{1}=u_{1}, f_{2}=u_{3}, f_{3}=\bar{f}_{1}=u_{5}$, and $f_{4}=\bar{f}_{2}=u_{7}$. One finds

$$
\begin{aligned}
\exp (1 / 2) \sum_{k} R f_{k} \wedge f_{k}= & 1-\left(1-c_{1}^{-1}\right) f_{1} \wedge f_{2}-\left(s_{1} / c_{1}\right)\left(f_{2} \wedge \bar{f}_{1}+f_{1} \wedge \bar{f}_{2}\right) \\
& +\left(1+c_{1}^{-1}\right) \bar{f}_{1} \wedge \bar{f}_{2}
\end{aligned}
$$

Making use of the fact that $F\left(f_{1}\right)=(1 / 2)(p(1 / 2)-i q(1 / 2))$ and $F\left(f_{2}\right)=$ $(1 / 2)(p(-1 / 2)-i q(-1 / 2))$ it follows that

$$
\begin{align*}
\theta \exp (1 / 2) \sum_{k} R f_{k} \wedge f_{k}= & 1+\left(2 c_{1}\right)^{-1}(p(1 / 2) p(-1 / 2)-q(\mathrm{I} / 2) q(-1 / 2)) \\
& +(i / 2)\left(1-s_{1} / c_{1}\right) p(1 / 2) q(-1 / 2) \\
& +(i / 2)\left(1+s_{1} / c_{1}\right) q(1 / 2) p(-1 / 2) \tag{2.1}
\end{align*}
$$

Now multiply the right hand side of (2.1) by $c_{1}$ and express the resulting element of the Clifford algebra in terms of $P(k)$ and $Q(k)$. One obtains

$$
\begin{aligned}
c_{1}+i s_{1} Q(1 / 2) P(-1 / 2) & =\operatorname{ch} 2 K^{*}+\operatorname{sh} 2 K^{*} i Q(1 / 2) P(-1 / 2) \\
& =\exp \left[2 K^{*} i Q(1 / 2) P(-1 / 2)\right]
\end{aligned}
$$

We have then

$$
\begin{align*}
V \sigma_{0} V^{-1} & =\sigma_{0}^{2} V \sigma_{0} V^{-1} \\
& =(\text { const. }) \sigma_{0} \exp \left[2 K^{*} i Q(1 / 2) P(-1 / 2)\right] . \tag{2.2}
\end{align*}
$$

By comparing the squares of both sides of this last equation we find (const.) $= \pm 1$. In Section 3 we will compute a normal ordered form for the right-hand side which will establish that (const.) $=+1$.

Since $V$ is a bounded self adjoint operator, $V^{-1}$ is a densely defined self-adjoint operator, and $V \sigma_{0} V^{-1}$ extends to a bounded operator, it follows that $V^{i} \sigma_{0} V=\left(V \sigma_{0} V^{1}\right)^{*}$ is a bounded operator (i.e., the range of $\sigma_{0} V$ is contained in the domain of $V^{-1}$ ). If we take the adjoint of both sides of (2.2) we find

$$
\begin{equation*}
V^{-1} \sigma_{0} V=\exp \left[2 K^{*} i Q^{*}(1 / 2) P^{*}(-1 / 2)\right] \sigma_{0} . \tag{2.3}
\end{equation*}
$$

Next we compute $V^{-1} P(-1 / 2) V$ using $V^{-1} F(x) V=F\left(T^{-1} x\right)$ :

$$
\begin{equation*}
V^{-1} P(-1 / 2) V=c_{1} P^{*}(-1 / 2)+i s_{1} Q^{*}(1 / 2) . \tag{2.4}
\end{equation*}
$$

Combining (2.3) and (2.4) it follows that

$$
V^{-1} P(-1 / 2) \sigma_{0} V=P^{*}(-1 / 2) \sigma_{0} .
$$

This finishes the proof of Theorem 2.0.
Repeated application of Theorem 2.0 (3) shows that $\sigma(a)=V^{a_{2}} \sigma_{a_{1}} V^{-a_{2}}$ is a bounded operator for any choice of integers $a_{1}, a_{2}$. Thus $\sigma(a) \in$ $G(W, Q)$.

In this section we compute normal ordered forms for the wave function $w_{j}^{s}(x, a)$ and for the identities established in Section 2. These forms will be useful in establishing scaling limit results, and will prove essential for the identification of local expansion coefficients. We begin with the definition of a normal ordered product from [23]:

Definition. If $g$ is a factorable element of $G\left(W, Q\right.$ ) ( $\sigma_{0}$ in our case) and $w_{l} \in W(l=1, \ldots, m)$ then $n\left[w_{1} \cdots w_{m} g\right]$ is defined inductively by

$$
\begin{aligned}
n\left[w_{1} \cdots w_{m} g\right]= & F\left(Q_{+} w_{1}\right) n\left[w_{2} \cdots w_{m} g\right] \\
& +(-1)^{m-1} n\left[w_{2} \cdots w_{m} g\right] F\left(Q_{-} w_{1}\right), n[g]=g
\end{aligned}
$$

(note: we use $n[\cdot]$ instead of : : to avoid a sea of dots). It is useful to introduce the notation $T(a)$ for the operator on $l^{2}\left(\mathbf{Z}_{1 / 2}, \mathbf{C}^{2}\right)$ whose action on the basis elements $e_{j}(k)$ is given by $T(a) e_{j}(k)=T^{a_{2}} e_{j}\left(k+a_{1}\right)$. We also write $V(a)=\Gamma(T(a))$. The quantities we are interested in may all be expressed in terms of the following "elementary" forms:

Definition. Suppose $u, v \in W=l^{2}\left(\mathbf{Z}_{1 / 2}, \mathbf{C}^{2}\right)$, then we write

$$
\begin{aligned}
& N_{i j}(a \mid u, v)=\frac{\left\langle\mathscr{T} n\left[T\left(a_{i}\right) u \sigma\left(a_{i}\right)\right] n\left[T\left(a_{j}\right) v \sigma\left(a_{j}\right)\right] \prod_{l \neq i, j} \sigma\left(a_{l}\right)\right\rangle}{\left\langle\mathscr{T} \prod_{l} \sigma\left(a_{l}\right)\right\rangle}, \\
& N_{i i}(a \mid u, v)=\frac{\left\langle\mathscr{T} n\left[T\left(a_{i}\right) u T\left(a_{i}\right) v \sigma\left(a_{i}\right)\right] \prod_{l \neq i} \sigma\left(a_{l}\right)\right\rangle}{\left\langle\mathscr{T} \prod_{l} \sigma\left(a_{l}\right)\right\rangle}
\end{aligned}
$$

For brevity we will often suppress the dependence on " $a$ " writing $N_{i j}(u, v)$. We turn now to calculation of normal forms for $w_{j}^{s}(x, a)$. In fact, it will prove useful to give not one formula but $2 r$ different formulas for $w_{j}^{s}(x, a)$. Each of the formulas will single out a "branch point" $a_{l}$, and will have advantages for the analysis of the local behavior of $w_{j}^{s}(x, a)$ for $x$ near $a_{l}$. For each point $a_{l}$ there are two situations we distinguish. If $x$ is near $a_{l}(l \neq s)$, then either $\psi_{j}(x) \sigma\left(a_{l}\right)$ or $\sigma\left(a_{l}\right) \psi_{j}(x)$ occurs in the vacuum expectation defining $w_{j}^{s}(x, a)$. If $x$ is near $a_{s}$, then either $\psi_{j}(x) \mu\left(a_{s}\right)$ or $\mu\left(a_{s}\right) \psi_{j}(x)$ occurs in the product defining $w_{j}^{s}(x, a)$. The formulas we give will depend on which of these factors occurs. The following lemma will yield normal forms for each of these factors.
Lemma 3.0. Suppose $w_{1}, w_{2} \in W$. Then if $T<T_{c}$ we have
(1) $F\left(w_{1}\right) \sigma_{m}=n\left[u_{1} \sigma_{m}\right]$, where $u_{1}=\left(Q_{+}+s_{m} Q_{-}\right)^{-1} w_{1}$,
(2) $\sigma_{m} F\left(w_{2}\right)=n\left[u_{2} \sigma_{m}\right]$, where $u_{2}=\left(Q_{-}+s_{m} Q_{+}\right)^{-1} w_{2}$,
(3) $F\left(w_{1}\right) n\left[u_{2} \sigma_{m}\right]=n\left[u_{1} u_{2} \sigma_{m}\right]+\left\langle Q_{+} u_{2}, \bar{w}_{1}\right\rangle \sigma_{m}$, where $u_{1}=\left(Q_{+}+\right.$ $\left.s_{m} Q_{-}\right)^{-1} w_{1}$,
(4) $n\left[u_{2} \sigma_{m}\right] F\left(w_{1}\right)=-n\left[u_{1} u_{2} \sigma_{m}\right]+\left\langle Q_{-} u_{2}, \bar{w}_{1}\right\rangle \sigma_{m}$, where $u_{1}=\left(Q_{-}+\right.$ $\left.s_{m} Q_{+}\right)^{-1} w_{1}$.

Proof. If $u \in W$, then by definition $n\left[u \sigma_{m}\right]=F\left(Q_{+} u\right) \sigma_{m}+\sigma_{m} F\left(Q_{-} u\right)$ $=F\left(\left(Q_{+}+s_{m} Q_{-}\right) u\right) \sigma_{m}$. Thus to write $F(w) \sigma_{m}$ in normal ordered form, we must solve $w=\left(Q_{+}+s_{m} Q_{-}\right) u$ for $u$. This proves (1) and (2) follows in a similar fashion.
Next we consider (3). Again by definition

$$
\begin{equation*}
n\left[u_{1} u_{2} \sigma_{m}\right]=F\left(Q_{+} u_{1}\right) n\left[u_{2} \sigma_{m}\right]-n\left[u_{2} \sigma_{m}\right] F\left(Q_{-} u_{1}\right) . \tag{3.1}
\end{equation*}
$$

Using the Clifford relations for $F(\cdot)$ and $\sigma_{m} F(u)=F\left(s_{m} u\right) \sigma_{m}$, one finds:

$$
\begin{aligned}
n\left[u_{2} \sigma_{m}\right] F\left(Q_{-} u_{1}\right) & =F\left(Q_{+} u_{2}\right) \sigma_{m} F\left(Q_{-} u_{1}\right)+\sigma_{m} F\left(Q_{-} u_{2}\right) F\left(Q_{-} u_{1}\right) \\
& =F\left(Q_{+} u_{2}\right) F\left(s_{m} Q_{-} u_{1}\right) \sigma_{m}-F\left(s_{m} Q_{-} u_{1}\right) \sigma_{m} F\left(Q_{-} u_{2}\right) \\
& =-F\left(s_{m} Q_{-} u_{1}\right) n\left[u_{2} \sigma_{m}\right]+\left\langle Q_{+} u_{2}, \overline{\left.s_{m} Q_{-} u_{1}\right\rangle \sigma_{m} .}\right.
\end{aligned}
$$

Substituting this in (3.1) one finds

$$
\begin{equation*}
n\left[u_{1} u_{2} \sigma_{m}\right]=F\left(\left(Q_{+}+s_{m} Q_{-}\right) u_{1}\right) n\left[u_{2} \sigma_{m}\right]-\left\langle Q_{+} u_{2}, \overline{s_{m} Q_{-} u_{1}}\right\rangle \sigma_{m} . \tag{3.2}
\end{equation*}
$$

Since $P$ and $Q$ anticommute, $\overline{Q_{+} u_{1}}=Q_{-} \overline{u_{1}}$. Thus $\left\langle Q_{+} u_{2}, \overline{s_{m} Q_{-} u_{1}}\right\rangle=$ $\left\langle Q_{+} u_{2}, \overline{\left(Q_{+}+s_{m} Q_{-}\right) u_{1}}\right\rangle$, since $\left\langle Q_{+} u_{2}, Q_{-} \bar{u}_{1}\right\rangle=0$. If we substitute this in (3.2), and let $w_{1}=\left(Q_{+}+s_{m} Q_{-}\right) u_{1}$, we obtain (3). Precisely the same considerations lead to (4).

We consider the application of this lemma to the products $\psi_{j}(x) \sigma\left(a_{l}\right)$ and $\psi_{j}(x) \mu\left(a_{s}\right)$. Since $\psi_{j}(x)=F\left(T^{x_{2}} f_{j}\left(x_{1}\right)\right)$, we may write

$$
\psi_{j}(x) \sigma(a)=F\left(T^{x_{2}} f_{j}\left(x_{1}\right)\right) V(a) \sigma_{0} V(a)^{-1}
$$

If we move $V(a)$ past $F(\cdot)$ in this last equality, we obtain

$$
\psi_{j}(x) \sigma(a)=V(a) F\left(T(x-a) f_{j}\right) \sigma_{0} V(a)^{-1}
$$

Now let

$$
\begin{equation*}
w_{j}^{ \pm}(x)=\left(Q_{\mp}+s_{0} Q_{ \pm}\right)^{-1} T(x) f_{j} . \tag{3.3}
\end{equation*}
$$

Then it follows from Lemma 3.0 that

$$
\begin{align*}
\psi_{j}(x) \sigma(a) & =V(a) n\left[w_{j}^{-}(x-a) \sigma_{0}\right] V(a)^{-1}  \tag{3.4}\\
& =n\left[T(a) w_{j}^{-}(x-a) \sigma(a)\right] .
\end{align*}
$$

Now consider $\psi_{j}(x) \mu(a)$. Since $\mu(a)=V(a) P(-1 / 2) \sigma_{0} V(a)^{-1}$ it is useful to observe that Lemma (3.0) implies $P(-1 / 2) \sigma_{0}=n\left[u \sigma_{0}\right]$, where

$$
\begin{equation*}
u=\sqrt{2}\left(Q_{+}+s_{0} Q_{-}\right)^{-1}\left((\operatorname{ch} K) e_{1}(-1 / 2)+i(\operatorname{sh} K) e_{2}(-1 / 2)\right) . \tag{3.5}
\end{equation*}
$$

Proceeding as above:

$$
\begin{align*}
\psi_{j}(x) \mu(a)= & n\left[T(a) w_{j}^{-}(x-a) \cdot T(a) u \sigma(a)\right] \\
& +\left\langle T(x-a) Q_{-} f_{j}, \bar{u}\right\rangle \sigma(a) . \tag{3.6}
\end{align*}
$$

If we substitute (3.4) and (3.6) in the formulas for $w_{j}^{s}(x, a)$ and analogous results for $\sigma(a) \psi_{j}(x)$ and $\mu(a) \psi_{j}(x)$, we obtain:

Theorem 3.1. Suppose $T<T_{c}$ and $\pi_{2}\left(a_{1}\right)<\pi_{2}\left(a_{2}\right) \cdots<\pi_{2}\left(a_{r}\right)$, then

1) if $l<s$

$$
\begin{array}{ll}
w_{j}^{s}(x, a)=N_{l s}\left(w_{j}^{-}\left(x-a_{l}\right), u\right), & \pi_{2}\left(a_{l-1}\right) \leqslant x_{2} \leqslant \pi_{2}\left(a_{l}\right), \\
w_{j}^{s}(x, a)=N_{l s}\left(w_{j}^{+}\left(x-a_{l}\right), u\right), & \pi_{2}\left(a_{l}\right) \leqslant x_{2} \leqslant \pi_{2}\left(a_{l+1}\right) ;
\end{array}
$$

2) if $l=s$

$$
\begin{aligned}
w_{j}^{s}(x, a)= & N_{s s}\left(w_{j}^{-}\left(x-a_{s}\right), u\right) \quad \pi_{2}\left(a_{s-1}\right) \leqslant x_{2} \leqslant \pi_{2}\left(a_{s}\right) \\
& +\left\langle T\left(x-a_{s}\right) Q_{-} f_{j}, \bar{u}\right\rangle, \\
w_{j}^{s}(x, a)= & N_{s s}\left(w_{j}^{+}\left(x-a_{s}\right), u\right) \quad \pi_{2}\left(s_{s}\right) \leqslant x_{2} \leqslant \pi_{2}\left(a_{s+1}\right) \\
& -\left\langle T\left(x-a_{s}\right) Q_{+} f_{j}, \bar{u}\right\rangle ;
\end{aligned}
$$

3) if $l>s$

$$
\begin{array}{ll}
w_{j}^{s}(x, a)=-N_{s l}\left(u, w_{j}^{-}\left(x-a_{l}\right)\right), & \pi_{2}\left(a_{l-1}\right) \leqslant x_{2} \leqslant \pi_{2}\left(a_{l}\right) \\
w_{j}^{s}(s, a)=-N_{s l}\left(u, w_{j}^{+}\left(x-a_{l}\right)\right), & \pi_{2}\left(a_{l}\right) \leqslant x_{2} \leqslant \pi_{2}\left(a_{l+1}\right)
\end{array}
$$

We now apply Lemma 3.0 to the identities in Section 2. First define

$$
\begin{equation*}
v=\sqrt{2}\left(Q_{+}+s_{0} Q_{-}\right)^{-1}\left((\operatorname{ch} K) e_{2}(1 / 2)-i(\operatorname{sh} K) e_{1}(1 / 2)\right) \tag{3.7}
\end{equation*}
$$

The following result is an elementary consequence of Lemma 3.0:

$$
\left(c_{1}-i s_{1} Q(1 / 2) P(-1 / 2)\right) \sigma_{0}=\sigma_{0}+i s_{1} n\left[u v \sigma_{0}\right]
$$

Since the vacuum expectation of the normal ordered product, $n(\cdot)$, is zero, and $\left\langle V \sigma_{0} V^{-1}\right\rangle=\left\langle\sigma_{0}\right\rangle$, this identity also demonstrates that the normalization in Theorem 2.0 is correct for $V \sigma_{0} V^{-1}$.

It is useful to introduce the notation

$$
\begin{equation*}
e_{j}^{ \pm}=\sqrt{2}\left(Q_{+}+s_{0} Q_{-}\right)^{-1} e_{j}( \pm 1 / 2) \tag{3.8}
\end{equation*}
$$

The following theorem summarizes the difference identities we will later need, and is a straightforward consequence of Theorem (2.0) and Lemma (3.0).

Theorem 3.2. Suppose $T<T_{c}$, then
(1) $\sigma_{1}-\sigma_{0}=\operatorname{in}\left[e_{2}^{+} e_{1}^{+} \sigma_{0}\right]$,
(2) $\mu_{1 / 2}-\mu_{-1 / 2}=-n\left[(i v+u) \sigma_{0}\right]$,
(3) $V \sigma_{0} V^{-1}-\sigma_{0}=i s_{1} n\left[u v \sigma_{0}\right]$,
(4) $\mu_{-1 / 2}-V^{-1} \mu_{-1 / 2} V=2 i(\operatorname{sh} K) n\left[e_{2}^{-} \sigma_{0}\right]$.

In the remainder of this section we compute $u, v$, and $e_{j}^{ \pm}$in coordinates that are convenient for scaling. It will suffice to illustrate the method for $e_{j}^{ \pm}$.

