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EMBEDDED ELLIPTIC CURVES AND THE YANG–BAXTER EQUATIONS†

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The completely $\mathbf{Z}_n$ symmetric S-matrix defined by Belavin is shown to satisfy the Yang–Baxter equations. In the projective space of Boltzmann weights, the curves on which there exist commuting transfer matrices are shown to be embedded elliptic curves. Explicit polynomial equations for these curves are given. For $n = 2$ these results reduce to the results of Baxter for the symmetric eight-vertex model.

1. Introduction

Most two-dimensional lattice models in statistical mechanics and one-dimensional lattice quantum models are "exactly solvable" only in the case that the Yang–Baxter equations [1, 21] possess a nontrivial solution. The reader is referred to Baxter [2], Takhtadzhan and Faddeev [19], and Thacker [20] for a review of these equations and their role in exactly solvable models.

It is the purpose of this paper to examine one class of vertex models that are natural generalizations of Baxter's symmetric eight-vertex model [1, 2], the so-called completely $\mathbf{Z}_n$ symmetric vertex models first introduced by Belavin [4] and Chudnovsky and Chudnovsky [7]. There are other interesting generalizations of the Baxter model and the reader is referred to Perk and Schultz [17] and references therein. In Belavin [4] a formula is given for the vertex weights in terms of theta functions (see (4.1) below). With this parametrization the claim is that the Yang–Baxter equations are satisfied. I say claim, since there is some confusion on what was proved and what was conjectured (see, for example, Öttinger and Honerkamp [16]). These questions were further explored by Cherednik [6] who concluded the Yang–Baxter equations are satisfied in the elliptic case†. This paper, which is partly expository, gives an account of these matters. It was felt that the elegance of Belavin's result required further explanation. In particular, I have emphasized the role embedded elliptic curves play in these models.

In section 2, the $\mathbf{Z}_n$ symmetric vertex model is defined and a representation for a $\mathbf{Z}_n$ symmetric S-matrix is given. The material here can be found in either Belavin [4] or Chudnovsky and Chudnovsky [7]. Section 3 summarizes the results needed from the theory of theta functions. The books by Mumford [15] and by Krazer [13] are my main sources and wherever possible I follow Mumford's notation. As do these authors, I stress the Riemann theta formulae. In section 4, a proof that the Yang–Baxter equations are satisfied is given, assuming the Belavin parametrization in terms of elliptic theta functions. The proof for arbitrary n is different in its details than that in Cherednik [6] but similar in spirit, i.e., function theoretic as opposed to algebraic. For the case $n = 2$ I point out that a special case of the Riemann quartic theta identity is precisely the Yang–Baxter equation. In some sense this is well known, but the

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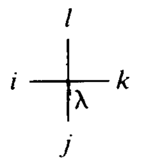
†See remarks at end of section 7.

details do not seem readily available. Section 5 takes up the problem of finding explicit homogeneous polynomials in the Boltzmann coordinates w_α that give the embedded curve on which there will exist commuting transfer matrices (we actually never discuss the relation to transfer matrices, but assume the reader understands this connection, see [2, 19, 20]). This problem is equivalent to finding a set of homogeneous polynomials whose common zero set give the embedded elliptic curve. As the reader can imagine, the 19th century literature is enormous. Given below are n^2 homogeneous polynomials that the embedded curve must satisfy. Certainly at least two of these equations are dependent, but I do not have a proof (for $n > 2$) that there are $n^2 - 2$ independent equations. The difficulty here is that the occurrence of too many theta nulls makes it unclear what is dependent and what is independent (the theta nulls themselves satisfy identities). All of these polynomials come from a particular set of Riemann theta formulae. The problem of reducing the number of theta nulls for $n = 3$ is discussed in considerable detail in section 6. A few remarks about the general case are given in section 7. The point in these sections is that to obtain the polynomial equations for the w_α in $\mathbb{P}^{n^2-1}$, one finds polynomials for the elliptic normal curve in $\mathbb{P}^{n-1}$. This mathematics is classical and our sources are Klein and Fricke [11], Krazer [13], and Bianchi [5]. In these sections the Heisenberg group H_n, previously introduced in sections 3 and 4, plays a prominent role in the determination of these homogeneous polynomials. An additional point to be emphasized is the appearance of elliptic modular functions of level n. I expect this function theory to play an important part in the statistical mechanics of these models. This will be taken up in another paper.