In view of (3.8) it is useful to work in the $Q$ representation where it is easy to compute the inverse of $\left(Q_{+}+s Q_{-}\right)$. In the $Q$ representation the spin operator is $D\left[\begin{array}{cc}s_{0} & 0 \\ 0 & s_{0}\end{array}\right] D^{*}=\left[\begin{array}{cc}A & B \\ B & A\end{array}\right]$. Thus $Q_{+}+s_{0} Q_{-}=\left[\begin{array}{cc}I & B \\ 0 & A\end{array}\right]$ and since $A^{-1} B=-B A^{-1},\left(Q_{+}+s_{0} Q_{-}\right)^{-1}=\left[\begin{array}{cc}I & A^{-1} B \\ 0 & A^{-1}\end{array}\right]$. Formulas for $A^{-1} B$ and $A^{-1}$ can be found in (2.9) of [24]. In particular the formula for $A^{-1}$ is

$$
A^{-1}=\left[\begin{array}{cc}
-a_{+} s_{+} a_{+}^{-1}+a_{-} s_{-} a_{-}^{-1} & 0 \\
0 & -a_{+}^{-1} s_{+} a_{+}+a_{-}^{-1} s_{-} a_{-}
\end{array}\right]
$$

where $s_{+}\left(s_{-}\right)$is the projection on the subspace of $l_{2}\left(\mathbb{Z}_{1 / 2}, \mathbb{R}\right)$ with vanishing negative (positive) half integer Fourier coefficients. The maps $a_{ \pm}$are multiplication by $a_{ \pm}\left(e^{i \theta}\right)$ in the Fourier transform variables, where $e^{i \alpha(\theta)}=$ $a_{+}\left(e^{i \theta}\right) a_{-}\left(e^{i \theta}\right)$ is the Wiener-Hopf factorization of $e^{i \alpha(\theta)}\left(T<T_{c}\right)$. Since $D e_{j}(1 / 2)=2^{-1 / 2}\left[\begin{array}{l}e_{j}(1 / 2) \\ e_{j}(1 / 2)\end{array}\right]$ we apply $\left[\begin{array}{cc}I & A^{-1} B \\ 0 & A^{-1}\end{array}\right]$ to this vector to obtain the $Q$ representation for $\left(Q_{+}+s_{0} Q_{-}\right)^{-1} e_{j}(1 / 2)$ :

$$
\begin{aligned}
2^{-1 / 2}\left[\begin{array}{cc}
I & A^{-1} B \\
0 & A^{-1}
\end{array}\right]\left[\begin{array}{l}
e_{j}(1 / 2) \\
e_{j}(1 / 2)
\end{array}\right] & =2^{-1 / 2}\left[\begin{array}{c}
e_{j}(1 / 2)+A^{-1} B e_{j}(1 / 2) \\
A^{-1} e_{j}(1 / 2)
\end{array}\right] \\
& =2^{-1 / 2}\left[\begin{array}{c}
-A^{-1} e_{j}(1 / 2) \\
A^{-1} e_{j}(1 / 2)
\end{array}\right]
\end{aligned}
$$

This last equality follows from $I+A^{-1} B=A^{-1}(A+B)=A^{-1} s_{0}$ and $s_{0} e_{j}(1 / 2)=-e_{j}(1 / 2)$. We now compute $A^{-1} e_{j}(1 / 2)$. Since $a_{+}^{ \pm 1}$ leave the range of $s_{+}$invariant and $e^{i \theta / 2}$ is in the range of $s_{+}$we have

$$
a_{+} s_{+} a_{+}^{-1} e^{i \theta / 2}=a_{+} a_{+}^{-1} e^{i \theta / 2}=e^{i \theta / 2} \quad \text { and } \quad a_{+}^{-1} s_{+} a_{+} e^{i \theta / 2}=e^{i \theta / 2}
$$

To finish the calculation we consider $a_{-} s_{-} a_{-}^{-1} e^{i \theta / 2}$ and $a_{-}^{-1} s_{-} a_{-} e^{i \theta / 2}$. Since $a_{-}^{-1}\left(e^{i \theta}\right)$ has an analytic extension outside the unit disk its Fourier series may be written

$$
a_{-}^{-1}\left(e^{i \theta}\right)=\sum_{n=0}^{\infty} b_{n} e^{-i n \theta}
$$

Thus the half-integer Fourier transform of $a_{-}^{-1} e^{i \theta / 2}$ is $\sum_{n=0}^{\infty} b_{n} e^{-i(n-1 / 2) \theta}$. The application of $s_{-}$produces $\sum_{n=1}^{\infty} b_{n} e^{-i(n-1 / 2) \theta}=e^{i \theta / 2}\left(a_{-}^{-1}\left(e^{i \theta}\right)-b_{0}\right)$. A residue calculation shows that $b_{0}=1$ and consequently $a_{-} s_{-} a_{-}^{-1} e^{i \theta / 2}=$
$e^{i \theta / 2}\left(1-a_{-}\left(e^{i \theta}\right)\right)$. Precisely analogous calculations show that

$$
\begin{aligned}
& a_{ \pm} s_{ \pm} a_{ \pm}^{-1} e^{\mp i \theta / 2}=e^{\mp i \theta / 2}\left(1-a_{ \pm}\left(e^{i \theta}\right)\right), \\
& a_{ \pm}^{-1} s_{ \pm} a_{ \pm} e^{\mp i \theta / 2}=e^{\mp i \theta / 2}\left(1-a_{ \pm}^{-1}\left(e^{i \theta}\right)\right) .
\end{aligned}
$$

Using these results it is a simple matter to verify that

$$
\begin{aligned}
& A^{-1} e_{1}( \pm 1 / 2)=\mp e^{ \pm i \theta / 2} a_{\mp}\left(e^{i \theta}\right)\left[\begin{array}{l}
1 \\
0
\end{array}\right], \\
& A^{-1} e_{2}( \pm 1 / 2)=\mp e^{ \pm i \theta / 2} a_{\mp}^{-1}\left(e^{i \theta}\right)\left[\begin{array}{l}
0 \\
1
\end{array}\right] .
\end{aligned}
$$

Substitution now gives the $Q$ representation for the vectors $e_{j}^{ \pm}$:

$$
\begin{aligned}
& D e_{1}^{ \pm}=e^{ \pm i \theta / 2} a_{\mp}\left(e^{i \theta}\right)\left[\begin{array}{l}
1 \\
0
\end{array}\right] \oplus\left[\begin{array}{c}
\mp 1 \\
0
\end{array}\right], \\
& D e_{2}^{ \pm}=e^{ \pm i \theta / 2} a_{\mp}^{-1}\left(e^{i \theta}\right)\left[\begin{array}{l}
0 \\
1
\end{array}\right] \oplus\left[\begin{array}{c}
0 \\
\mp 1
\end{array}\right] .
\end{aligned}
$$

To proceed we introduce a change of coordinates. Let $\mathcal{H}(\delta, T)$ denote the complex Hilbert space of functions $f:[-\pi / \delta, \pi / \delta] \rightarrow C$ with inner product

$$
\langle f, g\rangle=(2 \pi)^{-1} \int_{-\pi / \delta}^{\pi / \delta} f(p) \overline{g(p)}[\operatorname{sh} \gamma(\delta p)]^{-1} \delta d p
$$

(the parameter $\delta>0$ is introduced here because it will be useful later in the discussion of scaling). We now define a unitary map $U_{T}$ from the Hilbert space $\mathscr{K}=l^{2}\left(\mathbf{Z}_{1 / 2}, \mathbb{R}^{2}\right)$ with complex structure $\Lambda$ onto $\mathscr{K}(1, T)$ with complex structure $i$. If $f(\theta)=\left[\begin{array}{l}f_{1}(\theta) \\ f_{2}(\theta)\end{array}\right]$ is the Fourier transform of $f \in \mathscr{K}$, then

$$
\begin{equation*}
U_{T} f(\theta)=(\operatorname{sh} \gamma(\theta))^{1 / 2}\left(e^{-i \alpha(\theta) / 2} f_{1}(\theta)+i e^{i \alpha(\theta) / 2} f_{2}(\theta)\right) \tag{3.9}
\end{equation*}
$$

We leave it to the reader to check that $\Lambda$ is transformed into $i$ and that the map is unitary (the reader is cautioned that this map is slightly different from the map $U_{T}$ defined in [24]).
We transform $W=\mathscr{K} \oplus \mathscr{K}$ by $U_{T} \oplus U_{T}$. The results for $U_{T} \oplus U_{T} D e_{j}^{ \pm}$ are

$$
\begin{align*}
& e_{1}^{ \pm}=e^{ \pm i \theta / 2}(\operatorname{sh} \gamma)^{1 / 2}\left(a_{+} a_{-}\right)^{-1 / 2} a_{\mp}[1 \oplus \mp 1], \\
& e_{2}^{ \pm}=e^{ \pm i \theta / 2}(\operatorname{sh} \gamma)^{1 / 2}\left(a_{+} a_{-}\right)^{1 / 2} a_{\mp}^{-1}[i \oplus \mp i], \tag{3.10}
\end{align*}
$$

where we have written $e_{j}^{ \pm}$for $U_{T} \oplus U_{T} D e_{j}^{ \pm}$to unburden the notation. We now make use of some identities established in the remarks following (4.1)
in [24]. In particular

$$
\left[a_{+} / a_{-}\right]^{1 / 2}=(2 \operatorname{th} K / \operatorname{sh} \gamma)^{1 / 2} \operatorname{ch}(\gamma / 2)
$$

and

$$
\left[a_{-} / a_{+}\right]^{1 / 2}=(2 /(\operatorname{th} K) \operatorname{sh} \gamma)^{1 / 2} \operatorname{sh}(\gamma / 2)
$$

Substituting these results in (3.10) we obtain

$$
\begin{align*}
& e_{1}^{ \pm}=2^{-1 / 2} e^{ \pm i \theta / 2}(\operatorname{th} K)^{\mp 1 / 2}\left(e^{\gamma / 2} \mp e^{-\gamma / 2}\right)[1 \oplus \mp 1]  \tag{3.11}\\
& e_{2}^{ \pm}=2^{-1 / 2} e^{ \pm i \theta / 2}(\operatorname{th} K)^{ \pm 1 / 2}\left(e^{\gamma / 2} \pm e^{-\gamma / 2}\right)[i \oplus \mp i]
\end{align*}
$$

Employing the same abuse of notation $z=U_{T} \oplus U_{T} D z$, we find as a consequence of (3.11)

$$
\begin{align*}
& u=(\operatorname{sh} 2 K)^{1 / 2} e^{-i \theta / 2}\left[e^{-\gamma / 2} \oplus e^{\gamma / 2}\right] \\
& v=(\operatorname{sh} 2 K)^{1 / 2} e^{i \theta / 2}\left[i e^{-\gamma / 2} \oplus-i e^{\gamma / 2}\right] \tag{3.12}
\end{align*}
$$

In this section we establish the scaling limit results we require. We introduce a lattice spacing $\delta>0$ in the construction of the functions $w_{j}^{s}(x, a)$ and at the same time we make the temperature $T(\delta)$ depend on $\delta$ in such a fashion that the correlation length for the Ising model on the scaled lattice is asymptotic to 1 as $\delta \rightarrow 0$. We define $\gamma(p, \delta)$ by

$$
\operatorname{ch} \gamma(p, \delta)=2+\left(\delta^{2} / 2\right)-\cos \delta p, \quad p \in[-\pi / \delta, \pi / \delta]
$$

The temperature $T(\delta)$ is now implicitly defined by $[\gamma(\delta p)]_{T=T(\delta)}=$ $\gamma(p, \delta)$. This is shown to make sense in the discussion preceding Lemma 4.1 in [24].

We define:

$$
w_{j}^{s, \delta}(x, a)=\delta^{-1 / 2}\left[w_{j}^{s}\left(\delta^{-1} x, \delta^{-1} a\right)\right]_{T=T(\delta)}
$$

where $x, a_{s} \in \delta\left(\mathbf{Z}_{1 / 2} \times \mathbf{Z}\right)$ and $a_{l} \in \delta \mathbf{Z}^{2}(l \neq s)$. The scaling limit is

$$
w_{j}^{s}(x, a)=\lim _{\delta \rightarrow 0} w_{j}^{s, \delta}(x, a)
$$

Since we require formulas for $w_{j}^{s}(x, a)$, we review the constructions used in [24] to understand this limit. Recall the Hilbert space $\mathcal{K}(\delta, T)$ and the
unitary map $U_{T}$ from $\mathscr{H}$ to $\mathscr{K}(1, T)$ introduced in Section 2 . The principal reason for introducing these coordinates is the simplification which results for the kernels of $A^{-1}$ and $A^{-1} B$. In the following, we shall write $M$ for $A^{-1} B$ and $N$ for $A^{-1}$. Define

$$
\begin{align*}
& M\left(\theta, \theta^{\prime}\right)=i \frac{\operatorname{sh}\left[\left(\gamma(\theta)-\gamma\left(\theta^{\prime}\right)\right) / 2\right]}{\sin \left[\left(\theta+\theta^{\prime}\right) / 2\right]}, \\
& N\left(\theta, \theta^{\prime}\right)=-i \frac{\operatorname{sh}\left[\left(\gamma(\theta)+\gamma\left(\theta^{\prime}\right)\right) / 2\right]}{\sin \left[\left(\theta-\theta^{\prime}\right) / 2\right]} \tag{4,1}
\end{align*}
$$

Then it is a consequence of 4.2 in [24] that the action of $M$ and $N$ in the $\mathscr{K}(1, T)$ representation of $\mathscr{H}$ is

$$
\begin{align*}
& M f(\theta)=\int_{-\pi}^{\pi} M\left(\theta, \theta^{\prime}\right) \overline{f\left(\theta^{\prime}\right)}\left(2 \pi \operatorname{sh} \gamma\left(\theta^{\prime}\right)\right)^{-1} d \theta^{\prime} \\
& N f(\theta)=\int_{-\pi}^{\pi} N\left(\theta, \theta^{\prime}\right) f\left(\theta^{\prime}\right)\left(2 \pi \operatorname{sh} \gamma\left(\theta^{\prime}\right)\right)^{-1} d \theta^{\prime} \tag{4.2}
\end{align*}
$$

The integral defining $N$ is understood in the principal value sense (note: the map $\left[\begin{array}{rr}1 & 0 \\ 0 & -1\end{array}\right]$ on $\mathscr{K}$ is transformed into the map which takes $f(\theta)$ to $\overline{f(-\theta)}$ on $\mathcal{K}(1, T)$ ).
We introduce a unitary scale transformation from $\mathscr{H}(1, T)$ to $\mathscr{K}(\delta, T)$ by $f(\theta) \rightarrow f(\delta p)(p \in[-\pi / \delta, \pi / \delta])$. Under similarity transformation by this scaling operator, the kernels of $M$ and $N$ become (relative to the measure $\left.(2 \pi \operatorname{sh} \gamma(\delta p))^{-1} \delta d p\right)$

$$
\begin{aligned}
& M\left(p, p^{\prime}\right)=i \frac{\operatorname{sh}\left[\left(\gamma(\delta p)-\gamma\left(\delta p^{\prime}\right)\right) / 2\right]}{\sin \left[\delta\left(p+p^{\prime}\right) / 2\right]} \\
& N\left(p, p^{\prime}\right)=-i \frac{\operatorname{sh}\left[\left(\gamma(\delta p)+\gamma\left(\delta p^{\prime}\right)\right) / 2\right]}{\sin \left[\delta\left(p-p^{\prime}\right) / 2\right]}
\end{aligned}
$$

where $|p| \leqslant \pi / \delta$ and $\left|p^{\prime}\right| \leqslant \pi / \delta$.
To obtain the correct temperature dependence in the kernels for $M$ and $N$ we define $M_{T(\delta)}\left(p, p^{\prime}\right)$ and $N_{T(\delta)}\left(p, p^{\prime}\right)$ to be the kernels $M\left(p, p^{\prime}\right)$ and $N\left(p, p^{\prime}\right)$ evaluated at $T=T(\delta)$ (that is, $\gamma(\delta p)$ and $\gamma\left(\delta p^{\prime}\right)$ are replaced by $\gamma(p, \delta)$ and $\gamma\left(p^{\prime}, \delta\right)$ ). For $s, t>0$ we let $M_{\delta}(s)$ denote the Schmidt class operator on $\mathscr{H}(\delta)^{\text {def }}=\mathscr{H}(\delta, T(\delta))$ with kernel $e^{-s \gamma(p, \delta) / \delta} M_{T(\delta)}\left(p, p^{\prime}\right)$ $e^{-s \gamma\left(p^{\prime}, \delta\right) / \delta}$. The action of $M_{\delta}(s)$ on $f \in \mathscr{F}(\delta)$ is obtained by integrating
 $s, t>0$ let $N_{\delta}(s, t)$ denote the bounded operator on $\mathscr{H}(\delta)$ with kernel $e^{-s \gamma(p, \delta) / \delta} N_{T(\delta)}\left(p, p^{\prime}\right) e^{-t \gamma\left(p^{\prime}, \delta\right) / \delta}$. The action of $N_{\delta}(s, t)$ on $f \in \mathscr{K}(\delta)$ is
given by integrating this kernel against $f\left(p^{\prime}\right)\left(2 \pi \operatorname{sh} \gamma\left(p^{\prime}, \delta\right)\right)^{-1} \delta d p^{\prime}$ from $-\pi / \delta$ to $\pi / \delta$.

Let $\mathcal{H}(0)$ denote the Hilbert space of functions $F: \mathbb{R} \rightarrow \mathbb{C}$ with complex structure $i$ and inner product

$$
\langle f, g\rangle=\int_{-\infty}^{\infty} f(p) \overline{g(p)}(2 \pi \omega(p))^{-1} d p
$$

where $\omega(p)^{2}=1+p^{2}$. Define an isometric injection $i_{8}: \mathscr{K}(\delta) \rightarrow \mathscr{K}(0)$ by

$$
\begin{aligned}
i_{\delta} f(p) & =i_{\delta}(p) f(p), & & |p| \leqslant \pi / \delta, \\
& =0, & & |p|>\pi / \delta,
\end{aligned}
$$

where $i_{\delta}(p)=(\delta \omega(p) / \operatorname{sh} \gamma(p, \delta))^{1 / 2}$. Observe that $i_{\delta}^{*}: \mathscr{K}(0) \rightarrow \mathscr{K}(\delta)$ is multiplication by $\chi_{\delta}(p) i_{\delta}(p)^{-1}$, where

$$
\begin{array}{rlrl}
\chi_{\delta}(p) & =1, & & |p| \leqslant \pi / \delta, \\
& =0, & |p|>\pi / \delta
\end{array}
$$

Since $\lim _{\delta \rightarrow 0} \delta^{-1} \operatorname{sh} \gamma(p, \delta)=\omega(p)$, the space $\mathscr{H}(0)$ will provide a convenient arena for describing scaling results. Let $s, t>0$. We define a Schmidt class operator $M(s)$ and a bounded operator $N(s, t)$ on $\mathscr{H}(0)$ by

$$
\begin{aligned}
M(s) f(p) & =i \int_{-\infty}^{\infty} e^{-s \omega(p)} \frac{\omega(p)-\omega\left(p^{\prime}\right)}{p+p^{\prime}} e^{-s \omega\left(p^{\prime}\right)} \overline{f\left(p^{\prime}\right)}\left(2 \pi \omega\left(p^{\prime}\right)\right)^{-1} d p^{\prime} \\
N(s, t) f(p) & =-i \int_{-\infty}^{\infty} e^{-s \omega(p)} \frac{\omega(p)+\omega\left(p^{\prime}\right)}{p-p^{\prime}} \\
& \cdot e^{-t \omega\left(p^{\prime}\right)} f\left(p^{\prime}\right)\left(2 \pi \omega\left(p^{\prime}\right)\right)^{-1} d p^{\prime} .
\end{aligned}
$$

The integral defining $N(s, t)$ is understood in the principal value sense. The significance of these operators is that $i_{\delta} M_{\delta}(s) i_{\delta}^{*}$ converges in Schmidt norm to $M(s)$ as $\delta \rightarrow 0$, and the difference $\chi_{\delta} N(s, t) \chi_{\delta}-i_{\delta} N_{\delta}(s, t) i_{\delta}^{*}$ converges to zero in operator norm as $\delta \rightarrow 0$ (see Lemmas (4.2) and (4.3) in [24]). Before we state our principal result on scaling, it will be convenient to introduce some further terminology. Let $W(\delta)=\mathscr{F}(\delta) \oplus \mathscr{H}(\delta)$ and $W(0)$ $=\mathscr{F}(0) \oplus \mathscr{F}(0)$. The reader should note that the complex structure on these spaces relative to which $F(\cdot)$ is linear, is $i \oplus(-i)$. Let

$$
Q_{\delta}\left(\varepsilon_{1}, \varepsilon_{2}\right)=Q_{+} e^{-\varepsilon_{1} \gamma(p, \delta) / \delta}+Q_{-} e^{-\varepsilon_{2} \gamma(p, \delta) / \delta}
$$

and

$$
Q\left(\varepsilon_{1}, \varepsilon_{2}\right)=Q_{+} e^{-\varepsilon_{1} \omega(p)}+Q_{-} e^{-\varepsilon_{2} \omega(p)} .
$$

We will say a measurable function $f: \mathbb{R} \rightarrow \mathbb{C}^{2}$ is $\left(\varepsilon_{1}, \varepsilon_{2}\right)$ almost in $W(0)$ if
the function $Q\left(\varepsilon_{1}, \varepsilon_{2}\right) f(p)$ is in $W(0)$. If $f_{\delta} \in W(\delta)$ is such that $i_{\delta} Q_{\delta}\left(\varepsilon_{1}, \varepsilon_{2}\right) f_{\delta}$ converges in $W(0)$ as $\delta \rightarrow 0$, then we will say $f_{\delta}$ is ( $\varepsilon_{1}, \varepsilon_{2}$ ) convergent. If $f_{\delta}$ is $\left(\varepsilon_{1}, \varepsilon_{2}\right)$ convergent then it is not hard to see that there is a function $f$, $\left(\varepsilon_{1}, \varepsilon_{2}\right)$ almost in $W(0)$, such that $i_{\delta} Q_{\delta}\left(\varepsilon_{1}, \varepsilon_{2}\right) f_{\delta}$ converges in $W(0)$ to $Q\left(\varepsilon_{1}, \varepsilon_{2}\right) f$ as $\delta \rightarrow 0$ (see Lemma 4.1 in [24]). We will say that $f_{\delta}$ is ( $\varepsilon_{1}, \varepsilon_{2}$ ) convergent to $f$. Finally, if $f$ is $\left(\varepsilon_{1}, \varepsilon_{2}\right)$ almost in $W(0)$ and $f_{n}$ is a sequence $\left(\varepsilon_{1}, \varepsilon_{2}\right)$ almost in $W(0)$, we will say that $f_{n}$ is $\left(\varepsilon_{1}, \varepsilon_{2}\right)$ convergent to $f$ if $Q\left(\varepsilon_{1}, \varepsilon_{2}\right) f_{n}$ converges to $Q\left(\varepsilon_{1}, \varepsilon_{2}\right) f$ in $W(0)$ as $n \rightarrow \infty$.

Consider $N_{i j}(a \mid u, v)$ as a function of " $a$ ". Since $V\left(a_{i}\right)$ has a natural continuous extension to $a_{l} \in \mathbb{R}^{2}$ with $\pi_{2}\left(a_{l}\right)>0$, we may regard $N_{i j}(a \mid u, v)$ as a continuous function of $a \in \mathbb{R}^{2 r}$ in the domain $\pi_{2}\left(a_{1}\right)<\pi_{2}\left(a_{2}\right) \cdots<$ $\pi_{2}\left(a_{r}\right)$. Suppose $u$ and $v$ are elements of $W(\delta)$, then $u\left(\delta^{-1} \cdot\right)$ and $v\left(\delta^{-1} \cdot\right)$ are in $W(T=T(\delta)$ ) and we define

$$
N_{i j}^{\delta}(a \mid u, v)=N_{i j}\left(\delta^{-1} a \mid u\left(\delta^{-1} \cdot\right), v\left(\delta^{-1} \cdot\right)\right)_{T=T(\delta)} .
$$

Theorem 4.1. Suppose $\pi_{2}\left(a_{1}\right)<\pi_{2}\left(a_{2}\right) \cdots<\pi_{2}\left(a_{r}\right)$ with $a \in \mathbb{R}^{2 r}$ and $i<j$. Suppose $u_{\delta} \in W(\delta)$ is $\left(\varepsilon_{i}, \varepsilon_{i+1}\right)$ convergent to $u$ for some $\varepsilon_{l}<c_{l}\left(\pi_{2}\left(a_{l}\right)\right.$ $\left.-\pi_{2}\left(a_{l-1}\right)\right)(l=i, i+1)$ and $v_{\delta} \in W(\delta)$ is $\left(\varepsilon_{j}, \varepsilon_{j+1}\right)$ convergent to $v$ for some $\varepsilon_{l}<d_{l}\left(\pi_{2}\left(a_{l}\right)-\pi_{2}\left(a_{l-1}\right)\right)(l=j, j+1)$. The constants $c_{l}=d_{l}=1$, except when $j=i+1$, in which case there is the single constraint $c_{i+1}+d_{j}=$ $\pi_{2}\left(a_{i+1}\right)-\pi_{2}\left(a_{i}\right)$. In these circumstances $\lim _{\delta \rightarrow 0} N_{i j}^{\delta}\left(a \mid u_{\delta}, v_{\delta}\right)$ converges and we write

$$
\lim _{\delta \rightarrow 0} N_{i j}^{\delta}\left(a \mid u_{\delta}, v_{\delta}\right)=N_{i j}(a \mid u, v)
$$

Suppose $u_{n}$ is a sequence of functions $\left(\varepsilon_{i}, \varepsilon_{i+1}\right)$ almost in $W(0)$ which is $\left(\varepsilon_{i}, \varepsilon_{i+1}\right)$ convergent to $u$. Suppose $v_{n}$ is a sequence of functions $\left(\varepsilon_{j}, \varepsilon_{j+1}\right)$ almost in $W(0)$ which is $\left(\varepsilon_{j}, \varepsilon_{j+1}\right)$ convergent to $v$. Then

$$
\lim _{n \rightarrow \infty} N_{i j}\left(a \mid u_{n}, v_{n}\right)=N_{i j}(a \mid u, v)
$$

Proof. This result is a consequence of Theorem 3.2 in [23]. In order to apply this theorem, it is necessary to write the correlations in the definition of $N_{i j}(u, v)$ as the expectations of products in $G(W, Q)$. This may be done in many ways, but for scaling results it turns out to be useful to take advantage of the "smoothing" properties of positive powers of the transfer matrix. To simplify notation, we now let $m_{l}=\pi_{1}\left(a_{l}\right), n_{l}=\pi_{2}\left(a_{l}\right)$. Since $n_{l}<n_{l+1}$, we may choose $\lambda_{l}$, such that $n_{l}<\lambda_{l}<n_{l+1}$. We also choose $\lambda_{0}<n_{1}$ and $\lambda_{r}>n_{r}$. Now define $\tilde{\sigma}\left(a_{l}\right)=V^{n_{l}-\lambda_{l-1}} \boldsymbol{\sigma}_{m} V^{\lambda_{l}-n_{l}}$. Since the powers of the transfer matrix which occur in $\tilde{\sigma}\left(a_{t}\right)$ are all positive, it is clear
that $\tilde{\boldsymbol{\sigma}}\left(a_{l}\right) \in G(W, Q)$. We introduce

$$
\begin{align*}
& u\left(\delta, a_{l}\right)=T\left(m_{l}, 0\right)\left(Q_{+} T^{n_{l}-\lambda_{l-1}}+Q_{-} T^{n_{l}-\lambda_{l}}\right) u_{\delta}\left(\delta^{-1} \cdot\right), \\
& v\left(\delta, a_{l}\right)=T\left(m_{l}, 0\right)\left(Q_{+} T^{n_{l}-\lambda_{l-1}}+Q_{-} T^{n_{l}-\lambda_{l}}\right) v_{\delta}\left(\delta^{-1} \cdot\right) . \tag{4.3}
\end{align*}
$$

It is now an elementary calculation that

$$
\begin{equation*}
N_{i j}^{\delta}\left(a \mid u_{\delta}, v_{\delta}\right)=\frac{\left\langle\mathscr{T} n\left[u\left(\delta, b_{i}\right) \tilde{\sigma}\left(b_{i}\right)\right] n\left[v\left(\delta, b_{j}\right) \tilde{\sigma}\left(b_{j}\right)\right] \prod_{l \neq i, j} \tilde{\sigma}\left(b_{l}\right)\right\rangle_{T(\delta)}}{\left\langle\mathscr{T} \prod_{l} \tilde{\sigma}\left(b_{l}\right)\right\rangle_{T(\delta)}} \tag{4.4}
\end{equation*}
$$

where $b_{l}=\delta^{-1} a_{l}$ and $i \neq j$. The case $i=j$ is analogous. For brevity, we mainly confine our attention to $i \neq j$. Theorem 3.2 in [23] applies directly to (4.4) with $g_{l}=\tilde{\sigma}\left(\delta^{-1} a_{l}\right)$. The formula for the right-hand side of (4.4) which we shall present is based on Hilbert space constructions on the space $W(T=T(\delta))$. Since we wish to work in $W(\delta)$ we will make the unitary scale transformation in these constructs as they are introduced rather than repeat the formulas. The operators $L\left(g_{l}\right)$ and $\Delta R\left(g_{l}\right)$ in Theorem (3.2) of [23] are easily computed. Their action on $W(\delta)$ is

$$
\begin{align*}
L_{\delta}\left(g_{l}\right) & =\left[\begin{array}{cc}
N_{m_{i} \delta}\left(n_{l}-\lambda_{l-1}, \lambda_{l}-n_{l}\right) & 0 \\
0 & N_{m_{\delta} \delta}^{*-1}\left(n_{l}-\lambda_{l-1}, \lambda_{l}-n_{l}\right)
\end{array}\right], \\
\Delta R_{\delta}\left(g_{l}\right) & =\left[\begin{array}{cc}
0 & -M_{m_{l} \delta}\left(n_{l}-\lambda_{l-1}\right) \\
M_{m_{l} \delta}\left(\lambda_{l}-n_{l}\right) & 0
\end{array}\right], \tag{4.5}
\end{align*}
$$

where $X_{m \delta}(s)=e^{\mathrm{i} m p} X_{\delta}(s) e^{-\mathrm{i} m p}$. Let $W^{r}(\delta)$ denote the direct sum of $r$ copies of $W(\delta)$. Define $\Delta R_{\delta}$ to be the $r \times r$ block diagonal matrix on $W^{r}(\delta)$ with entries $\left(\Delta R_{\delta}\right)_{i j}=\delta_{i j} \Delta R_{\delta}\left(g_{i}\right)$, and define $L_{\delta}$ to be the $r \times r$ block matrix on $W^{r}(\delta)$ with entries

$$
\begin{align*}
L_{i j}^{(\delta)} & =-Q_{+} L_{\delta}\left(g_{i+1}\right) \cdots L_{\delta}\left(g_{j-1}\right), & & j>i+1, \\
& =-Q_{+}, & & j=i+1, \\
& =0, & & j=i,  \tag{4.6}\\
& =Q_{-}, & & j=i-1, \\
& =Q_{-} L_{\delta}^{-1}\left(g_{i-1}\right) \cdots L_{\delta}^{-1}\left(g_{j+1}\right), & & j<i-1 .
\end{align*}
$$

Now let $u_{\delta}\left(a_{i}\right)$ and $v_{\delta}\left(a_{j}\right)$ denote the elements of $W(\delta)$ given by the unitary scale transforms of $u\left(\delta, \delta^{-1} a_{i}\right)$ and $v\left(\delta, \delta^{-1} a_{j}\right)$. The formula for the
right-hand side of (4.4) as given by Theorem 3.2 in [23] is

$$
\begin{equation*}
N_{i j}^{\delta}\left(a \mid u_{\delta}, v_{\delta}\right)=-\left\langle\left(I+L_{\delta} \Delta R_{\delta}\right)^{-1} L_{\delta} I_{i}\left(u_{\delta}\left(a_{i}\right)\right), I_{j} \overline{\left(v_{\delta}\left(a_{j}\right)\right)}\right\rangle \tag{4.7}
\end{equation*}
$$

where $I_{k}$ is the injection of $W(\delta)$ into the $i$ th slot in $W^{r}(\delta)$, the conjugation $\overline{u_{1} \oplus u_{2}}=u_{2} \oplus u_{1}$ (on $W(\delta)$ ), and the inner product is the standard Hermitian symmetric one on $W^{r}(\delta)$. The operator $I+L_{\delta} \Delta R_{\delta}$ is shown to be invertible in [24].

We now use the isometric injection $i_{\delta}$ to transform the inner product in (4.6) into an inner product on $W^{r}(0)$. The argument in Theorem 4.2 of [24] then shows that $i_{\delta}\left(I+L_{\delta} \Delta R_{\delta}\right)^{-1} L_{\delta} i_{\delta}^{*}$ converges strongly to the operator on $W^{r}(0)$, which one obtains by replacing $M_{\delta}(s)$ and $N_{\delta}(s, t)$ in (4.4) by $M(s)$ and $N(s, t)$. We write $L$ and $\Delta R$ for the operators on $W^{r}(0)$, which are obtained from $L_{\delta}$ and $\Delta R_{\delta}$ by such substitution. We now note that the exponential factors in the transfer matrix in (4.3) and the fact that $u_{\delta}$ and $v_{\delta}$ are $\left(\varepsilon_{l}, \varepsilon_{l+1}\right)$ convergent are enough to guarantee that $i_{\delta} u_{\delta}\left(a_{i}\right)$ and $i_{\delta} v_{\delta}\left(a_{j}\right)$ do converge in $W(0)$. The idea is that the intermediate points $\lambda_{l}$ may be chosen arbitrarily with one exception. That exception is $\lambda_{i}$ when $j=i+1$. In this event, $\left(n_{j}-\lambda_{j-1}\right)+\left(\lambda_{i}-n_{i}\right)=n_{i+1}-n_{i}$ is a constraint. The hypothesis $c_{i+1}+d_{j}=n_{i+1}-n_{i}$ ensures the convergence of $i_{\delta} u_{\delta}\left(a_{i}\right)$ and $i_{\delta} v_{\delta}\left(a_{i}\right)$ for the appropriate choice of $\lambda_{i}$ between $n_{i}$ and $n_{i+1}$. We write $u_{i}(p)=e^{\mathrm{i} m_{i} p} Q\left(n_{i}-\lambda_{i-1}, \lambda_{i}-n_{i}\right) u(p)$ and $v_{j}(p)=e^{\mathrm{i} m_{j} p} Q\left(n_{j}-\lambda_{j-1}\right.$, $\left.\lambda_{j}-n_{j}\right) v(p)$ for the limits of $i_{\delta} u_{\delta}\left(a_{i}\right)$ and $i_{\delta} v_{\delta}\left(a_{i}\right)$. We have demonstrated

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} N_{i j}^{\delta}\left(a \mid u_{\delta}, v_{\delta}\right)=-\left\langle(I+L \Delta R)^{-1} L I_{i}\left(u_{i}\right), I_{j}\left(\bar{v}_{j}\right)\right\rangle \tag{4.8}
\end{equation*}
$$

The exponentials $e^{-s \omega}$, which relate $u_{i}$ and $v_{i}$ to $u$ and $v$, make it quite clear that the limiting functional on the right-hand side of (4.8) is continuous relative to ( $\varepsilon_{l}, \varepsilon_{l+1}$ ) convergence in $u$ and $v$. This finishes the proof of Theorem 4.1.

We are now prepared to consider the scaling limit of $w_{j}^{5, \delta}(x, a)$. Let $w_{j, \delta}^{ \pm}(x) \in W(\delta)$ and $u_{\delta} \in W(\delta)$ denote the scale transforms of $\delta^{-1 / 2} w_{j}^{ \pm}(x, \delta)$ and $u$ (see Section 3.3 and 3.5). Consulting Theorem 3.1 we find that for $l<s$ :

$$
\begin{array}{ll}
w_{j}^{s, \delta}(x, a)=N_{l l}^{\delta}\left(a \mid w_{j, \delta}^{-}\left(x-a_{l}\right), u_{\delta}\right), & n_{l-1} \leqslant x_{2} \leqslant n_{l}, \\
w_{j}^{s, \delta}(x, a)=N_{l s}^{\delta}\left(a \mid w_{j, \delta}^{+}\left(x-a_{l}\right), u_{\delta}\right), & n_{l} \leqslant x_{2} \leqslant n_{l+1} .
\end{array}
$$

According to (3.12), $u_{\delta}(p)=(\operatorname{sh} 2 K)^{1 / 2} e^{-i \delta p / 2}\left[e^{-\gamma(p, \delta) / 2} \oplus e^{\gamma(p, \delta) / 2}\right]$. Recalling Lemma 4.1 of [24] it is evident that $u_{\delta}$ is $(\varepsilon, \varepsilon)$ convergent to $1 \oplus 1$ for any $\varepsilon>0$.

In order to apply Theorem 4.1 we must show that $w_{j, \delta}^{ \pm}\left(x-a_{l}\right)$ is $\left(n_{l}-\lambda_{l-1}, \lambda_{l}-n_{l}\right)$ convergent as $\delta \rightarrow 0$. The vector $Q_{\delta}\left(n_{l}-\lambda_{l-1}\right.$, $\left.\lambda_{l}-n_{l}\right) w_{j, \delta}^{-}\left(x-a_{l}\right)$ is given by

$$
\left[\begin{array}{cc}
X_{11} & X_{12} \\
0 & X_{22}
\end{array}\right] f_{j, \delta}(x-a)
$$

where

$$
\begin{gather*}
X_{11}=e^{\left(\lambda_{t-1}-n_{l}\right) \gamma(p, \delta) / \delta}, \\
X_{12}=e^{\left(\lambda_{l-1}-n_{l}\right) \gamma(p, \delta) / \delta} M_{\delta}  \tag{4.9}\\
X_{22}=e^{\left(n_{l}-\lambda_{l}\right) \gamma(p, \delta) / \delta} N_{\delta} \\
f_{j, \delta}(y)=e^{i y_{l} p} Q_{\delta}\left(y_{2},-y_{2}\right)\left[g_{j, \delta} \oplus g_{j, \delta}\right]
\end{gather*}
$$

and

$$
\begin{aligned}
g_{1, \delta}(p) & =(2 \delta)^{-1 / 2}(\operatorname{sh} \gamma(p, \delta))^{1 / 2}\left(i e^{i \alpha(p, \delta) / 2}-e^{-i \alpha(p, \delta) / 2}\right) \\
g_{-1, \delta}(p) & =(2 \delta)^{-1 / 2}(\operatorname{sh} \gamma(p, \delta))^{1 / 2}\left(i e^{i \alpha(p, \delta) / 2}+e^{-i \alpha(p, \delta) / 2}\right)
\end{aligned}
$$

If we suppose $\lambda_{l-1}<x_{2}<n_{l}$, then the convergence results for $M_{\delta}(s)$ and $N_{\delta}(s, t)$ noted earlier in this section show that to settle the ( $n_{1}-$ $\left.\lambda_{l-1}, \lambda_{l}-n_{l}\right)$ convergence of $w_{j, \delta}^{-}\left(x-a_{l}\right)$ it suffices to prove the $(\varepsilon, \varepsilon)$ convergence of $g_{j, \delta} \oplus g_{j, \delta}$ for any $\varepsilon>0$. The reader should observe that if $x_{2}$ is in the open interval in $\left(n_{l-1}, n_{l}\right)$, then $\lambda_{l-1}$ may always be chosen so that $\lambda_{l-1}<x_{2}$ since the $(\varepsilon, \varepsilon)$ convergence of the vector $u_{\delta}$ imposes no constraint on the choice of intermediate points $\lambda_{I}$. It is a simple calculation to show that $\lim _{\delta \rightarrow 0} \delta^{-1} \operatorname{sh} \gamma(p, \delta) e^{i \alpha\left(\delta_{p}\right)}=1+i p$. Furthermore, Lemma 4.1 of [24] shows that this convergence is dominated in such a fashion as to imply the $(\varepsilon, \varepsilon)$ convergence of $g_{j, \delta} \oplus g_{j, \delta}$. The limits $g_{j}(p)=$ $\lim _{\delta \rightarrow 0} g_{j, \delta}(p)$ are then easily computed using the fact that the square root, $\sqrt{1+i p}$, which arises in this computation is given by $2^{-1 / 2}\left(\xi^{1 / 2} e^{i \pi / 4}+\right.$ $\xi^{-1 / 2} e^{-i \pi / 4}$ ), where $\xi=p+\sqrt{1+p^{2}}$. The results are

$$
\begin{aligned}
g_{1}(p) & =-\xi^{1 / 2} e^{-i \pi / 4} \\
g_{-1}(p) & =\xi^{-1 / 2} e^{i \pi / 4}
\end{aligned}
$$

We have assembled enough information to show that Theorem 4.1 applies to the convergence of $w_{j}^{s, \delta}(x, a)$ for $n_{l-1}<x_{2}<n_{l}$. We have also shown the reader how to obtain the scaling limit analogue of Theorem 3.1. Before we state such a result, it is useful at this point to make a systematic introduction of the mass shell coordinate $\xi=p+\sqrt{1+p^{2}}$. Under this
change of coordinates, the complex functions square integrable on $\mathbb{R}$ with respect to $(2 \pi \omega(p))^{-1} d p$ are mapped unitarily onto the space of complex functions on $(0, \infty)$ square integrable with respect to $(2 \pi \xi)^{-1} d \xi$. The kernels $i\left(\omega(p)-\omega\left(p^{\prime}\right)\right) /\left(p+p^{\prime}\right)$ and $-i\left(\omega(p)-\omega\left(p^{\prime}\right)\right) /\left(p-p^{\prime}\right)$ become in the new coordinates $i\left(\xi-\xi^{\prime}\right) /\left(\xi+\xi^{\prime}\right)$ and $-i\left(\xi+\xi^{\prime}\right) /\left(\xi-\xi^{\prime}\right)$. The following theorem is the scaling analog of Theorem 3.1. To avoid introducing more notation, we write $N_{i j}(a \mid u, v)$ for the function introduced in Theorem 4.1, and $w_{j}^{s}(x, a)$ will now denote the scaled wave function.

Theorem 4.2. Suppose $\pi_{2}\left(a_{1}\right)<\pi_{2}\left(a_{2}\right) \cdots<\pi_{2}\left(a_{r}\right)$, then
(1) if $l<s$

$$
\begin{array}{ll}
w_{j}^{s}(x, a)=N_{l s}\left(a \mid w_{j}^{-}\left(x-a_{l}\right), u\right), & n_{l-1}<x_{2}<n_{l} \\
w_{j}^{s}(x, a)=N_{l s}\left(a \mid w_{j}^{+}\left(x-a_{l}\right), u\right), & n_{l}<x_{2}<n_{l+1}
\end{array}
$$

(2) if $l=s$

$$
\begin{aligned}
w_{j}^{s}(x, a)= & N_{s s}\left(a \mid w_{j}^{-}\left(x-a_{s}\right), u\right), \quad n_{s-1}<x_{2}<n_{s} \\
& +\left\langle T\left(x-a_{s}\right) Q_{-} f_{j}, \bar{u}\right\rangle \\
w_{j}^{s}(x, a)= & N_{s s}\left(a \mid w_{j}^{+}\left(x-a_{s}\right), u\right), \quad n_{s}<x_{2}<n_{s+1} \\
& -\left\langle T\left(x-a_{s}\right) Q_{+} f_{j}, \bar{u}\right\rangle
\end{aligned}
$$

(3) if $l>s$

$$
\begin{array}{ll}
w_{j}^{s}(x, a)=-N_{s l}\left(a \mid u, w_{j}^{-}\left(x-a_{l}\right)\right), & n_{l-1}<x_{2}<n_{l} \\
w_{j}^{s}(x, a)=-N_{s l}\left(a \mid u, w_{j}^{+}\left(x-a_{l}\right)\right), & n_{l}<x_{2}<n_{l+1}
\end{array}
$$

We write $w_{j}^{ \pm}(y)=w_{j}^{ \pm}(r, \varphi), y=r(\cos \varphi, \sin \varphi)$. The functions $w_{j}^{ \pm}(r, \varphi)$ in mass shell coordinates $\xi$ are

$$
\begin{align*}
\left(w_{j}^{-}(r, \varphi)\right)_{1}= & e^{-(r / 2)\left(\xi(\varphi)+\xi(\varphi)^{-1}\right)} g_{j}(\xi) \\
& +i \int_{0}^{\infty} \frac{\xi-\xi^{\prime}}{\xi+\xi^{\prime}} e^{(r / 2)\left(\xi^{\prime}(\varphi)+\xi^{\prime}(\varphi)^{-1}\right)} \overline{g_{j}\left(\xi^{\prime}\right)} \frac{d \xi^{\prime}}{2 \pi \xi^{\prime}} \\
\left(w_{j}^{-}(r, \varphi)\right)_{2}= & -i \int_{0}^{\infty} \frac{\xi+\xi^{\prime}}{\xi-\xi^{\prime}} e^{-(r / 2)\left(\xi^{\prime}(-\varphi)+\xi^{\prime}(-\varphi)^{-1}\right)} g_{j}\left(\xi^{\prime}\right) \frac{d \xi^{\prime}}{2 \pi \xi^{\prime}} \\
\left(w_{j}^{+}(r, \varphi)\right)_{1}= & -i \int_{0}^{\infty} \frac{\xi+\xi^{\prime}}{\xi-\xi^{\prime}} e^{-(r / 2)\left(\xi^{\prime}(\varphi)+\xi^{\prime}(\varphi)^{-1}\right)} g_{j}\left(\xi^{\prime}\right) \frac{d \xi^{\prime}}{2 \pi \xi^{\prime}},  \tag{4.10}\\
\left(w_{j}^{+}(r, \varphi)\right)_{2}= & e^{-(r / 2)\left(\xi(-\varphi)+\xi(-\varphi)^{-1}\right)} g_{j}(\xi) \\
& +i \int_{0}^{\infty} \frac{\xi-\xi^{\prime}}{\xi+\xi^{\prime}} e^{(r / 2)\left(\xi^{\prime}(-\varphi)+\xi^{\prime}(-\varphi)^{-1}\right)} \frac{d \xi^{\prime}}{g_{j}\left(\xi^{\prime}\right)} \frac{d \xi^{\prime}}{2 \pi \xi^{\prime}}
\end{align*}
$$

where $\xi(\varphi)=-i e^{i \varphi} \xi, g_{1}(\xi)=-e^{-i \pi / 4} \xi^{1 / 2}, g_{-1}(\xi)=e^{i \pi / 4} \xi^{-1 / 2}$ and the inner products are

$$
\begin{align*}
& \left\langle T\left(x-a_{s}\right) Q_{-} f_{j}, \bar{u}\right\rangle=\int_{0}^{\infty} e^{\left(r_{s} / 2\right)\left(\xi\left(\varphi_{s}\right)+\xi\left(\varphi_{s}\right)^{-1}\right)} \frac{g_{j}(\xi)}{} \frac{d \xi}{2 \pi \xi},  \tag{4.11}\\
& \left\langle T\left(x-a_{s}\right) Q_{+} f_{j}, \bar{u}\right\rangle=\int_{0}^{\infty} e^{-\left(r_{s} / 2\right)\left(\xi\left(\varphi_{s}\right)+\xi\left(\varphi_{s}\right)^{-1}\right)} g_{j}(\xi) \frac{d \xi}{2 \pi \xi},
\end{align*}
$$

where $x-a_{s}=r_{s}\left(\cos \varphi_{s}, \sin \varphi_{s}\right)$.
We now scale the difference identities in Theorem 3.2 to obtain formulas for the derivatives of scaled $n$-point functions. Suppose $\pi_{2}\left(a_{1}\right)<\pi_{2}\left(a_{2}\right) \cdots$ $<\pi_{2}\left(a_{r}\right)$, then we write

$$
\begin{align*}
\left\langle\sigma_{\delta}(a)\right\rangle & =\langle\sigma\rangle_{T(\delta)}^{-r}\left\langle\sigma \prod_{l=1}^{r} \sigma\left(\delta^{-1} a_{l}\right)\right\rangle_{T(\delta)}, \\
\left\langle\sigma_{\delta}^{i j}(a)\right\rangle & =\langle\sigma\rangle_{T(\delta)}^{-r}\left\langle\mathscr{T} \mu\left(\delta^{-1} a_{i}\right) \mu\left(\delta^{-1} a_{j}\right) \prod_{l \neq i, j} \sigma\left(\delta^{-1} a_{l}\right)\right\rangle_{T(\delta)}, \\
\langle\sigma(a)\rangle & =\lim _{\delta \rightarrow 0}\left\langle\sigma_{\delta}(a)\right\rangle,  \tag{4.12}\\
\left\langle\sigma^{i j}(a)\right\rangle & =\lim _{\delta \rightarrow 0}\left\langle\sigma_{\delta}^{i j}(a)\right\rangle .
\end{align*}
$$