2. Completely $\mathbf{Z}_n$ symmetric vertex model

Consider a square lattice Λ with M rows and N columns with periodic boundary conditions. Denote by $\mathbf{Z}_n$ the group defined by the set $\{0, 1, \dots, n-1\}$ with addition modulo n. A configuration ω is specified by assigning to each bond in Λ an element of $\mathbf{Z}_n$. Denote by Ω_Λ the collection of all such configurations. To a given vertex $\lambda \in \Lambda$ (see fig. 1)



$i, j, k, l \in \mathbf{Z}_n$

Fig. 1.

is assigned an energy ε_{ij}^{kl} and a corresponding Boltzmann weight $S_{ij}^{kl} \equiv \exp(-\beta\varepsilon_{ij}^{kl})$ where $0 < \beta < \infty$ is proportional to the inverse temperature. The physical values of S_{ij}^{kl} are nonnegative real numbers. The energy E_Λ: $\Omega_\Lambda \to \mathbf{R} \cup \{\infty\}$ of a configuration ω is defined to be the sum of the energies ε_{ij}^{kl} of each vertex of Λ. The probability of configuration $\omega \in \Omega_\Lambda$ is given by the Gibbs measure

$$P_\Lambda(\omega) = \exp(-\beta E_\Lambda(\omega))/Z_\Lambda(\beta),$$

where $Z_\Lambda(\beta) = \sum_{\omega\in\Omega_\Lambda} \exp(-\beta E_\Lambda(\omega))$. To obtain the thermodynamics in the $\Lambda \to \mathbf{Z}^2$ limit one must evaluate $\lim_{\Lambda\to\mathbf{Z}^2}(1/MN)\log Z_\Lambda(\beta)$.

To proceed further one must make restrictions on the Boltzmann weights S_{ij}^{kl}. The above vertex model is said to be completely $\mathbf{Z}_n$ symmetric if the following two conditions are satisfied [4, 7]

$$\text{(i)} \quad S_{ij}^{kl} = 0, \qquad \text{unless } i + j = k + l(n),$$

$$\text{(ii)} \quad S_{i+p\,j+p}^{k+p\,l+p} = S_{ij}^{kl}, \qquad \text{for all } p \in \mathbf{Z}_n \text{ and all } i, j, k, l \in \mathbf{Z}_n.$$

For $n = 2$ this defines the symmetric 8-vertex model solved by Baxter [1, 2]. Thus we call the completely $\mathbf{Z}_n$ symmetric vertex model the $\mathbf{Z}_n$ Baxter model.

We want to rephrase properties (i) and (ii). To do this we consider the vector space $V = \mathbf{C}^n$ and denote its standard basis by $\{e_m\}_{m\in\mathbf{Z}_n}$. We define linear maps g and h by

$$ge_i = \omega^i e_i, \quad he_i = e_{i-1}, \tag{2.1}$$

where $\omega = \exp[(2\pi\sqrt{-1})/n]$. The Boltzmann weights S_{ij}^{kl} define a linear map $S: V \otimes V \to V \otimes V$ in the basis $\{e_i \otimes e_j\}_{i,j \in \mathbf{Z}_n}$ for $V \otimes V$.

Lemma 2.1. Property (i) holds if and only if S satisfies

$$S(g \otimes g) = (g \otimes g)S. \tag{2.2}$$

Proof. We assume (2.2). Applying $(g^{-1} \otimes g^{-1}) S(g \otimes g)$ to $e_k \otimes e_l$ gives $\Sigma_{i,j \in \mathbf{Z}_n} S_{ij}^{kl} \omega^{k+l-i-j} e_i \otimes e_j$. This must equal $\Sigma_{i,j \in \mathbf{Z}_n} S_{ij}^{kl} e_i \otimes e_j$. Thus $S_{ij}^{kl} = 0$ unless $i + j = k + l(n)$. Reversing the above steps establishes the converse. □

Lemma 2.2. Property (ii) holds if and only if S satisfies

$$S(h \otimes h) = (h \otimes h)S. \tag{2.3}$$

Proof. To show (2.3) implies (ii) we apply $(h^{-1} \otimes h^{-1})S(h \otimes h)$ to $e_k \otimes e_l$ and equate the result to the result of applying S to $e_k \otimes e_l$. This establishes (ii) for $p = 1$. To obtain the general case apply $(h^{-p} \otimes h^{-p})S(h^p \otimes h^p)$ to $e_k \otimes e_l$. Reversing these steps gives the converse. □

The matrices g and h have a group-theoretic meaning. To see this let

$$G_n \equiv \mathbf{Z}_n^2 = \{(\alpha_1, \alpha_2) | \alpha_1, \alpha_2 \in \mathbf{Z}_n\} \tag{2.4}$$

and

$$H_n \equiv \{(\lambda, \alpha) | \lambda \in \mu_n, \alpha \in G_n\}, \tag{2.5}$$

where μ_n denotes the multiplicative group of the nth roots of unity. With the multiplication rule

$$(\lambda, \alpha)(\lambda', \beta) = (\lambda\lambda'\omega^{-\alpha_2\beta_1}, \alpha + \beta)$$

H_n becomes the Heisenberg group [15]. If we define

$$I_\alpha = h^{\alpha_1} g^{\alpha_2}, \quad \alpha \in G_n,$$

then a calculation shows

$$I_\alpha I_\beta = \omega^{-\alpha_2\beta_1} I_{\alpha+\beta}. \tag{2.6}$$

Thus the map $H_n \ni (\lambda, \alpha) \mapsto \lambda I_\alpha$ defines a representation of H_n.