For collections of sites $\left\{a_{l}\right\}$ with noncoincident second coordinates, the convergence of these limits was established in [24]. We now require more detailed information about this convergence. Suppose $a_{l}(l \neq i)$ are considered fixed, and $a_{i}$ is permitted to range over a compact region $C_{i}$ with the second coordinates of $C_{i}$ strictly between $\pi_{2}\left(a_{i-1}\right)$ and $\pi_{2}\left(a_{i+1}\right)$. Let $a_{i}(k)$ $=\pi_{k}\left(a_{i}\right)$. The result we require is that $\partial\left\langle\sigma_{\delta}(a)\right\rangle / \partial a_{i}(k)$ and $\partial\left\langle\boldsymbol{\sigma}_{\delta}^{i j}(a)\right\rangle / \partial a_{i}(k)$ converge to $\partial\langle\boldsymbol{\sigma}(a)\rangle / \partial a_{i}(k)$ and $\partial\left\langle\boldsymbol{\sigma}^{i j}(a)\right\rangle \partial a_{i}(k)$ uniformly for $a_{i} \in C_{i}$ as $\delta \rightarrow 0$. We sketch how to obtain such a result for $\partial\left\langle\sigma_{\delta}(a)\right\rangle / \partial a_{i}(k)$ (the other case may be treated analogously). In [24] it was shown that $\left\langle\sigma_{\delta}(a)\right\rangle=\operatorname{det}_{2}\left(I+G_{\delta}(a)\right)$ (where $\left.G_{\delta}(a)=L_{\delta} \Delta R_{\delta}\right)$ and the convergence in (4.12) was proved by demonstrating that $i_{\delta} G_{\delta} i_{\delta}^{*}$ converged in Schmidt norm to a Schmidt class operator $G(a)$. Using Lemmas 4.1-4.3 in [24] it is not hard to see that the difference quotients for $G_{\delta}(a)$ in $a_{i}(k)$ converge to $\partial G_{\delta}(a) / \partial a_{i}(k)$ in Schmidt norm. Differentiating $\operatorname{det}_{2}(\cdot)$ one finds

$$
\begin{aligned}
\partial\left\langle\sigma_{\delta}(a)\right\rangle / \partial a_{i}(k)= & -\operatorname{det}_{2}\left(I+G_{\delta}(a)\right) \\
& \times \operatorname{Tr}\left(G_{\delta}(a)\left(I+G_{\delta}(a)\right)^{-1} \partial G_{\delta}(a) / \partial a_{i}(k)\right) .
\end{aligned}
$$

Pointwise, convergence to $\partial\langle\sigma(a)\rangle / \partial a_{i}(k)$ as $\delta \rightarrow 0$ is an immediate conse-
quence of the convergence of $i_{\delta} \partial G_{\delta}(a) / \partial a_{i}(k) i_{\delta}^{*}$ to $\partial G(a) / \partial a_{i}(k)$ in Schmidt norm, and it is not hard to make the estimates for the convergence uniform provided the differences $\pi_{2}\left(a_{l}\right)-\pi_{2}\left(a_{l-1}\right)$ are uniformly bounded away from zero. The essential point is that differentiating $G_{\delta}(a)$ with respect to $a_{i}(k)$ brings down factors, ip or $-\gamma(p, \delta) / \delta$, which always occur in conjunction with, and are controlled by, the presence of factors $e^{-\left(n_{l}-n_{i-1}\right) \gamma(p, \delta) / \delta}$ (see Lemma 4.1 of [24]).

The uniform convergence of the derivatives has the following useful consequence, which for clarity we state for functions of a single real variable " $a$." Suppose $f_{\delta}^{\prime}(a)$ is continuous and converges locally uniformly to $f^{\prime}(a)$ as $\delta \rightarrow 0$. Then $\delta^{-1}\left(f_{\delta}(a+\delta)-f_{\delta}(a)\right)$ converges to $f^{\prime}(a)$ as $\delta \rightarrow 0$. The proof is an elementary consequence of the mean value theorem. This simple observation coupled with Theorem 3.2 and Theorem 4.1 yields the following result:

Theorem 4.3. Suppose $\pi_{2}\left(a_{1}\right)<\cdots<\pi_{2}\left(a_{r}\right)$. Then
(4) $\langle\sigma(a)\rangle^{-1} \partial\left\langle\sigma^{i j}(a)\right\rangle / \partial a_{i}(2)=-N_{i j}(-\omega(p) \oplus \omega(p), 1 \oplus 1)$, $i<j$.

Proof. The proof is a simple scaling calculation and is left to the reader. It is helpful for this calculation to replace $u(i v)$ in (3) of Theorem 3.2 by $u(u+i v)$ before proceeding. One should also remember that $i F(x)=$ $F(i \oplus(-i) x)$. The continuity result (4.1) makes it straightforward to justify the limits which are encountered.

In this section we show that $\left[\begin{array}{c}w_{1}^{s}(x, a) \\ w_{-1}^{s}(x, a)\end{array}\right]$ is a multivalued solution of the Euclidean Dirac equation with branch points at $a_{l}(l=1, \ldots, r)$, and we compute local expansion coefficients for these solutions at the branch points. The Euclidean Dirac equation satisfied by $w_{j}^{s}(x, a)$ is

$$
\left[\begin{array}{cc}
-1 & \partial / \partial x_{1}-i \partial / \partial x_{2}  \tag{5.1}\\
\partial / \partial x_{1}+i \partial / \partial x_{2} & -1
\end{array}\right]\left[\begin{array}{c}
w_{1}^{s}(x, a) \\
w_{-1}^{s}(x, a)
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] .
$$

It is easy to prove this when $x_{2}$ is strictly between $n_{l}$ and $n_{l+1}$. Consulting Theorem 4.2 it is enough to show that $\left[\begin{array}{l}w_{1}^{ \pm}(r, \varphi) \\ w_{-1}^{ \pm}(r, \varphi)\end{array}\right]$ is a vector valued solution
of (5.1). For this purpose it is convenient to introduce $x=(1 / 2)\left(x_{1}+i x_{2}\right)$ and replace $e^{-(r / 2)\left(\xi(\varphi)+\xi(\varphi)^{-1}\right)}$ with $e^{i \xi x-i \xi^{-1} \bar{x}}$, and so on. The differential operators $\partial / \partial x_{1} \pm i \partial / \partial x_{2}$ acting on such exponentials bring down factors $i \xi$ and $(i \xi)^{-1}$ which convert $g_{-1}(\xi)$ into $g_{1}(\xi)$ and vice versa. In each case this is all that is needed to show that $\left[\begin{array}{c}w_{1}^{ \pm}(r, \varphi) \\ w_{-}^{ \pm}(r, \varphi)\end{array}\right]$ is a vector valued solution of (5.1). We remind the reader that the complex structure in the differential operators $\partial / \partial x_{1} \pm i \partial / \partial x_{2}$ acts in $W(0)$ as $i \oplus(-i)$.

We next present arguments to show that the different expressions for $\left[\begin{array}{c}w_{1}^{s}(x, a) \\ w_{-1}^{s}(x, a)\end{array}\right]$ piece together to give a solution to the Dirac equation everywhere except on the horizontal rays to the right of the branch points $a_{i}$. On these rays the upper and lower limits differ by a sign. In order to show that the function $\left[\begin{array}{c}w_{1}^{s}(s, a) \\ w_{-1}^{s}(x, a)\end{array}\right]$ satisfies the Dirac equation on the rays to the left of the branch points $a_{l}$ it is enough to show that it is continuous on these rays. Once continuity is established a straightforward integration by parts shows that $\left[\begin{array}{c}w_{1}^{s}(x, a) \\ w_{-1}^{s}(x, a)\end{array}\right]$ is a local distribution solution of (5.1). Elliptic regularity then implies it is actually a $C^{\infty}$ solution.

We turn next to the continuity and monodromy properties for $w_{j}^{s}(x, a)$. Theorem 4.1 shows that to prove these results (for $\left.a_{l}(l \neq s)\right)$ it is enough to show that

$$
\begin{align*}
\lim _{\varphi \rightarrow \pi-} w_{j}^{+}(r, \varphi) & =\lim _{\varphi \rightarrow \pi^{+}} w_{j}^{-}(r, \varphi) \\
\lim _{\varphi \rightarrow 0+} w_{j}^{+}(r, \varphi) & =-\lim _{\varphi \rightarrow 2 \pi-} w_{j}^{-}(r, \varphi) \tag{5.2}
\end{align*}
$$

where the convergence is ( $\varepsilon_{l}, \varepsilon_{l+1}$ ) convergence in $W(0)$. When $a_{l}=a_{s}$ one needs in addition the appropriate boundary values for the inner products $\left\langle T\left(x-a_{s}\right) Q_{ \pm} f_{j}, \bar{u}\right\rangle$. Since these inner products are simple to deal with we consider them first. In the integral

$$
\int_{0}^{\infty} e^{\left(r_{s} / 2\right)\left(\xi\left(\varphi_{s}\right)+\xi\left(\varphi_{s}\right)^{-1}\right)} \overline{g_{j}(\xi)}(2 \pi \xi)^{-1} d \xi
$$

introduce the complex integration variable $z=-\xi\left(\varphi_{s}\right)$ and then deform the contour of integration to the positive real axis. The result is

$$
\begin{align*}
\left\langle T\left(x-a_{s}\right) Q_{-} f_{1}, \bar{u}\right\rangle & =e^{-i \varphi_{s} / 2} \int_{0}^{\infty} e^{-\left(r_{s} / 2\right)\left(z+z^{-1}\right)} z^{1 / 2} \frac{d z}{2 \pi z} \\
\left\langle T\left(x-a_{s}\right) Q_{-} f_{-1}, \bar{u}\right\rangle & =-e^{i \varphi_{s} / 2} \int_{0}^{\infty} e^{-\left(r_{s} / 2\right)\left(z+z^{-1}\right)} z^{-1 / 2} \frac{d z}{2 \pi z} \tag{5.3}
\end{align*}
$$

Introducing $z=\xi\left(\varphi_{s}\right)$ in the integral

$$
\int_{0}^{\infty} e^{-\left(r_{s} / 2\right)\left(\xi\left(\varphi_{s}\right)+\xi\left(\varphi_{s}\right)^{-1}\right)} g_{j}(\xi)(2 \pi \xi)^{-1} d \xi
$$

and deforming the contour to the positive real axis one obtains

$$
\begin{align*}
\left\langle T\left(x-a_{s}\right) Q_{+} f_{1}, \bar{u}\right\rangle & =-e^{i \varphi_{s} / 2} \int_{0}^{\infty} e^{-\left(r_{s} / 2\right)\left(z+z^{-1}\right)} z^{1 / 2} \frac{d z}{2 \pi z}, \\
\left\langle T\left(x-a_{s}\right) Q_{+} f_{-1}, \bar{u}\right\rangle & =e^{i \varphi_{s} / 2} \int_{0}^{\infty} e^{-\left(r_{s} / 2\right)\left(z+z^{-1}\right)} z^{-1 / 2} \frac{d z}{2 \pi z} \tag{5.4}
\end{align*}
$$

Recalling Theorem 4.2, it is evident that these inner products have the appropriate continuity and monodromy. In order to prove the continuity results (5.2), it is convenient to transform the expressions for $w_{j}^{ \pm}(r, \varphi)$. In each of the integrals in (4.10), introduce the complex change of coordinates $z= \pm \xi( \pm \varphi)$. The choice of $z$ is made so that in each case $e^{-(r / 2)\left(z+z^{-1}\right)}$ appears in the integrand. Now rotate the integration contour to the positive real axis. The integrals containing the principal value kernel

$$
\frac{\xi+\xi^{\prime}}{\xi-\xi^{\prime}}=\frac{1}{2}\left[\frac{\xi+\xi^{\prime}}{\xi-\xi^{\prime}+i 0}+\frac{\xi+\xi^{\prime}}{\xi-\xi^{\prime}-i 0}\right]
$$

will pick up contributions from poles under this change of contours. (Except, of course, when no rotation of contour is needed at $\varphi=\pi / 2,3 \pi / 2$ ). One finds

$$
\begin{align*}
\left(w_{1}^{ \pm}(r, \varphi)\right)_{1}= & -e^{-i \varphi / 2} \xi(\varphi)^{1 / 2} e^{-(r / 2)\left(\xi(\varphi)+\xi(\varphi)^{-1}\right)} \\
& -e^{-i \varphi / 2} \int_{0}^{\infty} \frac{z-i e^{i \varphi} \xi}{z+i e^{i \varphi} \xi} z^{1 / 2} e^{-(r / 2)\left(z+z^{-1}\right)} \frac{d z}{2 \pi z}, \\
\left(w_{1}^{ \pm}(r, \varphi)\right)_{2}= & e^{i \varphi / 2} \xi(-\varphi)^{1 / 2} e^{-(r / 2)\left(\xi(-\varphi)+\xi(-\varphi)^{-1}\right)} \\
& +i e^{i \varphi / 2} \int_{0}^{\infty} \frac{z-i e^{-i \varphi} \xi}{z+i e^{-i \varphi} \xi} z^{1 / 2} e^{-(r / 2)\left(z+z^{-1}\right)} \frac{d z}{2 \pi z}, \\
\left(w_{-1}^{ \pm}(r, \varphi)\right)_{1}= & e^{i \varphi / 2} \xi(\varphi)^{-1 / 2} e^{-(r / 2)\left(\xi(\varphi)+\xi(\varphi)^{-1}\right)}  \tag{5.5}\\
& +i e^{i \varphi / 2} \int_{0}^{\infty} \frac{z-i e^{i \varphi} \xi}{z+i e^{i \varphi} \xi} z^{-1 / 2} e^{-(r / 2)\left(z+z^{-1}\right)} \frac{d z}{2 \pi z},
\end{align*}
$$

and

$$
\begin{aligned}
\left(w_{-}^{ \pm}(r, \varphi)\right)_{2}= & -e^{-i \varphi / 2} \xi(-\varphi)^{-1 / 2} e^{-(r / 2)\left(\xi(-\varphi)+\xi(-\varphi)^{-1}\right)} \\
& -i e^{-i \varphi / 2} \int_{0}^{\infty} \frac{z-i e^{-i \varphi} \xi}{z+i e^{-i \varphi \xi}} z^{-1 / 2} e^{-(r / 2)\left(z+z^{-1}\right)} \frac{d z}{2 \pi z},
\end{aligned}
$$

where the square roots $\xi( \pm \varphi)^{ \pm 1 / 2}$ are all computed by taking $\arg \xi( \pm \varphi) \in$ $(0,2 \pi)$. The continuity and monodromy results (5.2) are immediate consequences of (5.5) and this square root convention. The reader should note that $\xi( \pm \varphi)^{1 / 2}$ is discontinuous at $\varphi=\pi / 2,3 \pi / 2$ (not at $\varphi=\pi, 2 \pi$ ). At these points the formulas in (5.5) are not correct; the singular integrals which occur should be interpreted in the principal value sense and the "pole terms" associated with these integrals should be dropped.

We are now prepared to compute the coefficients in the local expansions of $w_{j}^{s}(x, a)$. Define

$$
e_{1}(\varphi, n)=\left(e^{i(n-1 / 2) \varphi}, 0\right), e_{2}(\varphi, n)=\left(0, e^{i(n+1 / 2) \varphi}\right),
$$

and

$$
\begin{aligned}
& v(r, \varphi, n)=I_{n-1 / 2}(r) e_{1}(\varphi, n)+I_{n+1 / 2}(r) e_{2}(\varphi, n) \\
& \bar{v}(r, \varphi, n)=I_{n+1 / 2}(r) e_{1}(\varphi,-n)+I_{n-1 / 2}(r) e_{2}(\varphi,-n)
\end{aligned}
$$

Here $I_{\nu}(r)$ is the modified Bessel function of order $\nu$. A contour integral representation of $I_{\nu}(r)$ which is particularly useful for our purposes is

$$
I_{\nu}(r)=e^{i(\nu+1) \pi} \int_{C} z^{-\nu} e^{-(r / 2)\left(z+z^{-1}\right)} \frac{d z}{2 \pi i z},
$$

where the contour $C$ starts at $+\infty$ on the positive real axis just in the lower half plane, winds clockwise around the origin, and finishes at $+\infty$ on the positive real axis just in the upper half plane.

It is shown in [26] that a two valued solution, $f(x)$, of the Euclidean Dirac equation with a branch point at $x=a$ has an expansion

$$
\begin{equation*}
f(x)=\sum_{n=-\infty}^{\infty}[c(n,+) v(r, \varphi, n)+c(n,-) \bar{v}(r, \varphi, n)] \tag{5.6}
\end{equation*}
$$

where $(x-a)=(r \cos \varphi, r \sin \varphi), \varphi \in[0,2 \pi)$. Our goal is to calculate the coefficients $c_{l}^{s}(n, \pm)\left(w_{1}^{s}(x, a), w_{-1}^{s}(x, a)\right)$ about each branch point $a_{l}$. In particular, we will show that $c_{l}^{s}(n, \pm)=0$ for $n<0$. This will have consequences for the order of growth at the singularities $a_{l}$ and is an important part of the SMJ analysis. The reader should have no trouble seeing that the
calculation of the coefficients $c_{l}^{s}(n, \pm)$ reduces to the calculation of halfinteger Fourier coefficients for the vector valued functions $w_{j}^{ \pm}\left(r_{l}, \varphi_{l}\right)$ and the inner products $\left\langle T\left(x-a_{s}\right) Q_{ \pm} f_{j}, \bar{u}\right\rangle$. The Fourier coefficients for these inner products are obvious from Eqs. (5.3) and (5.4), and the other Fourier coefficients in question are

$$
\begin{align*}
f_{j k}= & \frac{1}{2 \pi} \int_{0}^{\pi}\left(e^{i k \varphi} \oplus e^{-i k \varphi}\right) w_{j}^{+}(r, \varphi) d \varphi \\
& +\frac{1}{2 \pi} \int_{\pi}^{2 \pi}\left(e^{i k \varphi} \oplus e^{-i k \varphi}\right) w_{j}^{-}(r, \varphi) d \varphi \tag{5.7}
\end{align*}
$$

where $k \in \mathbb{Z}_{1 / 2}$. The factor $e^{i k \varphi} \oplus e^{-i k \varphi}$ appears because $i F(x)=F(i \oplus$ $(-i) x$ ). In order to calculate the Fourier coefficients it is useful to work with the representation for $w_{j} \pm(r, \varphi)$ in (5.5). Perhaps the simplest way to extract the Fourier coefficients from the integrals is to use the geometric series expansions:

$$
\begin{array}{ll}
\frac{x-\xi}{x+\xi}=1+2 \sum_{n=1}^{\infty}(-\xi)^{n} x^{-n}, & \xi<x, \\
\frac{x-\xi}{x+\xi}=-1-2 \sum_{n=1}^{\infty}(-\xi)^{-n} x^{n}, & \xi>x .
\end{array}
$$

Once this is done and the other $\varphi$ integrals are converted to contour integrals on the circle $|z|=\xi$, the $z$ integrals which result are naturally contour integrals over two sorts of contours:


The first contour is immediately identifiable as a Bessel function; the second contour may be deformed to the unit circle and subjected to the change of variable $z \rightarrow z^{-1}$, at which point it, too, is evidently a Bessel function. The result of these calculations are

$$
\begin{aligned}
f_{1 k} & =I_{k}(r)\left[\begin{array}{c}
-i(i \xi)^{-n} \\
i(i \xi)^{-n}
\end{array}\right], & k=n+1 / 2, n=1,2, \ldots, \\
& =\left(I_{1 / 2}(r)+I_{-1 / 2}(r)\right)\left[\begin{array}{c}
-i / 2 \\
i / 2
\end{array}\right], & k=1 / 2, \\
& =I_{-k}(r)\left[\begin{array}{c}
-i(i \xi)^{n} \\
i(i \xi)^{n}
\end{array}\right], & k=-(n-1 / 2), n=1,2, \ldots,
\end{aligned}
$$

$$
\begin{array}{rlrl}
f_{2 k} & =I_{k}(r)\left[\begin{array}{c}
-i(i \xi)^{-n} \\
i(i \xi)^{-n}
\end{array}\right], & & k=n-1 / 2, n=1,2, \ldots, \\
& =\left(I_{1 / 2}(r)+I_{-1 / 2}(r)\right)\left[\begin{array}{c}
-i / 2 \\
i / 2
\end{array}\right], & & k=-1 / 2, \\
& =I_{-k}(r)\left[\begin{array}{c}
-i(i \xi)^{n} \\
i(i \xi)^{n}
\end{array}\right], & k=-(n+1 / 2), n=1,2, \ldots
\end{array}
$$

We may use these results to calculate the coefficients $c_{l}^{s}(n, \pm)$. Before we record the results, it is convenient to make a number of observations. In order to calculate $c_{s}^{s}(0, \pm)$, it is useful to note that

$$
\int_{0}^{\infty} z^{ \pm 1 / 2} e^{-(r / 2)\left(z+z^{-1}\right)} \frac{d z}{2 \pi z}=\frac{1}{2}\left(I_{-1 / 2}(r)-I_{1 / 2}(r)\right) .
$$

The operator $X=(I+L \Delta R)^{-1} L$ which appears in (4.8) is $P$ skew-symmetric. That is $P X^{*} P=-X$. This follows directly from the fact that $L$ and $\Delta R$ are both $P$-skew symmetric, a fact which is easily verified. One consequence of the $P$-skew symmetry of $X$ is that $N_{s s}(u, u)=0$. The reader should note that because $N_{s s}(u, u)=0$, the coefficients $c_{s}^{s}(0, \pm)$ are determined completely by contributions from the integrals (5.3) and (5.4).

Finally, if we introduce the notation $\partial / \partial a_{i}=\partial / \partial a_{i}(1)-i \partial / \partial a_{i}(2)$ and $\partial / \partial \bar{a}_{i}=\partial / \partial a_{i}(1)+i \partial / \partial a_{i}(2)$ then the results of Theorem (4.3) may be summarized

$$
\begin{align*}
& \langle\sigma(a)\rangle^{-1} \partial\langle\sigma(a)\rangle / \partial a_{i}=-i N_{i i}((i \xi) u, u), \\
& \langle\sigma(a)\rangle^{-1} \partial\langle\sigma(a)\rangle / \partial \bar{a}_{i}=i N_{i i}\left((i \xi)^{-1} u, u\right), \\
& \langle\sigma(a)\rangle^{-1} \partial\left\langle\sigma^{i j}(a)\right\rangle / \partial a_{i}=N_{i j}((i \xi) u, u), \\
& \langle\sigma(a)\rangle^{-1} \partial\left\langle\sigma^{i j}(a)\right\rangle / \partial \bar{a}_{i}=N_{i j}\left((i \xi)^{-1} u, u\right) . \tag{5.9}
\end{align*}
$$

Here we use the conventions $u=1 \oplus 1$ and

$$
(i \xi)^{n} u=(i \xi)^{n} \oplus(i \xi)^{n} .
$$

We are now prepared to state the main result of this paper:
Theorem 5.1. Suppose $\pi_{2}\left(a_{1}\right)<\pi_{2}\left(a_{2}\right) \cdots<\pi_{2}\left(a_{r}\right)$. Then the scaled Euclidean wavefunctions $w_{j}^{s}(x, a)$ satisfy the Dirac equation (5.1), except on the horizontal rays to the right of the points $a_{l}$ where they have upper and lower boundary values differing by a sign. The expansion coefficients $c_{l}^{s}(n, \pm)$ in the
expansion (5.6) are given by

$$
\begin{array}{ll}
c_{l}^{s}(n, \pm)=0, & n<0 \\
c_{l}^{s}(0, \pm)= \pm \delta_{l s} / 2-\left(\varepsilon_{l}^{s} i / 2\right) N_{l s}(a \mid u, u), & \varepsilon_{l}^{s}=\left\{\begin{array}{cc}
+1 & l \leqslant s \\
-1 & l>s
\end{array}\right. \\
c_{l}^{s}(n, \pm)=-\varepsilon_{l}^{s} i N_{l s}\left(a \mid(i \xi)^{n} u, u\right), & n=1,2, \ldots
\end{array}
$$

The coefficients $c_{l}^{s}(1, \pm)$ may be expressed in terms of derivatives of scaled n-point functions as follows:

$$
\begin{aligned}
& c_{s}^{s}(1,+)=\langle\sigma(a)\rangle^{-1} \partial\langle\sigma(a)\rangle / \partial a_{s} \\
& c_{s}^{s}(1,-)=-\langle\sigma(a)\rangle^{-1} \partial\langle\sigma(a)\rangle / \partial \bar{a}_{s} \\
& c_{l}^{s}(1,+)=-i \varepsilon_{l}^{s}\langle\sigma(a)\rangle^{-1} \partial\left\langle\sigma^{l s}(a)\right\rangle / \partial a_{l}, \quad l \neq s \\
& c_{l}^{s}(1,-)=-i \varepsilon_{l}^{s}\langle\sigma(a)\rangle^{-1} \partial\left\langle\sigma^{l s}(a)\right\rangle / \partial \bar{a}_{l}, \quad l \neq s
\end{aligned}
$$

The functions $w_{j}^{s}(x, a)$ are exponentially small as $|x| \rightarrow \infty$.
Proof. The only part of this theorem which is not a direct consequence of Theorem 4.2 and Eqs. (5.6), (5.8), and (5.9) is the very last statement of the theorem.

In order to prove that the functions $w_{j}^{s}(x, a)$ are exponentially small for $x$ near infinity, it suffices to show that these functions are bounded for $|x|$ sufficiently large (see Proposition 3.1.5 in SMJ [26]). In view of the representation for $w_{j}^{s}(x, a)$ in Theorem 4.2, we seek bounds for $N_{l s}\left(w_{j}^{ \pm}\left(x-a_{l}\right), u\right)$, $N_{s l}\left(u, w_{j}^{ \pm}\left(x-a_{l}\right)\right)$, and $\left\langle T\left(x-a_{s}\right) Q_{ \pm} f_{j}, \bar{u}\right\rangle$ for $x$ in an appropriate region. The expressions (5.3) and (5.4) for the inner products $\left\langle T\left(x-a_{s}\right) Q_{ \pm} f_{j}, \bar{u}\right\rangle$ are exponentially small for large values of $\left|x-a_{s}\right|$ so that we may concentrate on the other terms.

First consider $N_{l s}\left(w_{j}^{-}\left(x-a_{l}\right), u\right)$ for $l=2,3, \ldots, r$. Choose $\varepsilon_{k}(k=$ $1, \ldots, r$ ) as in Theorem 4.1. Since $e^{-\epsilon \omega} u$ is in $\mathcal{H}$ for arbitrarily small $\varepsilon>0$, we may choose $\varepsilon_{l}$ subject only to the inequality $\varepsilon_{l}<\pi_{2}\left(a_{l}-a_{l-1}\right)$. We will establish a bound on $N_{l s}\left(w_{j}^{-}\left(x-a_{l}\right), u\right)$ for $x$ in the strip $\pi_{2}\left(a_{l}\right)-\varepsilon_{l} \leqslant x_{2}$ $\leqslant \pi_{2}\left(a_{l}\right)$ with $\left|x-a_{l}\right|$ sufficiently large. Since $N_{l s}\left(w_{j}^{-}\left(x-a_{l}\right), u\right)$ depends linearly on $w_{j}^{-}\left(x-a_{l}\right)$, the continuity result in Theorem 4.1 shows that we need only bound the norm of $Q\left(\varepsilon_{l}, \varepsilon_{l+1}\right) w_{j}^{-}\left(x-a_{l}\right)$ in $W(0)$. For this purpose the representation in (5.5) for $w_{j}^{-}\left(x-a_{l}\right)$ is useful. Let $x-a_{1}=$ ( $r_{l} \cos \varphi_{l}, r_{l} \sin \varphi_{l}$ ) and recall $2 \omega=\xi+\xi^{-1}$. Making use of the inequality

$$
\frac{z-i e^{i \varphi} \xi}{z+i e^{i \varphi} \xi} \leqslant \sqrt{\frac{1+\sin \varphi}{1-\sin \varphi}}=\tan \varphi+\sec \varphi, \quad \varphi \in[\pi, 2 \pi]
$$

and (5.5), one easily finds that the absolute value of the function
$e^{-\varepsilon, \omega} Q_{+} w_{j}^{-}\left(x-a_{f}\right)$ of $\xi$ is dominated by

$$
\begin{align*}
& \xi^{j / 2} e^{-\left(x_{2}-\pi_{2}\left(a_{l}\right)+\varepsilon_{l}\right)\left(\xi+\xi^{-1}\right) / 2} \\
& \quad+\left(\tan \varphi_{l}+\sec \varphi_{l}\right) e^{-\varepsilon_{l}\left(\xi+\xi^{-1}\right) / 2} \int_{0}^{\infty} z^{j / 2} e^{-r_{l}\left(z+z^{-1}\right) / 2} \frac{d z}{2 \pi z} \tag{5.10}
\end{align*}
$$

The norm of $\xi^{j / 2} e^{-\left(x_{2}-\pi_{2}\left(a_{l}\right)+\varepsilon_{l}\right)\left(\xi+\xi^{-1}\right) / 2}$ in $\mathscr{K}$ is uniformly bounded, provided $x_{2}-\pi_{2}\left(a_{l}\right)+\varepsilon_{l} \geqslant \varepsilon>0$. On the other hand, if we restrict $\varphi_{l}$ to the union of the intervals [ $\pi, 5 \pi / 4]$ and $[7 \pi / 4,2 \pi]$, and take $r_{l} \geqslant 1$, then $\tan \varphi_{l}+\sec \varphi_{l}<\sqrt{2}$ and

$$
\int_{0}^{\infty} z^{j / 2} e^{-\left(r_{1} / 2\right)\left(z+z^{-1}\right)} \frac{d z}{2 \pi z} \leqslant \int_{0}^{\infty} z^{j / 2} e^{-\left(z+z^{-1}\right) / 2} \frac{d z}{2 \pi z}
$$

It follows that the norm in $\mathcal{K}$ of the second term on the right-hand side of (5.10) is uniformly bounded for $x$ in the region determined by $\varphi_{1} \in$ $[\pi, 5 \pi / \pi] \cup[7 \pi / 4,2 \pi]$ with $r_{l} \geqslant 1$. Now consider the function $e^{-\varepsilon_{l+1} \omega} Q_{-} w_{j}^{-}\left(x-a_{l}\right)$. Making use of the inequality

$$
\frac{z-i e^{-i \varphi} \xi}{z+i e^{-i \varphi} \xi} \leqslant \sqrt{\frac{1-\sin \varphi}{1+\sin \varphi}}=\sec \varphi-\tan \varphi, \quad \varphi \in[\pi, 2 \pi]
$$

and consulting (5.5), one sees that the function $e^{-\varepsilon_{l+1}{ }^{\omega}} Q_{-} w_{j}^{-}\left(x-a_{l}\right)$ is dominated by

$$
\begin{align*}
& \xi^{j / 2} e^{-\varepsilon_{l+1}\left(\xi+\xi^{-1}\right) / 2} \\
& \quad+\left(\sec \varphi_{l}-\tan \varphi_{l}\right) e^{-\varepsilon_{l+1}\left(\xi+\xi^{-1}\right) / 2} \int_{0}^{\infty} z^{j / 2} e^{\left.-r_{l\left(z+z^{-1}\right.}\right) / 2} \frac{d z}{2 \pi z} \tag{5.11}
\end{align*}
$$

If we again require that $\varphi_{l} \in[\pi, 5 \pi / 4] \cup[7 \pi / 4,2 \pi]$ and $r_{l} \geqslant 1$, then it is clear that the norm of (5.11) in $\mathscr{H}$ is uniformly bounded. It is evident that precisely the same considerations bound $N_{s l}\left(u, w_{j}^{-}\left(x-a_{i}\right)\right)$. Putting these results together we conclude that the function $w_{j}^{s}(x, a)$ is uniformly bounded in the strip $\pi_{2}\left(a_{l}\right)-\varepsilon_{l} \leqslant x_{2} \leqslant \pi_{2}\left(a_{l}\right)$ with $\left|x-a_{l}\right|$ sufficiently large, $l=$ $2,3, \ldots, r$, and $\varepsilon_{l}<\pi_{2}\left(a_{t}-a_{t-1}\right)$.
If we now consider the representation of the functions $w_{j}^{s}(x, a)$ in terms of $N_{l s}\left(w_{j}^{+}\left(x-a_{t}\right), u\right)$ and $N_{s l}\left(u, w_{j}^{+}\left(x-a_{t}\right)\right)$, then the same estimates on (5.5) show that $w_{j}^{s}(x, a)$ is uniformly bounded in the strips $\pi_{2}\left(a_{1}\right) \leqslant x_{2} \leqslant$ $\pi_{2}\left(a_{l}\right)+\varepsilon_{l+1}$ for $\left|x-a_{l}\right|$ sufficiently large, $l=1,2, \ldots, r-1$, and $\varepsilon_{l+1}<$ $\pi_{2}\left(a_{l+1}-a_{l}\right)$. Putting these results together we conclude that the function $w_{j}^{s}(x, a)$ is uniformly bounded for all $x$ such that $\pi_{2}\left(a_{1}\right) \leqslant x_{2} \leqslant \pi_{2}\left(a_{r}\right)$ with $|x|$ sufficiently large. Thus we concentrate further attention on the cases
$x_{2} \leqslant \pi_{2}\left(a_{1}\right)$ and $x_{2} \geqslant \pi_{2}\left(a_{r}\right)$. For $x_{2} \leqslant \pi_{2}\left(a_{1}\right)$ the function $w_{j}^{s}(x, a)$ is given by $N_{1 s}\left(w_{j}^{-}\left(x-a_{1}\right), u\right)$. In this formula $w_{j}^{-}\left(x-a_{1}\right)$ may be replaced by $Q_{-} w_{j}^{-}\left(x-a_{1}\right)$. This follows from the formula (4.8) for $N_{l s}(u, v)$, but may also be understood in the following manner. Before the scaling limit is taken, the operator $\psi_{j}(x) \sigma\left(a_{1}\right)$ (or $\psi_{j}(x) \mu\left(a_{1}\right)$ ) occurs in the vacuum expectation defining $w_{j}^{s}(x, a)$ for $x_{2} \leqslant \pi_{2}\left(a_{1}\right)$. When $\psi_{j}(x) \sigma\left(a_{1}\right)$ (or $\left.\psi_{j}(x) \mu\left(a_{1}\right)\right)$ is expressed as a normal ordered product as in (3.4) or (3.6) the function $Q_{+} w_{j}^{-}\left(x-a_{1}\right)$ appears in a creation operator to the left of all the spin operators. The adjoint of this operator acting on the vacuum gives 0 . Thus only $Q_{-} w_{j}^{-}\left(x-a_{1}\right)$ makes a nontrivial contribution to the lattice wave function. Of course, this will persist in the scaling limit. We are thus reduced to proving that $e^{-\varepsilon_{1} \omega} Q_{-} w_{j}^{-}\left(x-a_{1}\right)$ is uniformly bounded in $\mathscr{H}$ norm for $x_{2} \leqslant \pi_{2}\left(a_{1}\right)$ and $\left|x-a_{1}\right|$ sufficiently large. For the region $\varphi_{1} \in$ $[\pi, 5 \pi / 4] \cup[7 \pi / 4,2 \pi]$ and $r_{1} \geqslant 1$ the estimates given above suffice. For the region $\varphi_{1} \in(5 \pi / 4,7 \pi / 4)$ and $r_{1} \geqslant 1$, it is useful to employ the representation (4.10) for $Q_{-} w_{j}^{-}\left(x-a_{1}\right)$. As a function of $\xi$, the vector $e^{-e \omega}$. $Q_{-} w_{j}^{-}\left(x-a_{1}\right)$ is given by

$$
\begin{equation*}
e^{-\varepsilon_{1}\left(\xi+\xi^{-1}\right) / 2} \int_{0}^{\infty} \frac{\xi+\xi^{\prime}}{\xi-\xi^{\prime}} F_{j}\left(\xi^{\prime}\right) d \xi^{\prime} \tag{5.12}
\end{equation*}
$$

where $F_{j}(\xi)=(2 \pi i \xi)^{-1} g_{j}(\xi) e^{-r_{1}\left(\xi\left(-\varphi_{1}\right)+\xi\left(-\varphi_{1}\right)^{-1}\right)}$. Since $\left(\xi+\xi^{\prime}\right) /\left(\xi-\xi^{\prime}\right)$ $=\left(2 \xi /\left(\xi-\xi^{\prime}\right)-1\right.$, it follows that the function in (5.12) is

$$
\begin{equation*}
e^{-\varepsilon_{1}\left(\xi+\xi^{-1}\right) / 2}(2 \xi) \int_{0}^{\infty} \frac{F_{j}\left(\xi^{\prime}\right)}{\xi-\xi^{\prime}} d \xi^{\prime}-e^{-\varepsilon_{1}\left(\xi+\xi^{-1}\right) / 2} \int_{0}^{\infty} F_{j}\left(\xi^{\prime}\right) d \xi^{\prime} \tag{5.13}
\end{equation*}
$$

Since $\left|F_{j}(\xi)\right| \leqslant(2 \pi \xi)^{-1} \xi^{j / 2} e^{-r_{1}\left(\xi+\xi^{-1}\right) / 2 \sqrt{2}}$ for $\varphi_{1} \in(5 \pi / 4,7 \pi / 4)$, it follows that the second term in (5.13) is uniformly bounded in $\mathcal{K}$ norm for $\varphi_{1} \in(5 \pi / 4,7 \pi / 4)$ and $r_{1} \geqslant 1$. Now let $G_{j}(\xi)=\int_{0}^{\infty}\left(F\left(\xi^{\prime}\right) /\left(\xi-\xi^{\prime}\right) d \xi^{\prime}\right.$. Then the square of the $\mathcal{H}$ norm of the first term in (5.13) is given by

$$
\begin{equation*}
4 \int_{0}^{\infty} \xi e^{-\varepsilon_{1}\left(\xi+\xi^{-1}\right)} G_{j}^{2}(\xi) \frac{d \xi}{2 \pi} \tag{5.14}
\end{equation*}
$$

An elementary application of calculus shows that

$$
\xi e^{-\varepsilon_{1}\left(\xi+\xi^{-1}\right)} \leqslant\left(2 \varepsilon_{1}\right)^{-1}\left(1+\sqrt{1+4 \varepsilon_{1}^{2}}\right) e^{-\sqrt{1+4 \varepsilon_{1}^{2}}}
$$

Using this, and the fact that the truncated Hilbert transform is bounded on $L^{2}(d \xi)$, it follows that the norm in $\mathfrak{K}$ of the first term in (5.13) is bounded by a constant times the $L^{2}(d \xi)$ norm of $F_{j}(\xi)$. The estimate of $F_{j}(\xi)$ above shows that this norm is uniformly bounded in the region $\varphi_{1} \in[5 \pi / 4,7 \pi / 4]$
with $r_{1} \geqslant 1$. When $x_{2} \geqslant \pi_{2}\left(a_{r}\right)$, one may deal with $Q_{+} w_{j}^{+}\left(x-a_{r}\right)$ in an analogous fashion. This completes the proof of Theorem 5.1.

## 6

In this section, we review the SMJ analysis of monodromy preserving deformations of the Euclidean Dirac equation and its application to the Ising correlations. Let $\gamma_{2}=\left[\begin{array}{cc}0 & -i \\ i & 0\end{array}\right]$ and $\gamma_{1}=\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$. The Euclidean Dirac equation is

$$
\begin{equation*}
\left(\gamma_{2} \frac{\partial}{\partial x_{2}}+\gamma_{1} \frac{\partial}{\partial x_{1}}-m\right) w=0 \tag{6.0}
\end{equation*}
$$

where $w=\left(w_{1}, w_{-1}\right) \in \mathbb{C}^{2}$. We are interested in spaces of multivalued solutions to this equation with isolated singularities $a_{j} \in \mathbb{R}^{2}(j=1, \ldots, r)$ and specified monodromy at each point $a_{j}$. It is convenient to introduce complex coordinates $z=1 / 2\left(x_{1}+i x_{2}\right), \bar{z}=1 / 2\left(x_{1}-i x_{2}\right)$ and differential operators $\partial=\partial / \partial x_{1}-i \partial / \partial x_{2}$ and $\bar{\partial}=\partial / \partial x_{1}+i \partial / \partial x_{2}$. We also write
$A=(1 / 2)\left[\begin{array}{ccc}a_{1}(1)+i a_{1}(2) & & 0 \\ & \ddots & \\ 0 & a_{r}(1)+i a_{r}(2)\end{array}\right] \quad$ and $\quad L=\left[\begin{array}{lll}l_{1} & & 0 \\ & \ddots & \\ 0 & & l_{r}\end{array}\right]$,
where $l_{j}(j=1, \ldots, r)$ are real numbers, and we will use the notation

$$
\begin{array}{rlrl}
\partial_{j} & =\partial / \partial a_{j}(1)-i \partial / \partial a_{j}(2), \quad \bar{\partial}_{j}=\partial / \partial a_{j}(1)+i \partial / a_{j}(2), \\
d A_{j} & =(1 / 2)\left(d a_{j}(1)+i d a_{j}(2)\right), \quad \text { and } \\
d \overline{A_{j}} & =(1 / 2)\left(d a_{j}(1)-i d a_{j}(2)\right) . &
\end{array}
$$

Let $R_{j}(\theta)$ denote the rotation by $\theta$ about $a_{j}$; that is

$$
R_{j}(\theta) x=a_{j}+R(\theta)\left(x-a_{j}\right), \quad \text { where } \quad R(\theta)=\left[\begin{array}{rr}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right] .
$$

Definition. $W(L, A)$ will denote the complex linear space of multivalued solutions $w$ to (6.0) with isolated singularities at $a_{j}(j=1, \ldots, r)$ satisfying:

1) $w\left(R_{j}(2 \pi-) x\right)=e^{2 \pi\left(l_{j}-1 / 2\right) i} w(x)$ for $x$ near $a_{j}$,
2) $\int_{\mathbf{R}^{2}}\left(w_{1} \bar{w}_{1}+w_{-1} \bar{w}_{-1}\right) d x<\infty$.

Because of the monodromy condition, the functions $w_{k} \bar{w}_{k}(k= \pm 1)$ are single valued on $\mathbb{R}^{2}$ and the integral in 2) is well defined. The monodromy
condition fixes $l_{j} \bmod \mathbf{Z}$ and our convention will be to choose each $l_{j}$ so that $-1 / 2<l_{j} \leqslant 1 / 2$.
Before we present any more details, we will outline the main features of the analysis in [26].
First the space $W(L, A)$ is shown to be finite dimensional with $\operatorname{dim} W(L, A)=r$, where $-1 / 2<l_{j}<1 / 2(j=1, \ldots, r)$. A "canonical" basis $\left\{W^{j}\right\}_{j=1}^{\gamma}$ is constructed which spans $W(L, A)$. Because $W(L, A)$ is finite dimensional and because there is a natural action of infinitesimal rotations on the $W^{j}$, it turns out that the Euclidean Dirac equation for each $W^{j}$ may be extended to a holonomic system:

$$
\begin{equation*}
d_{z, \bar{z}} W^{j}=\sum_{k=1}^{r} \Omega_{j k}^{(1)} W^{k}, \tag{6.1}
\end{equation*}
$$

where the $\Omega_{j k}^{(1)}$ are one forms in $d z$ and $d \bar{z}$ with coefficients that are unknown functions of $A, \bar{A}$ with explicit dependence on $z, \bar{z}$. Next, one supposes that the "canonical" basis $W^{j}$ depends differentiably on the points $a_{j}$ away from places of coincidence $a_{j}=a_{k}(j \neq k)$. Again using the finite dimensionality of $W(L, A)$ one sees that $d_{A,},{ }_{A} W^{j}$ may be expressed as a linear combination of $\partial W^{k}, \bar{\partial} W^{k}$ and $W^{k}(k=1, \ldots, r)$ with coefficients that depend on $A, \bar{A}$. However, because of (6.1) the $z$ and $\bar{z}$ derivatives of $W^{k}$ are linear combinations of $W^{j}(j=1, \ldots, r)$. Thus one has

$$
\begin{equation*}
d_{A, \bar{A}} W^{j}=\sum_{k=1}^{r} \Omega_{j k}^{(2)} W^{k}, \tag{6.2}
\end{equation*}
$$

where $\Omega_{j k}^{(2)}$ are one forms in $d A_{j}, d \bar{A}_{j}$ with coefficients that are unknown functions of $A, \bar{A}$ that depend on $z, \bar{z}$ in an explicit fashion. Combining (6.1) and (6.2) one has

$$
\begin{equation*}
d_{2, \bar{z}, A, \bar{A} W^{j}}-\sum_{k=1}^{r} \Omega_{j k} W^{k}, \tag{6.3}
\end{equation*}
$$

where $\Omega=\Omega^{(1)}+\Omega^{(2)}$. The consistency condition for this holonomic system is

$$
\begin{equation*}
d_{z, \bar{z}, A, \bar{A}} \Omega=\Omega \wedge \Omega \tag{6.4}
\end{equation*}
$$

Thus one may expect that the coefficients in the matrix valued one form $\Omega$ satisfy nonlinear differential equations. The assumption concerning the differentiability of $W^{j}(j=1, \ldots, r)$ in the variables $A, \bar{A}$ is disposed of in the following manner. First choose $A_{0}$ without coincident entries. Then construct the canonical basis $W^{j}\left(z, \bar{z}, A_{0}, \bar{A}_{0}\right)$ for $W\left(L, A_{0}\right)$. Using (6.1) this determines $\Omega^{(1)}$ at $A_{0}, \overline{A_{0}}$. The entries in $\Omega^{(1)}$ determine the entries in
$\Omega^{(2)}$. Thus one has $\Omega$ at $A_{0}, \bar{A}_{0}$ (and all $z \neq a$ diagonal entry of $A_{0}$ ). Use this $\Omega$ as initial data for (6.4). Local existence guarantees a smooth solution $\Omega(z, \bar{z}, A, \bar{A})$ for $A, \bar{A}$ near $A_{0}, \bar{A}_{0}$ and $z \neq a$ diagonal entry of $A$. Now substitute $\Omega(z, \bar{z}, A, \bar{A})$ in (6.3) and solve for $W^{j}$ with initial data $W^{j}\left(z, \bar{z}, A_{0}, \overline{A_{0}}\right)$. Local existence shows that $W^{j}(z, \bar{z}, A, \bar{A})$ will depend smoothly on the parameters $A, \bar{A}$ since $\Omega$ does. One may also prove that $\left\{W^{j}\right\}$ will remain a canonical basis if it started out as one at $A_{0}, \overline{A_{0}}$. Thus the canonical basis does depend smoothly on $A, \bar{A}$. (See the discussion on pp. 621 and 622 of [26] for the details of this argument.)

Before we make the connection with the Ising correlations, we must explain a complication which we have sloughed over so far. Rather than work with (6.4) SMJ introduce $r \times r$ matrices $F(A, \bar{A})$ and $G(A, \bar{A})$ whose entries satisfy a total system of differential equations in $A, \bar{A}$. The matrices $F$ and $G$ determine the coefficients of $\Omega$ (although not vice versa), and the differential equations for $F$ and $G$ are shown to imply the integrability conditions for (6.3). Finally SMJ introduce a one form $\omega$ in $d A_{j}$ and $d \overline{A_{j}}$ with coefficients that depend on $F$ and $G$, and which is closed when $F$ and $G$ satisfy the deformation equations.

To make the connection with the Ising model, first put $l_{1}=l_{2}=\cdots=l_{r}$ $=0$. The work of the first five sections of this paper may be used to calculate $F$ and $G$ in terms of scaled order-disorder correlations for the Ising model. These calculations also show that the closed one-form $\omega$ is related to the logarithmic derivative of the scaled $n$ point function $\langle\sigma\rangle$. Thus

$$
\begin{equation*}
d \ln \langle\sigma\rangle=\frac{1}{2} \omega+\frac{1}{4} d \operatorname{Tr}\left(\ln (1+G)\left(1+G^{-1}\right)\right) \tag{6.5}
\end{equation*}
$$

Now we describe how this equation might be used to calculate $\langle\sigma\rangle$. First integrate the deformation equations for $F, G$. To get a definite result, one needs the value of $F$ and $G$ at a single point $A, \bar{A}$. In principle, since $F$ and $G$ are determined by correlations for which we have formulas, this may be done. In practice, these formulas are hard to compute with. Alternatively, the formulas for the correlations do yield the precise asymptotic behavior for the particular solution of interest at large separation in the $a_{j}$. For $r=2$ this asymptotic behavior does single out a particular solution for $F$ and $G$ [19]. Finally once $F$ and $G$ are known, (6.5) may be integrated to give the scaled correlation $\langle\sigma\rangle$. Thus far only the two point function (and a mass zero limit of the scaled 3 point [13]) has yielded to such analysis. In [26] SMJ show that for $r=2$ the deformation equations for $F$ and $G$ reduce to a Painlevé equation. In [19] solutions of this Painlevé equation are analysed and a connection formula is developed which relates the coefficients in the asymptotic expansion at small separation with the precise asymptotic behavior at large separation. This analysis is quite delicate and has thus far only been carried out modulo some technical assumptions concerning asymptotic
expansions. With this reservation the subsequent calculation of $\langle\sigma\rangle$ using (6.5) yields the only demonstration so far that the scaled two point function has a singularity at coincidence asymptotic to $r^{-1 / 4}$, where $r=\left\|a_{2}-a_{1}\right\|$.

We return now to a more detailed account. The principal tool in the SMJ analysis is the use of local "Fourier" expansions for functions in $W(L, A)$. Suppose $-1 / 2<l_{0} \leqslant 1 / 2$ is given and $w$ is a multivalued solution of (6.0) in a neighborhood of 0 which satisfies $w(R(2 \pi-) x)=e^{2 \pi\left(l_{0}-1 / 2\right) i} w(x)$. The operator $M=-i\left(x_{1} \partial / \partial x_{2}-x_{2} \partial / \partial x_{1}\right)+\left[\begin{array}{cc}1 / 2 & 0 \\ 0 & -1 / 2\end{array}\right]$ is an infinitesimal generator of rotations which commutes with $\gamma_{2} \partial / \partial x_{2}+\gamma_{1} \partial / \partial x_{1}-m$. It is natural to expand the function $w$ in eigenfunctions for $M$ with the appropriate monodromy. The general solution of $M e(\theta)=k e(\theta)$ is $f_{1}(r) e_{1}(\theta, k)+f_{2}(r) e_{2}(\theta, k)$, where $f_{j}(r)$ are functions of $r=\|x\|, e_{1}(\theta, k)$ $=\left(e^{i(k-1 / 2) \theta}, 0\right)$, and $e_{2}(\theta, k)=\left(0, e^{i(k+1 / 2) \theta}\right)$. These eigenfunctions have the appropriate monodromy if and only if $k \equiv l_{0} \bmod \mathbb{Z}$. Thus we seek an expansion for $w$ :

$$
\begin{equation*}
w(x)=\sum_{k \equiv l_{0}[\mathbf{Z}]} f_{1}(r, k) e_{1}(\theta, k)+f_{2}(r, k) e_{2}(\theta, k) \tag{6.6}
\end{equation*}
$$

To prove the existence and uniqueness of this expansion, we merely note that $e^{-i l_{0} \theta} w(x)$ is single valued near 0 as a consequence of the monodromy property of $w$, and is smooth in a punctured neighborhood of 0 since $w$ satisfies the elliptic equation (6.0). Thus each component of $e^{-l_{0} \theta} w(r, \theta)$ has a convergent Fourier expansion as a function of $\theta$. It is easy to write this expansion in the form (6.6). Since $w(r, \theta)$ is a smooth (multivalued) function of $(r, \theta)$, its Fourier expansion may be differentiated term by term in $(r, \theta)$. Applying the Dirac operator to (6.6), one finds that $w$ is a solution of (6.0) if and only if

$$
\begin{aligned}
& \partial f_{1}(r, k) / \partial r-r^{-1}(k-1 / 2) f_{1}(r, k)-m f_{2}(r, k)=0 \\
& \partial f_{2}(r, k) / \partial r+r^{-1}(k+1 / 2) f_{2}(r, k)-m f_{1}(r, k)=0
\end{aligned}
$$

When $k \not \equiv 1 / 2[\mathbb{Z}]$ two independent solutions of this system are given by $\left(f_{1}, f_{2}\right)=\left(I_{k-1 / 2}(m r), I_{k+1 / 2}(m r)\right)$, and $\left(f_{1}, f_{2}\right)=$ $\left(I_{-k+1 / 2}(m r), I_{-k-1 / 2}(m r)\right)$. If $k \equiv 1 / 2[\mathbb{Z}]$, the second solution is the same as the first, and we replace it with the independent solution $\left(f_{1}, f_{2}\right)=$ ( $-K_{k-1 / 2}(m r), K_{k+1 / 2}(m r)$ ).

One may verify that these are solutions by using the recursion relations

$$
\begin{align*}
r \frac{d}{d r} I_{k}(r) \pm k I_{k}(r) & =r I_{k \mp 1}(r) \\
r \frac{d}{d r} K(r) \pm k K_{k}(r) & =-r K_{k \mp 1}(r) \tag{6.7}
\end{align*}
$$

Now define

$$
\begin{aligned}
v(x, k) & =I_{k-1 / 2}(m r) e_{1}(\theta, k)+I_{k+1 / 2}(m r) e_{2}(\theta, k) \\
\bar{v}(x,-k) & =I_{-k+1 / 2}(m r) e_{1}(\theta, k)+I_{-k-1 / 2}(m r) e_{2}(\theta, k), \\
\tilde{v}(x, k) & =-K_{k-1 / 2}(m r) e_{1}(\theta, k)+K_{k+1 / 2}(m r) e_{2}(\theta, k)
\end{aligned}
$$

We use the notation $\bar{v}=\left(\bar{v}_{2}, \bar{v}_{1}\right)$, where $\bar{v}_{j}$ is the usual complex conjugate of $v_{j}$. The map $v \rightarrow \bar{v}$ is a conjugation which leaves the space of solutions to (6.0) invariant.

The local expansion (6.6) of $w$ takes the form

$$
\begin{align*}
& w(x)=\sum_{k \equiv l_{0}[\mathbf{Z}]} c(k,+) v(x, k)+c(-k,-) \bar{v}(x,-k), \quad l_{0} \neq 1 / 2, \\
& w(x)=\sum_{k \equiv 1 / 2[\mathbf{z}]} c(k,+) v(x, k)+c(k,-) \tilde{v}(x, k), \quad l_{0}=1 / 2 \tag{6.8}
\end{align*}
$$

We will now show that $\operatorname{dim} W(L, A) \leqslant n$ by exploiting these local expansions. We write $v_{j}(x, k)=v\left(x-a_{j}, k\right), x-a_{j}=\left(r_{j} \cos \theta_{j}, r_{j} \sin \theta_{j}\right)$, and $D_{j}(\varepsilon)$ for the disk of radius $\varepsilon$ about $a_{j}$. For $\varepsilon$ small enough, $w \in W(L, A)$, and $x \in D_{j}(\varepsilon)$, we have

$$
\begin{aligned}
& w(x)=\sum_{k \equiv l_{j}[\mathbf{Z}]} c_{j}(k,+) v_{j}(x, k)+c_{j}(-k,-) \bar{v}_{j}(x,-k), \quad l_{j} \neq 1 / 2, \\
& w(x)=\sum_{k \equiv 1 / 2[\mathbf{Z}]} c_{j}(k,+) v_{j}(x, k)+c_{j}(k,-) \tilde{v}_{j}(x, k), \quad l_{j}=1 / 2
\end{aligned}
$$

Using these expansion to compute $\int_{-\pi}^{\pi}\left(w_{1} \bar{w}_{1}+w_{-1} \bar{w}_{-1}\right) d \theta_{j}$ in a neighborhood of each of the points $a_{j}$, one finds that $\left(\left|l_{j}\right|<1 / 2\right)$ the resulting function of $r_{j}$ is in $L^{1}\left(D_{j}(\varepsilon), r_{j} d r_{j}\right)$ if and only if $c_{j}(k, \pm)=0$ for $k \leqslant-1 / 2$. In case $l_{j}=1 / 2$, then $c_{j}(k,+)=0$ for $k \leqslant-1 / 2$, and $c_{j}(k,-)=0$ for all $k$. Observe that these last restrictions imply $w(x)$ is a single valued $C^{\infty}$ solutions of (6.0) in a neighborhood of any $a_{j}$ such that $l_{j}=1 / 2$. Evidently such points do not contribute to the structure of $W(L, A)$, and we shall assume from now on that $-1 / 2<l_{j}<1 / 2$ (see SMJ [26] for the analysis of a more general space of solutions than the space $W(L, A)$ we consider here).

Although it is not necessary for the result on finite dimensionality it will prove useful to note here the restriction on the expansion of an element $w \in W(L, A)$ in a neighborhood of $\infty$. Choose $l_{\infty} \in(-1 / 2,1 / 2]$ so that $l_{\infty}-1 / 2 \equiv\left(l_{1}+l_{2}+\cdots+l_{r}-r / 2\right)[\mathbf{Z}]$. Then for large enough $R, w$ has
the expansion

$$
w(x)=\sum_{k \equiv l_{\infty}[\mathbf{Z}]} c_{\infty}(k,+) v(x, k)+c_{\infty}(k,-) \tilde{v}(x, k), \quad\|x\|>R
$$

Reasoning as above, it is clear that for $w$ to be in $L^{2}$ near $\infty$ it is necessary that $c_{\infty}(k,+)=0$ for all $k$.

We are now ready to prove:
Theorem 6.0. Suppose $w \in W(L, A)\left(l_{j}<1 / 2\right)$. Then

$$
\begin{aligned}
w(x)= & \sum_{k-l_{j}=0,1,2 \ldots} c_{j}(k,+) v_{j}(x, k) \\
& +\sum_{k+l_{j}=0,1,2 . .} c_{j}(k,-) \bar{v}_{j}(x, k), \quad \text { x near } a_{j}
\end{aligned}
$$

and

$$
w(x)=\sum_{k \equiv l_{\infty}[\mathbf{Z}]} c_{\infty}(k) \tilde{v}(x, k), \quad x \text { near } \infty
$$

If $w^{\prime}$ is another element of $W(L, A)$ with coefficients $c_{j}^{\prime}(k, \pm)$ in its local expansions then

$$
\begin{aligned}
I\left(w, w^{\prime}\right) \stackrel{\operatorname{def}}{=} \int_{\mathbf{R}^{2}} \frac{m^{2}}{2}\left(w_{1} \bar{w}_{1}^{\prime}+w_{-1} \bar{w}_{-1}^{\prime}\right) & d x
\end{aligned}=-\sum_{j=1}^{r} c_{j}\left(l_{j},+\right) \overline{c_{j}^{\prime}\left(-l_{j},-\right)} \cos \pi l_{j} .
$$

Proof. Only the evaluation of the inner product remains to be demonstrated. Consider the one form $i\left(w, w^{\prime}\right)=w_{1} \bar{w}_{-1}^{\prime} d x_{2}-i w_{1} \bar{w}_{-1}^{\prime} d x_{1}$. When $w$ and $w^{\prime}$ are in $W(L, A)$ this one-form is single valued and one easily computes the exterior derivative $\operatorname{di}\left(w, w^{\prime}\right)=m\left(w_{1} \bar{w}_{1}^{\prime}+w_{-1} \bar{w}_{-1}^{\prime}\right) d x_{1} \wedge d x_{2}$. Thus by Stokes' theorem

$$
\begin{aligned}
I\left(w, w^{\prime}\right) & =-\lim _{\varepsilon \rightarrow 0} \sum_{j=1}^{r} \frac{m}{2} \int_{\partial D_{j}(\varepsilon)} i\left(w, w^{\prime}\right) \\
& =-\lim _{\varepsilon \rightarrow 0} \sum_{j=1}^{r} \frac{m}{2} \int_{-\pi}^{\pi} e^{i \theta_{j} w_{1} \bar{w}_{-1}^{\prime} \varepsilon d \theta_{j}}
\end{aligned}
$$

Using the local expansion in the statement of this theorem one finds

$$
\lim _{\varepsilon \rightarrow 0} \int_{\partial D_{j}(\varepsilon)} i\left(w, w^{\prime}\right)=2 c_{j}\left(l_{j},+\overline{c_{j}^{\prime}\left(-l_{j},-\right)} \cos \pi l_{j}\right.
$$

where we used $\Gamma(1 / 2+l) \Gamma(1 / 2-l)=\pi \sec \pi l$. To prove the second identity for $I\left(w, w^{\prime}\right)$, observe that

$$
d\left(w_{-1} \bar{w}_{1}^{\prime} d x_{2}+i w_{-1} \bar{w}_{1}^{\prime} d x_{1}\right)=m\left(w_{1} \bar{w}_{1}^{\prime}+w_{-1} \bar{w}_{-1}^{\prime}\right) d x_{1} \wedge d x_{2},
$$

from which it follows that

$$
I\left(w, w^{\prime}\right)=-\lim _{\varepsilon \rightarrow 0} \sum_{j=1}^{r} \frac{m}{2} \int_{-\pi}^{\pi} e^{-i \theta_{j} w_{-1} \bar{w}_{1}^{\prime} \varepsilon d \theta_{j} .}
$$

If one now inserts the local expansions for $w$ and $w^{\prime}$, then the limit gives the final identity in this theorem.
Theorem 6.0 shows that $I(w, w)=0$ if $c_{j}\left(l_{j},+\right)$ or $c_{j}\left(-l_{j},-\right)$ vanishes at $j=1, \ldots, r$. Thus $w$ is determined by the choice of $c_{j}\left(l_{j},+\right)$ or $c_{j}\left(-l_{j},-\right)$ for each $j$, and it follows that $\operatorname{dim} W(L, A) \leqslant r$. We will show that $\operatorname{dim} W(L, A)=r$ by constructing a basis.

Theorem 6.1. There exists a basis $W^{j}(j=1, \ldots, r)$ of $W(L, A)$ such that $c_{k}^{j}\left(l_{k},+\right)=\delta_{j k}$, where $c_{k}^{j}\left(l_{k},+\right)$ is the local expansion coefficient of $W^{j}$ at the $k$ th site $a_{k}$.

Proof. Multiplying the Euclidean Dirac equation (6.0) by $\gamma_{2} \partial / \partial x_{2}+$ $\gamma_{1} \partial / \partial x_{1}+m$ one finds $\left(-\Delta+m^{2}\right) w=0$, where $\Delta=\partial^{2} / \partial x_{2}^{2}+\partial^{2} / \partial x_{1}^{2}$. If we let $v=w_{1}$, then $m w_{-1}=\left(\partial / \partial x_{1}+i \partial / \partial x_{2}\right) v$ and $\left(-\Delta+m^{2}\right) v=0$. Evidently we may use this to identify $W(L, A)$ with the space of solutions $v(x)$ of $\left(-\Delta+m^{2}\right) v=0, x \neq a_{j}$, such that

$$
v\left(R_{j}(2 \pi-) x\right)=e^{2 \pi\left(l_{j}-1 / 2\right) i} v(x)
$$

and

$$
I(v, v)=\frac{1}{2} \int_{\mathbf{R}^{2}}\left\{m^{2}|v(x)|^{2}+|\partial v(x)|^{2}\right\} d x<\infty(\partial=\partial / \partial z) .
$$

Next we construct $v_{j}(x) \in W(L, A)$, which looks like $e^{i\left(l_{j}-1 / 2\right) \theta}$ $I_{l,-1 / 2}(m r)$ near $a_{j}$, but which is too well behaved near $a_{k}(k \neq j)$ to generate a nonzero coefficient $c_{k}\left(l_{k},+\right)$ in the local expansion of $W^{j} \stackrel{\text { def }}{=}\left(v_{j}, m^{-1} \partial v_{j}\right)$.

Suppose $\varphi(x)$ is a $C^{\infty}$ function on $\mathbb{R}^{2}$ which is identically 1 in a disk about $a_{j}$ and identically zero outside a larger disk which does not contain $a_{k}(k \neq j)$. Define $u_{j}(x)=\varphi_{j}(x) e^{i\left(l_{j}-1 / 2\right) \theta} I_{l_{j}-1 / 2}\left(m r_{j}\right)$ and let

$$
\begin{array}{rlrl}
g_{j}(x) & = & \left.-\Delta+m^{2}\right) u_{j}(x), & \\
& x \neq a_{j} \\
& 0, & & x=a_{j}
\end{array}
$$

It is obvious that $g_{j}(x)$ is a smooth multivalued function, of compact support (in a "single sheet" restriction to $\mathbb{R}^{2}$ ) and

$$
g_{j}\left(R_{k}(2 \pi-) x\right)=e^{2 \pi\left(l_{k}-1 / 2\right) i_{j}} g_{j}(x) \quad \text { for } k=1, \ldots, r
$$

Now define a Hilbert space, $H^{1}(L, A)$, by completion of the space of smooth multivalued functions which vanish identically in some neighborhood of each of the points $a_{1}, \ldots, a_{r}, \infty$, and have fixed monodromy $f\left(R_{k}(2 \pi-) x\right)=e^{2 \pi\left(l_{k}-1 / 2\right) i} f(x)$, in the norm determined by the inner product

$$
\langle f, g\rangle_{1}=\int_{\mathbf{R}^{2}}\left\{m^{2} f \bar{g}+\partial f / \partial x_{2} \partial \bar{g} / \partial x_{2}+\partial f / \partial x_{1} \partial \bar{g} / \partial x_{1}\right\} d x_{1} d x_{2} .
$$

Since $g_{j}(x)$ is smooth, has the appropriate monodromy, and is supported away from the points $a_{1}, \ldots, a_{r}, \infty$, it follows that $H^{1}(L, A) \ni v \rightarrow$ $\int_{\mathbf{R}^{2}} v(x) g_{j}(x) d x$ is a continuous linear functional on $H^{1}(L, A)$. By the Riesz representation theorem, there exists $h_{j} \in H^{1}(L, A)$ such that $f v(x) \overline{g_{j}(x)} d x=\left\langle v, h_{j}\right\rangle_{1}$. The function $h_{j}$ is thus a weak solution of $\left(-\Delta+m^{2}\right) h_{j}=g_{j}$ for smooth (multi-valued) test functions supported away from $a_{1}, \ldots, a_{r}, \infty$. A standard regularity result for elliptic equations implies that $h_{j}$ is $C^{\infty}$ away from $\left\{a_{1}, \ldots, a_{r}\right\}$ and satisfies $\left(-\Delta+m^{2}\right) h_{j}(x)=$ $g_{j}(x)$ in the usual sense for $x \neq a_{k}$.

If we now define $v_{j}(x)=u_{j}(x)-h_{j}(x)$, then it is easy to verify that $v_{j} \in W(L, A)$. Furthermore since $\varphi_{k}(x) c^{i\left(l_{k}-1 / 2\right) \theta_{k}} I_{I_{k}-1 / 2}\left(m r_{k}\right)$ is not in $H^{1}(L, A)$ for any $k$ it follows that $h_{k}$ cannot make a nonzero contribution to $C_{k}^{l}\left(l_{k},+\right)$ in the local expansion of $W^{j}=\left(v_{j}, m^{-1} i \partial v_{j}\right)$ for any $k$. Hence $C_{k}^{j}\left(l_{k},+\right)=\delta_{j k}$.

We shall write $p\left(\partial / \partial x_{1}, \partial / \partial x_{2}\right)$ for a polynomial in $\partial / \partial x_{1}$ and $\partial / \partial x_{2}$ with constant coefficients. Now define a space of multivalued functions $W^{n}(L, A)$ by

$$
W^{n}(L, A)=\left\{p\left(\partial / \partial x_{1}, \partial / \partial x_{2}\right) w \mid w \in W(L, A), \operatorname{deg} p \leqslant n\right\} .
$$

The following theorem gives a useful characterization of $W^{n}(L, A)$ in terms of their local expansions.

Theorem 6.2. The multivalued function $w$ is in $W^{n}(L, A)$ if and only if it satisfies (6.0) at points $x \neq a_{k}$ and
(1) $w\left(R_{j}(2 \pi-) x\right)=e^{2 \pi\left(l_{j}-1 / 2\right) i} w(x), x$ near $a_{j}$.
(2) In the local expansion of $w$ at $a_{j}$, the coefficients $c_{j}(k, \pm)$ vanish for $k \leqslant-n-1 / 2$.
(3) There exists $R>0$ such that for $|x|>R$

$$
w(x)=\sum_{k \equiv l_{\infty}[\mathbf{Z}]} c_{\infty}(k) \tilde{v}(x, k) .
$$

Proof. Suppose first that $w \in W^{n}(L, A)$. The fact that $w$ satisfy the monodromy condition and the Euclidean Dirac equation is obvious. In order to verify (2), observe that a polynomial in $\partial / \partial x_{1}$ and $\partial / \partial x_{2}$ may also be expressed as a polynomial of the same degree in $\partial$ and $\bar{\partial}$. Thus for some $u \in W(L, A)$ and polynomial $p$ of degree less than or equal to $n$, we have $w=p(\partial, \bar{\partial}) u$. This is useful for the local analysis of $w$ because of the following consequence of the recursion formulas (6.7)

$$
\begin{array}{rlrl}
\partial v_{j}(x, k) & =m v_{j}(x, k-1), & \bar{\partial} v_{j}(x, k)=m v_{j}(x, k+1) \\
\bar{\partial}_{j}(x, k) & =m \bar{v}_{j}(x, k-1), & & \partial \bar{v}_{j}(x, k)=m \bar{v}_{j}(x, k+1) . \tag{6.9}
\end{array}
$$

By theorem (6.0) the coefficients in the local expansion for $u$ vanish when $k \leqslant-1 / 2$. Combining this with (6.9), it is clear that the local expansion coefficients for $w$ vanish when $k \leqslant-n-1 / 2$.

To check (3) for $w$, simply differentiate the expansion for $u$ near $\infty$ using the following further consequence of (6.7):

$$
\begin{equation*}
\partial \tilde{v}(x, k)=\tilde{v}(x, k-1), \quad \bar{\partial} \tilde{v}(x, k)=m \tilde{v}(x, k+1) \tag{6.10}
\end{equation*}
$$

To finish the proof, suppose that $w$ is a solution of (6.0) satisfying (1), (2), and (3). Let $c_{j}\left(l_{j}-n,+\right) v_{j}\left(x, l_{j}-n\right)+c_{j}\left(-l_{j}-n,-\right) \bar{v}_{j}\left(x,-l_{j}-n\right)$ denote the first terms in the local expansion of $w$ at $a_{j}$. Define

$$
u_{n}(x)=m^{-n} \sum_{j=1}^{n}\left[c_{j}\left(l_{j}-n,+\right) \partial^{n} W^{j}+c_{j}\left(-l_{j}-n,-\right) \bar{\partial}^{n} V^{j}\right]
$$

where $V^{j}=\left(\cos \pi l_{j}\right)^{-1} W^{* j}$, and $W^{* j}$ is the basis of $W(L, A)$ dual to $W^{j}$ via the inner product $I(\cdot, \cdot)$. The property of $V^{k}$ which we use is that the local expansion coefficient $c_{j}^{k}\left(-l_{j},-\right)$ of $V^{k}$ at $a_{j}$ is equal to $\delta_{j k}$. One sees that $u_{n} \in W^{n}(L, A)$ by construction and using (6.9) one may check that the coefficients in the local expansion of $w-u_{n}$ vanish for $k \leqslant(-n-1 / 2)+1$. Repeating this process a finite number of times, it is possible to find $u=u_{n}+u_{n-1}+\cdots u_{1}$ such that $u \in W^{n}(L, A)$, and such that the coefficients in the local expansions of $w-u$ vanish when $k \leqslant-1 / 2$. To finish the proof, we need only show that $w-u \in W(L, A)$. Evidently $w-u$ is locally well behaved enough to be in $W(L, A)$; we must only check that it is in $L^{2}\left(\mathbb{R}^{2}\right)$ near $\infty$. Both $w$ and $u$ have expansions of type (3) near $\infty$, and we are thus reduced to showing that this implies $w$ and $u$ are in $L^{2}$ near $\infty$. This is certainly plausible because of the exponential decay of the functions $\tilde{v}(x, k)$, and the precise estimates can be found in Proposition 3.1.5 in [26].

It is useful at this point to introduce some further notation. Let $\left\{w^{j}\right\}$ denote an arbitrary basis of $W(L, A)$, and write $c_{k}^{j}(l, \pm)$ for the coefficients in the local expansion of $w^{j}$ at $a_{k}$. If we make a linear transformation from the basis $\left\{w^{j}\right\}$ to another basis of $W(L, A)$, then it is obvious that the coefficients $c_{k}^{j}(l, \pm)$ all transform in the same fashion. Let $c_{m}( \pm)$ denote the matrix with entries $\left(c_{m}( \pm)\right)_{j k}=c_{k}^{j}\left(m \pm l_{k}, \pm\right)$. Because $\left\{w^{j}\right\}$ is a basis $c_{0}( \pm)$ is invertible and following Kadanoff and Kohmoto [13], we define

$$
\begin{align*}
G & =-(\cos \pi L)^{-1} c_{0}(-)^{-1} c_{0}(+), \\
\Lambda^{+} & =c_{0}(+)^{-1} c_{1}(+), \\
\Lambda^{-} & =c_{0}(-)^{-1} c_{1}(-), \\
F & =\left[\Lambda^{+}, m A\right]-L . \tag{6.11}
\end{align*}
$$

Because of the transformation properties of the local expansion coefficients, the matrices $G, \Lambda^{+}, \Lambda^{-}$, and $F$ do not depend on the choice of basis $\left\{w^{j}\right\}$. In the following, the local expansion matrices for the canonical basis will be distinguished by using a capital $C$. In the canonical basis $\left\{W^{j}\right\}$, we have $C_{0}(+)=I$ and the matrix $G$ is the inverse of the kernel of the inner product $I$ and is thus symmetric and positive definite. We are now prepared to state a key result in the SMJ analysis:

Theorem 6.3. $M W^{j} \in W^{1}(L, A)$ for $j=1, \ldots, r$, and if we write $W=$ $\left[W_{1}, \ldots, W_{r}\right]^{i}$, then

$$
M W=A \partial W-\tilde{A} \bar{\partial} W+F W,
$$

where $\tilde{A}=G^{-1} \bar{A} G$.
Proof. We use the characterization of $W^{1}(L, A)$ in Theorem 6.2. Since $M$ commutes with the Dirac operator, it is clear that $M W^{j}$ is again a solution of the Dirac equation. It is also obvious that $M W^{j}$ has the correct monodromy. In order to understand the action of $M$ on the local expansion of $W^{j}$ at $a_{k}$, we rewrite $M$ as

$$
\begin{aligned}
M= & -i\left(\left(x_{1}-a_{k}(1)\right) \partial / \partial x_{2}-\left(x_{2}-a_{k}(2)\right) \partial / \partial x_{1}\right)+\left[\begin{array}{cc}
1 / 2 & 0 \\
0 & -1 / 2
\end{array}\right] \\
& -i\left(a_{k}(1) \partial / \partial x_{2}-a_{k}(2) \partial / \partial x_{1}\right) \\
= & -i \partial / \partial \theta_{k}+\left[\begin{array}{cc}
1 / 2 & 0 \\
0 & -1 / 2
\end{array}\right]+A_{k} \partial-\bar{A}_{k} \bar{\partial} \\
= & M_{k}+A_{k} \partial-\bar{A}_{k} \bar{\partial},
\end{aligned}
$$

where $A_{k}=1 / 2\left(a_{k}(1)+i a_{k}(2)\right)$. Now $M_{k} v_{k}(x, l)=k v_{k}(x, l)$, and the ac-
tion of $\partial$ and $\bar{\partial}$ on the local expansion of $W^{j}$ at $a_{k}$ is computable using (6.9). We see that coefficients in the local expansions of $M W^{j}$ vanish for $l \leqslant-3 / 2$. Condition (3) of Theorem 6.2 is verified by simply observing that $M \bar{v}(x, k)$ $=k \tilde{v}(x, k)$. Since $M W^{j}$ is in $W^{1}(L, A)$, it follows from Theorem 6.2 that there exist matrices $X, Y$, and $Z$ (depending on $A, \bar{A}$ but not on $z, \bar{z}$ ) such that

$$
\begin{equation*}
M W^{j}=\sum_{k=1}^{r}\left[X_{k}^{j} \partial W^{k}+Y_{k}^{j} \bar{\partial} W^{k}+Z_{k}^{j} W^{k}\right] \tag{6.12}
\end{equation*}
$$

To finish the proof, it remains to evaluate the matrices $X, Y$, and $Z$ in terms of $A, F$, and $G$. One may do this by simply computing the local expansions of both sides of (6.12) and comparing coefficients. The details may be found in Proposition 3.31 of SMJ [26]. We remark here that it is useful to write the analogue of (6.12) for a general basis $\left\{w^{j}\right\}$ and specialize to a "canonical" choice only after deriving general recursion relations.

The Dirac operator $\gamma_{2} \partial / \partial x_{2}+\gamma_{1} \partial / \partial x_{1}-m=\left[\begin{array}{cc}-m & \partial \\ \bar{\partial} & -m\end{array}\right]$. Thus the "canonical" basis $W^{j}$ satisfies the following extended system of linear differential equations

$$
\begin{align*}
{\left[\begin{array}{cc}
m & -\partial \\
-\bar{\partial} & m
\end{array}\right] W^{j} } & =0, \quad j=1, \ldots, r \\
M W & =A \partial W-\tilde{A} \bar{\partial} W+F W \tag{6.13}
\end{align*}
$$

If we partition $W$ in the following way, $W=\left[W_{+}^{1}, W_{+}^{2} \cdots W_{+}^{r}, W_{-}^{1}, \cdots\right.$ $\left.W_{-}^{r}\right]^{t}$, then $W$ satisfies the following Pfaffian system equivalent to (6.13):

$$
\begin{equation*}
d_{z, \bar{z}} W=\Omega^{(1)} W \tag{6.14}
\end{equation*}
$$

where $\Omega^{(1)}=P d z+P^{*} d \bar{z}$ and

$$
\begin{aligned}
P & =(z-A)^{-1}\left[\begin{array}{cc}
F-1 / 2 & m(z-\tilde{A}) \\
m(z-A) & 0
\end{array}\right] \\
P^{*} & =(\bar{z}-\tilde{A})^{-1}\left[\begin{array}{cc}
0 & m(\bar{z}-\tilde{A}) \\
m(z-A) & -F-1 / 2
\end{array}\right]
\end{aligned}
$$

with $\tilde{A}=G^{-1} \bar{A} G$ as above.
Theorem 6.4. If the "canonical" basis depends differentiably on $A, \bar{A}$, then

$$
\begin{equation*}
d_{A,-\bar{A}} W=-d A \cdot \partial W-(\widetilde{d \bar{A}}) \bar{\partial} W+\Theta W \tag{6.15}
\end{equation*}
$$

where $(\widetilde{d A})=G^{-1} d \bar{A} G$, and $\Theta=-\left[\Lambda^{+}, m d A\right]$.

Proof. Differentiating the local expansion for $W$ with respect to $A_{k}$ and making use of $v_{j}(x, k)=v\left(x-a_{j}, k\right)$, one sees that $\partial_{k} W^{j}, \bar{\partial}_{k} W^{j} \in$ $W^{1}(L, A)$. An equation of the general type of (6.15) follows from the characterization of $W^{1}(L, A)$ in Theorem 6.2, and the coefficients in this relation are determined by comparing local expansions. The details are in Proposition 3.3.3 of SMJ [26].
If one substitutes $P W$ and $P^{*} W$ from (6.14) for $\partial W$ and $\bar{\partial} W$ in (6.15), one obtains $d_{A, \bar{A}} W=\Omega^{(2)} W$ as mentioned in the introduction to this section. Both $\Omega^{(1)}$ and $\Omega^{(2)}$ contain $F$ as a coefficient matrix. Thus one is led to seek a differential equation for $F$. Comparing the local expansions for both sides of (6.15), one finds an equation for $d_{A,} \bar{A}^{+}$, and hence an equation for $d F=d\left[\Lambda^{+}, m A\right]=\left[d \Lambda^{+}, m A\right]+\left[\Lambda^{+}, m d A\right]$. The expression for $d F$ which results involves the higher-order local expansion coefficient $C_{2}(+)$. In Section 3.3 of [26], SMJ show how to eliminate $C_{2}(+)$ as it occurs in this equation in favor of lower order expansion coefficients by making use of relations among the expansion coefficients implied by (6.13). In the final result $d_{A, \bar{A}} F$ is expressed in terms of $A$ and $G$. Thus it is natural to seek an equation for $d_{A, \bar{A}} G$. Such an equation is found directly by examining the low order expansion coefficients in (6.15). In this fashion, SMJ prove the following result:
Theorem 6.5. The matrices $F$ and $G$ satisfy the following deformation equations

$$
\begin{align*}
& d F=[\Theta, F]+m^{2}([d A, \tilde{A}]+[A,(\widetilde{d A})] \\
& d G=-G \Theta-\Theta^{*} G \tag{6.16}
\end{align*}
$$

where $\Theta$ and $\Theta^{*}$ denote matrices of one-forms characterized by:

$$
\begin{aligned}
{[\Theta, A]+[F, d A] } & =0, & & \text { diagonal of } \Theta=0, \\
{[\Theta, \bar{A}]+\left[G \bar{F} G^{-1}, d \bar{\Lambda}\right] } & =0, & & \text { diagonal } \Theta^{*}=0
\end{aligned}
$$

(see SMJ [26, p. 614]).
SMJ go on to show that the deformation equations (6.16) are completely integrable, and further that if $F$ and $G$ satisfy the deformation equations, then the Pfaffian system $d_{z, \bar{\Sigma}, A, \bar{A}} W=\Omega W$ is integrable. SMJ also shows that the matrices $F$ and $G$ satisfy certain algebraic conditions:

$$
\begin{align*}
& G^{*}=G \text { and } G \text { is positive definite, } \\
& F^{*}=G F G^{-1}, \text { diagonal } F=-L, \tag{6.17}
\end{align*}
$$

where * denotes the transpose conjugate.
The identity $G^{*}=G$, and the positive definite character of $G$ follow from the connection $G$ has with the inner product $I$ as remarked earlier.

The second identity $F^{*}=G F G^{-1}$ is more involved. Comparing local expansions in (6.13) one finds

$$
\begin{equation*}
F=-m\left[C_{1}(-) C_{0}(-)^{-1}, C_{0}(-) \bar{A} C_{0}(-)^{-1}\right]-C_{0}(-) L C_{0}(-)^{-1} . \tag{6.18}
\end{equation*}
$$

Now consider the basis $\left\{V^{j}\right\}$ for $W(L, A)$ introduced in Theorem 6.2. Let $D_{m}( \pm)$ denote the local expansion matrices $c_{m}( \pm)$ in the basis $\left\{V^{j}\right\}$. Because $D_{0}(-)=I$ it is clear that $W^{j}=\Sigma_{k} C_{0}(-)_{j k} V^{k}$. Using this relation to compare local expansion coefficients, one finds: $C_{0}(-) D_{1}(-)=C_{1}(-)$. Now substitute $C_{0}(-) D_{1}(-)$ for $C_{1}(-)$ in (6.18) to obtain

$$
\begin{equation*}
F=-C_{0}(-)\left(m\left[D_{1}(-), \bar{A}\right]+L\right) C_{0}(-)^{-1} \tag{6.19}
\end{equation*}
$$

Finally we make use of an identity that may be found in the following manner. Compute $I\left(W^{j}, \bar{\partial} V^{k}\right)$ for $j \neq k$ by deleting $\varepsilon$ neighborhoods of each $a_{m}$ and then passing to the limit $\varepsilon \rightarrow 0$. The function $\bar{\partial} V^{k}$ is not in $W(L, A)$, but it is still a solution of (6.0). Thus we may use the method of Theorem (6.1) to calculate this limit in two different ways. The result is

$$
\begin{equation*}
{\overline{D_{1}(-)}}_{k j} \cos \pi l_{j}-C_{1}(+)_{j k} \cos \pi l_{k}=0, \quad j \neq k \tag{6.20}
\end{equation*}
$$

Since the matrix $\left[D_{1}(-), \bar{A}\right]$ does not have any diagonal elements of $D_{1}(-)$, we may use this last relation to deduce

$$
(\cos \pi L)^{-1}\left[D_{1}(-), \bar{A}\right] \cos \pi L=\left[C_{1}(+)^{*}, \bar{A}\right]
$$

Substituting this in (6.19) one finds $F=G^{-1} F^{*} G$.
There is one more algebraic relation which we shall make use of. This relation expresses the diagonal elements of $\Lambda^{+}$in terms of $G$ and the off diagonal part of $\Lambda^{+}$(which is expressible in terms of $F$ ). It is

$$
\begin{equation*}
\Lambda_{k k}^{+}=-m \sum_{j \neq k}\left(A_{k}-A_{j}\right) \Lambda_{k j}^{+} \Lambda_{j k}^{+}+m\left(\overline{A_{k}}-\left(G \bar{A} G^{-1}\right)_{k k}\right) \tag{6.21}
\end{equation*}
$$

This is a simple consequence of (3.3.11) in SMJ [26], and is derived from the diagonal part of the local expansion of (6.13) at the second order.

Rather than pursue the general analysis further, we will now examine the application to the Ising correlations in some detail. The case $L=0$ is relevant to the Ising correlations, and throughout the rest of this section we will assume $L=0$ and we will set $m=1$. We will use lower case " $c$ " in the rest of this section to distinguish the Ising coefficients. Theorem 5.1 shows that $c_{0}( \pm)=-1 / 2(T \mp 1)$, where $T$ is the skew symmetric matrix with entries $T_{j k}=i \varepsilon_{k}^{j} N_{k j}(u, u)$. Thus $G=(1+T)^{-1}(1-T)$, from which it
follows that $T=(1-G)(1+G)^{-1}$. Hence not only do the coefficients $c_{0}( \pm)$ determine the basis invariant matrix $G$, but what is more interesting, $G$ determines $c_{0}( \pm)$. Another immediate consequence of Theorem 5.1 is

$$
\begin{equation*}
d_{A, \bar{A}} \ln \langle\sigma\rangle=\sum_{j=1}^{r} c_{j}^{j}(1,+) d A_{j}-c_{j}^{j}(1,-) d \overline{A_{j}} . \tag{6.22}
\end{equation*}
$$

Following SMJ [27], we note that the diagonal elements $c_{j}^{j}(0 \pm)$ are determined in terms of $\Lambda^{ \pm}$and $G$ as follows:

$$
\begin{aligned}
& c_{1}(+)=c_{0}(+) \Lambda^{+}=(1 / 2)(1-T) \Lambda^{+} \\
& c_{1}(-)=c_{0}(-) \Lambda^{-}=-(1 / 2)(1+T) \Lambda^{-}
\end{aligned}
$$

We now make use of the fact that $\langle\sigma\rangle$ and $\left\langle\sigma^{l s}\right\rangle$ are real valued functions (in [24] it was shown that $\left\langle\sigma^{l s}\right\rangle$ is approximable by ratios of real partition functions on a finite lattice). It is a consequence of Theorem 5.1, and the fact that $\langle\sigma\rangle$ and $\left\langle\sigma^{l s}\right\rangle$ are real that $\overline{c_{0}(+)}=-c_{0}(-)$, and $\overline{c_{1}(+)}=$ $-c_{1}(-)$. Thus $\Lambda^{-}=\overline{\Lambda^{+}}$and

$$
\begin{align*}
& c_{1}(+)=(1 / 2)(1-T) \Lambda^{+} \\
& c_{1}(-)=-(1 / 2)(T+1) \bar{\Lambda}^{+} \tag{6.23}
\end{align*}
$$

This is useful, since the off diagonal part of $\Lambda^{+}$is determined directly by $F$, and the diagonal entries of $\Lambda^{+}$are determined by $F$ and $G$ using 6.21. Thus, the right-hand side of (6.22) is determined by the appropriate solution of the deformation equations. To simplify notation, we shall write $\Lambda=\Lambda^{+}$ henceforth. Using (6.23) Eq. (6.22) may be rewritten

$$
\begin{align*}
d_{A, \bar{A}} \ln \langle\sigma\rangle= & \frac{1}{2} \sum_{k}\left(\Lambda_{k k} d A_{k}+\bar{\Lambda}_{k k} d \overline{A_{k}}\right) \\
& -\frac{1}{2} \sum_{j, k} T_{j k}\left(\Lambda_{k j} d A_{j}-\bar{\Lambda}_{k j} d \overline{A_{j}}\right) \tag{6.24}
\end{align*}
$$

In the case $L=0$, the basis $V^{j}=\bar{W}^{j}$, where the conjugation $\left(\overline{w_{1}, w_{2}}\right)=$ ( $\bar{w}_{2}, \bar{w}_{1}$ ). This may be checked easily by comparing local expansions at the lowest order. The relation $V^{j}=\bar{W}^{j}$ implies $\overline{D_{1}(-)}=C_{1}(+)$. If we substitute this in (6.20), then $C_{1}(+)_{j k}=C_{1}(+)_{k j}(k \neq j)$. Since $\Lambda=C_{1}(+)$, it follows that $\Lambda^{T}=\Lambda$. We make use of this and the skew symmetry of $T$ in following manner:

$$
\begin{aligned}
\operatorname{Tr}(T \Lambda d A) & =\operatorname{Tr}\left(d A^{T} \Lambda^{T} T^{T}\right) \\
& =-\operatorname{Tr}(T d A \Lambda)=\frac{1}{2} \operatorname{Tr}(T[\Lambda, d A])
\end{aligned}
$$

where $\mathrm{Tr}=$ trace of $r \times r$ matrix. Similarly

$$
\operatorname{Tr}(T \bar{\Lambda} d \bar{A})=\frac{1}{2} \operatorname{Tr}(T[\bar{\Lambda}, d \bar{A}])
$$

Substituting these results in (6.24), one finds

$$
\begin{align*}
d_{A, \bar{A}} \ln \langle\sigma\rangle= & \frac{1}{2} \sum_{k}\left(\Lambda_{k k} d A_{k}+\bar{\Lambda}_{k k} d \overline{A_{k}}\right) \\
& -\frac{1}{4} \operatorname{Tr}(T[\Lambda, d A]-T[\bar{\Lambda}, d \bar{A}]) \tag{6.25}
\end{align*}
$$

This result is 4.6 .5 in SMJ [27]. Following Kadanoff and Kohmoto, we make one further observation concerning this formula. Comparing local expansions in (6.15) and making use of $\Lambda^{-}=\bar{\Lambda}^{+}$, one sees that the deformation equation for $G$ can be written $d G=[d \bar{A}, \bar{\Lambda}] G-G[d A, \Lambda]$. Making use of this and the fact that $T$ commutes with $G$, one finds

$$
\begin{aligned}
\operatorname{Tr}(T[\Lambda, d A]-T[\bar{\Lambda}, d \bar{A}]) & =\operatorname{Tr}\left(T G^{-1} d G\right) \\
& =-d_{A, A_{A}} \operatorname{Tr}\left[\ln (1+G)\left(1+G^{-1}\right)\right]
\end{aligned}
$$

Thus:

$$
\begin{align*}
d_{A, \bar{A}} \ln \langle\sigma\rangle= & \frac{1}{2} \sum_{k}\left(\Lambda_{k k} d A_{k}+\bar{\Lambda}_{k k} d \overline{A_{k}}\right) \\
& +\frac{1}{4} d_{A, \bar{A}} \operatorname{Tr}\left[\ln (1+G)\left(1+G^{-1}\right)\right] . \tag{6.26}
\end{align*}
$$

We now discuss two applications of these results. The first application is the verification of the rotational invariance for the $n$-point functions scaled from below $T_{c}$, and a proof of the continuity of $\langle\sigma(a)\rangle$ at noncoincident points. The formulas for the scaled correlations in [24] show that these functions are differentiable in $a_{i}$ in the neighborhood of configurations without coincidences among second coordinates $\left(\pi_{2}\left(a_{j}\right) \neq \pi_{2}\left(a_{k}\right)\right.$ if $\left.j \neq k\right)$. The correlation function $\langle\sigma(a)\rangle$ was also shown to be invariant under simultaneous rotation of the $a_{j}$ by $\pi / 2$. Thus the correlations are known to be smooth in a neighborhood of any configuration which does not possess a coincidence of horizontal coordinates for one pair of points, and a coincidence of vertical coordinates for another pair of points. Let $E$ denote the set of configurations $a \in \mathbb{R}^{2 r}$ such that under some simultaneous rotation of all the $a_{j}$, there is both a coincidence of horizontal coordinates for some pair of points $\left\{a_{i_{1}}, a_{i_{2}}\right\}$ and a coincidence of vertical coordinates for another pair of points $\left\{a_{k_{1}}, a_{k_{2}}\right\}$. Evidently $E$ is the union of a finite number of sets defined by the conditions $\left(a_{i_{2}}-a_{i_{1}}\right) \cdot\left(a_{k_{2}}-a_{k_{1}}\right)=0$. These subsets, and hence also $E$, evidently have measure zero in $\mathbb{R}^{2 r}$. For any configuration $a \notin E$, the functions $\langle\sigma(a)\rangle$ are smooth and (6.26) is valid. Now in

Proposition 3.38 in SMJ [26] it is shown that under simultaneous rotation $A_{k} \rightarrow e^{i \theta} A_{k}$, the matrix functions $F$ and $G$ are invariant. Since $\Lambda_{k j}=\left(A_{k}-\right.$ $\left.A_{j}\right)^{-1} F_{j k}(j \neq k)$, it follows that $\Lambda_{k j} \rightarrow e^{-i \theta} \Lambda_{k j}(k \neq j)$. Comparing this with (6.21) we see that $\Lambda_{k k} \rightarrow e^{-i \theta} \Lambda_{k k}$. Thus the differential forms $\Lambda_{k k} d A_{k}$ and $\bar{\Lambda}_{k k} d \bar{A}_{k}$ are invariant under simultaneous rotations. The rotational invariance of the right-hand side of (6.26) does not, of course, guarantee that every solution $\ln \langle\sigma\rangle$ will be rotationally invariant. However, for the particular solution of interest this is the case, essentially because the boundary condition defining this solution is invariant. To be more explicit, suppose $a \notin E$. Then for any $s>0 s a \notin E$. It follows from the formulas given for $\langle\sigma(a)\rangle$ in [24] that $\lim _{s \rightarrow \infty}\langle\sigma(s a)\rangle=1,(a \notin E)$. Thus $\lim _{s \rightarrow \infty} \ln \langle\sigma(s a)\rangle=0$. Let $\Gamma$ denote the path $[1, \infty) \ni s \rightarrow s a$ and write $\Gamma_{\theta}$ for the image of $\Gamma$ under simultaneous rotation of the $a_{j}$ by $\theta$ radians. Let $B$ denote the differential form on the right-hand side of (6.26). Because $\ln \langle\sigma(s a)\rangle$ vanishes at $s=\infty$, we have

$$
\ln \langle\sigma(a)\rangle=-\int_{\Gamma} B .
$$

Now let $R_{\theta}$ denote simultaneous rotation of the $a_{j}$ by $\theta$ radians. Then

$$
\ln \left\langle\sigma\left(R_{\theta} a\right)\right\rangle=-\int_{\Gamma_{\theta}} B=-\int_{\Gamma} R_{\theta}^{*} B=-\int_{\Gamma} B=\ln \langle\sigma(a)\rangle .
$$

This establishes rotational invariance for $\langle\sigma(a)\rangle$ for $a \notin E$. We now use this to obtain rotational invariance in the distribution sense. Suppose $f_{j}(x) \in \delta\left(\mathbb{R}^{2}\right)$. We write

$$
S^{-}\left(f_{1}, \ldots, f_{r}\right)=\int_{\mathbf{R}^{2} r}\langle\sigma(a)\rangle f_{1}\left(a_{1}\right) \cdots f_{r}\left(a_{r}\right) d a_{1} \cdots d a_{r} .
$$

Then since $\langle\sigma(a)\rangle$ is rotationally invariant off a set of measure zero in $\mathbb{R}^{2 r}$ and is an integrable function [24] it follows that the Schwinger distributions $S^{-}\left(f_{1}, \ldots, f_{r}\right)$ are invariant under the substitution $f_{j} \rightarrow f_{j} o R(\theta)$, where $R(\theta)$ is rotation by $\theta$ in $\mathbb{R}^{2}$.

A slight refinement of the argument also shows that $\langle\sigma(a)\rangle$ is smooth away from configurations with coincidences $a_{i}=a_{k}(i \neq k)$. As noted above, the only problem occurs for configurations in which there are simultaneous coincidences of vertical and horizontal coordinates for two pairs of points. Let " $a$ " denote such a configuration (with $a_{i} \neq a_{k}$ for $1 \neq k$, however) and write $R_{\theta} a=\left(R(\theta) a_{1}, R(\theta) a_{2}, \ldots, R(\theta) a_{r}\right)$. Then there exists $\varepsilon>0$ such that for all $\theta$ with $0<|\theta|<\varepsilon$ the configuration $R_{\theta} a$ does not possess a horizontal or vertical coincidence of coordinates. Suppose $0<\theta<\varepsilon$. Then we may consistently define $\langle\sigma(a)\rangle=\left\langle\sigma\left(R_{\theta} a\right)\right\rangle$ since $\left\langle\sigma\left(R_{\theta} a\right)\right\rangle$ does not depend on $\theta$ for $0<\theta<\varepsilon$. Now we consider rotational invariance for small rotations from " $a$." Clearly there is no problem for $\theta>0$ and small. Fix $\theta$ with $0<\theta<\varepsilon$. Choose a sequence $b_{k} \in U / E$ such that $\lim _{k} b_{k}=a$. Then by continuity,
by continuity,

$$
\langle\sigma(a)\rangle=\left\langle\sigma\left(R_{\theta} a\right)\right\rangle=\lim _{k}\left\langle\sigma\left(R_{\theta} b_{k}\right)\right\rangle .
$$

By rotational invariance off $E$ we have $\left\langle\sigma\left(R_{\theta} b_{k}\right)\right\rangle=\left\langle\sigma\left(R_{\theta} b_{k}\right)\right\rangle$ and by continuity $\lim _{k}\left\langle\sigma\left(R_{\theta} b_{k}\right)\right\rangle=\left\langle\sigma\left(R_{\theta} a\right)\right\rangle$. Thus $\langle\sigma(a)\rangle=\left\langle\sigma\left(R_{\theta} a\right)\right\rangle$ and this proves local rotational invariance. We see that $\left\langle\sigma\left(R_{\theta} a\right)\right\rangle$ will be independent of $\theta$ until we rotate sufficiently to encounter another simultaneous coincidence of vertical and horizontal coordinates. However, since the extension of $\langle\sigma(\cdot)\rangle$ to such points is consistent regardless of the direction in which they are approached, it is clear that local rotational invariance extends to global rotational invariance. We have extended $\langle\sigma(a)\rangle$ to configurations without coincidences ( $a_{1} \neq a_{k}, 1 \neq k$ ) in a rotationally invariant fashion. If " $a$ " is a configuration as above, we would like to show that $\langle\sigma(b)\rangle$ is smooth for $b$ near $a$. Suppose $\varepsilon>0$ is given such that $R_{\varepsilon} a$ is free from coincidence among horizontal or vertical coordinates. Then for some sufficiently small neighborhood $U$ of $a$ in $R^{2 n}$ the set $R_{\varepsilon} U$ consists entirely of configurations free from such coincidences. But then $\left\langle\sigma\left(R_{\varepsilon} b\right)\right\rangle=\langle\sigma(b)\rangle$ is a smooth function of $b \in U$. We have proved:

Theorem 6.6. If $\langle\sigma(a)\rangle$ denotes the Ising correlations scaled from below $T_{c}$, then $\left\langle\sigma\left(R_{\theta} a\right)\right\rangle=\langle\sigma(a)\rangle$ and $\langle\sigma(a)\rangle$ is smooth except where $a_{i}=a_{k}$ for $i \neq k$.

The second application we wish to consider is the SMJ derivation of the WMTB $[3,31,32]$ result that the scaled two point functions for the Ising model are expressible in terms of a Painleve transcendent. First we introduce the radial coordinate $r=\left|a_{2}-a_{1}\right|$ and make use of the fact that $\left\langle\sigma\left(a_{1}\right) \sigma\left(a_{2}\right)\right\rangle$ and $\left\langle\mu\left(a_{1}\right) \mu\left(a_{2}\right)\right\rangle=N_{12}(u, u)=$ two point function scaled from above $T_{c}$, are both known to be functions of $r$ alone [24, 32]. Let

$$
N(r)=\frac{\left\langle\mu\left(a_{1}\right) \mu\left(a_{2}\right)\right\rangle}{\left\langle\sigma\left(a_{1}\right) \sigma\left(a_{2}\right)\right\rangle}
$$

and define $\psi(r)$ by

$$
\operatorname{ch} \psi=\frac{1+N^{2}}{1-N^{2}}, \quad \operatorname{sh} \psi=\frac{2 N}{1-N^{2}} .
$$

Then it follows from Theorem 5.1 that

$$
\begin{align*}
F & =\left[\begin{array}{cc}
0 & \frac{i}{2} r \frac{\partial \psi}{\partial r} \\
-\frac{i}{2} r \frac{\partial \psi}{\partial r} & 0
\end{array}\right] \\
G & =\left[\begin{array}{cc}
\operatorname{ch} \psi & i \operatorname{sh} \psi \\
-i \operatorname{sh} \psi & \operatorname{ch} \psi
\end{array}\right] \tag{6.27}
\end{align*}
$$

where we made use of $\left(\overline{A_{2}}-\overline{A_{1}}\right)\left(A_{2}-A_{1}\right)=r^{2} / 4$. The deformation equation for $d G$ is identically satisfied by the forms (6.27) and contains no new information. The deformation equation for $F$ is

$$
d F=\frac{i \operatorname{sh} 2 \psi}{4} r\left[\begin{array}{rr}
0 & 1 \\
-1 & 0
\end{array}\right] d r
$$

from which one obtains

$$
\begin{equation*}
\frac{d}{d r}\left(r \frac{d \psi}{d r}\right)=\frac{r}{2} \operatorname{sh} 2 \psi \tag{6.28}
\end{equation*}
$$

The substitution $\eta=e^{-\psi}$ transforms this last equation into a Painlevé equation of the third kind [19]. We now use (6.27) and (6.21) to calculate the diagonal elements $\Lambda_{11}$ and $\Lambda_{22}$. We find

$$
\begin{align*}
& \Lambda_{11}=\left(\overline{A_{2}}-\overline{A_{1}}\right)\left(\operatorname{sh}^{2} \psi-\left(\frac{d \psi}{d r}\right)^{2}\right), \\
& \Lambda_{22}=-\Lambda_{11} \tag{6.29}
\end{align*}
$$

Finally, if we substitute (6.29) into (6.26) and integrate, we obtain

$$
\begin{equation*}
\left\langle\sigma\left(a_{1}\right) \sigma\left(a_{2}\right)\right\rangle=\operatorname{ch}(\psi / 2) \exp \frac{1}{4} \int_{r}^{\infty} s d s\left[\operatorname{sh}^{2} \psi-\left(\frac{d \psi}{d s}\right)^{2}\right] . \tag{6.30}
\end{equation*}
$$

Also since $N(r)=\operatorname{th}(\psi / 2)$ it follows that

$$
\begin{equation*}
\left\langle\mu\left(a_{1}\right) \mu\left(a_{2}\right)\right\rangle=\operatorname{sh}(\psi / 2) \exp \frac{1}{4} \int_{r}^{\infty} s d s\left[\operatorname{sh}^{2} \psi-\left(\frac{d \psi}{d s}\right)^{2}\right] . \tag{6.31}
\end{equation*}
$$

## Acknowledgments

The authors wish to acknowledge helpful conversations with T. Miwa and M. Jimbo. We also thank our secretaries, Nancy French and Helen Hanchett for their fine typing of this series of papers.

## References

1. M. Ablowitz and H. Segur, Exact linearization of a Painlevé transcendent, Phys. Rev. Lett. 38 (1977), 1103-1106.
2. R. Z. Bariev, Correlation functions of the semi-infinite two-dimensional Ising model. I. Local magnetization, Theoret. and Math. Phys. 40 (1979), 623-626.
3. E. Barouch, B. M. McCoy, and T. T. Wu, Zero-field susceptibility of the two-dimensional Ising model near Tc, Phys. Rev. Lett. 31 (1973), 1409-1411.
4. J. M. Cook, The mathematics of second quantization, Trans. Amer. Math. Soc. 74 (1953), 222-245.
5. D. Creamer, unpublished work.
6. D. Creamer, H. Thacker, and D. Wilkinson, Some exact results for the two-point function of an integrable quantum field theory, Phys. Rev. D 23 (1981), 3081-3084.
7. H. Flaschka and A. Newell, Monodromy- and spectrum-preserving deformations I., Comm. Math. Phys. 76 (1980), 65-116.
8. J. Glimm and A. Jaffe, "Quantum Physics: A Functional Integral Point of View," Springer-Verlag, New York, 1981.
9. M. Jimbo, M: Kashiwara, and T. Miwa, Monodromy preserving deformation of ramified solutions to $\left(\Delta-m^{2}\right) u=0, J$. Math. Phys. 22 (1981), 2581-2587.
10. M. Jimbo and T. Miwa, On the $1 / c$ expansion of the density matrix for the $\delta$-function Bose gas, RIMS-370 preprint.
11. M. Jimbo, T. Miwa, Y. Mori, and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painleve transcendent, Physica 1D (1980), 80-158.
12. M. Jimbo, T. Miwa, and M. Sato, Holonomic quantum fields-The unanticipated link between deformation theory of differential equations and quantum fields, in "Mathematical Problem in Theoretical Physics" (K. Osterwalder, Ed.), Lecture Notes in Physics No. 116, Springer-Verlag, Berlin, 1980.
13. L. D. Kadanoff and M. Конmoto, SMJ's analysis of Ising model correlation functions, Ann. Physics 126 (1980), 371-398.
14. B. Kaufman, Crystal statistics, II. Partition function evaluated by spinor analysis, Phys. Rev. 76 (1949), 1232-1243.
15. B. Kaufman and L. Onsager, Crystal statistics. III. Short-range order in a binary Ising lattice, Phys. Rev. 76 (1949), 1244-1252.
16. G. E. Latta, The solution of a class of integral equations, J. Rational Mech. Anal. 5 (1956), 821-834.
17. A. Lenard, Momentum distribution in the ground state of the one-dimensional system of impenetrable Bosons, J. Math. Phys. 5 (1964), 930-943.
18. E. H. Lieb and W. Liniger, Exact analysis of an interacting Bose gas. I. The general solution and the ground state, Phys. Rev. 130 (1963), 1605-1616.
19. B. M. McCoy, C. A. Tracy, and T. T. Wu, Painleve functions of the third kind, J. Math. Phys. 18 (1977), 1058-1092.
20, E. W. Montroll, R. B. Potts, and J. C. Ward, Correlations and spontaneous magnetization of the two-dimensional Ising model, J. Math. Phys. 4 (1963), 308-322.
20. J. Myers, Ph.D. thesis, Harvard University, 1962 (unpublished).
21. L. Onsager, Crystal statistics: I. A two-dimensional model with an order-disorder transition, Phys. Rev. 65 (1944), 117-149.
22. J. Palmer, Products in spin representations, Adv. in Appl. Math 2 (1981), 290-328.
23. J. Palmer and C. Tracy, Two-dimensional Ising correlations: Convergence of the scaling limit, Adv. in Appl. Math. 2 (1981), 329-388.
24. M. Sato, T. Miwa, and M. Jimbo, Studies on holonomic quantum fields I-IV, Proc. Japan Acad. 53A (1977), 6-10; 53A (1977), 147-152; 53A (1977), 153-158; 53A (1977), 183-185.
25. M. Sato, T. Miwa, and M. Jimbo, Holonomic quantum fields, III, Publ. RIMS, Kyoto Univ. 15 (1979), 577-629.
26. M. Sato, T. Miwa, and M. Jimbo, Holonomic quantum Fields, IV, Publ. RIMS, Kyoto Univ. 15 (1979), 871-972.
27. M. Sato, T. Miwa, and M. Jimbo, Holonomic quantum fields, V, Publ. RIMS, Kyoto Univ. 16 (1980), 531-584.
28. T. D. Schultz, Note on the one-dimensional gas of impenetrable point-particle bosons, $J$. Math. Phys. 4 (1963), 666-671,
29. D. Shale and F. Stinespring, Spinor representations of infinite orthogonal groups, $J$. Math. Mech. 14 (1965), 315-322.
30. C. A. Tracy and B. M. McCoy, Neutron scattering and the correlation functions of the Ising model near Tc, Phys. Rev. Lett. 31 (1973), 1500-1504.
31. T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region, Phys. Rev. B13 (1976), 316-374.

[^0]:    *Supported in part by the National Science Foundation under Grant MCS-8102536.