Proposition 2.3. An S-matrix is completely $\mathbf{Z}_n$-symmetric if and only if

$$S(I_\alpha \otimes I_\alpha) = (I_\alpha \otimes I_\alpha)S$$

for all $\alpha \in G_n$.

Proof. By the two previous lemmas we have $S(h^{\alpha_1} \otimes h^{\alpha_1}) = (h^{\alpha_1} \otimes h^{\alpha_1})S$ and $S(g^{\alpha_2} \otimes g^{\alpha_2}) = (g^{\alpha_2} \otimes g^{\alpha_2})S$ if and only if S is completely $\mathbf{Z}_n$ symmetric. Thus

$$\begin{aligned}(I_\alpha \otimes I_\alpha)S &= (h^{\alpha_1} g^{\alpha_2} \otimes h^{\alpha_1} g^{\alpha_2})S \\ &= (h^{\alpha_1} \otimes h^{\alpha_1})(g^{\alpha_2} \otimes g^{\alpha_2})S \\ &= (h^{\alpha_1} \otimes h^{\alpha_1})S(g^{\alpha_2} \otimes g^{\alpha_2}) \\ &= S(h^{\alpha_1} g^{\alpha_2} \otimes h^{\alpha_1} g^{\alpha_2}) = S(I_\alpha \otimes I_\alpha). \quad \square\end{aligned}$$

Proposition 2.4. The set $\{I_\alpha\}_{\alpha \in G_n}$ forms a basis for the vector space $M_n(\mathbf{C})$, the vector space of all $n \times n$ matrices with complex entries.

Proof. For $A, B \in M_n(\mathbf{C})$ we define the standard inner product

$$\langle A, B \rangle = \sum_{j,k} A_{jk} \bar{B}_{jk}.$$

Using $(I_\alpha)_{ij} = \delta_{i,j-\alpha_1} \omega^{j\alpha_2}$, a calculation shows

$$\begin{aligned}\langle I_\alpha, I_\beta \rangle &= \delta_{\alpha_1\beta_1} \sum_{k=0}^{n-1} \exp\left(\frac{2\pi i}{n} k(\alpha_2 - \beta_2)\right) \\ &= n\delta_{\alpha,\beta}.\end{aligned}$$

Thus the set $\{I_\alpha\}_{\alpha\in G_n}$ is a linearly independent set and since there are n^2 elements it follows that $\{I_\alpha\}_{\alpha\in G_n}$ is a basis for $M_n(\mathbf{C})$. □

Theorem 2.5. An S-matrix is completely $\mathbf{Z}_n$ symmetric if and only if S is of the form

$$S=\sum_{\alpha\in G_n} w_\alpha I_\alpha \otimes I_\alpha^{-1} \tag{2.7}$$

for some choice $w_\alpha\in\mathbf{C}$. We call w_α the Boltzmann coordinates.

Proof. Let $S\in M_{n^2}(\mathbf{C})$. Since $\{I_\alpha\otimes I_\beta\}_{\alpha,\beta\in G_n}$ is a basis of $M_{n^2}(\mathbf{C})$. S has the expansion

$$S=\sum_{\alpha,\beta\in G_n} w_{\alpha\beta} I_\alpha\otimes I_\beta, \quad w_{\alpha\beta}\in\mathbf{C}.$$

Assume that S is completely $\mathbf{Z}_n$-symmetric. Then $S(I_\sigma\otimes I_\sigma)=(I_\sigma\otimes I_\sigma)S$ for all $\sigma\in G_n$. A computation shows (using (2.6))

$$S(I_\sigma\otimes I_\sigma)=\sum_{\alpha,\beta} w_{\alpha\beta}\omega^{-\sigma_1(\alpha_2+\beta_2)} I_{\alpha+\sigma}\otimes I_{\beta+\sigma},$$

$$(I_\sigma\otimes I_\sigma)S=\sum_{\alpha,\beta} w_{\alpha\beta}\omega^{-\sigma_2(\alpha_1+\beta_1)} I_{\alpha+\sigma}\otimes I_{\beta+\sigma}.$$

Since these must be equal for all $\sigma\in G_n$ and the fact that $\{I_\alpha\otimes I_\beta\}_{\alpha,\beta\in G_n}$ is a basis, we conclude that $\alpha_1+\beta_1=0(n)$ and $\alpha_2+\beta_2=0(n)$. Hence

$$S=\sum_{\alpha\in G_n} w_{\alpha-\alpha} I_\alpha\otimes I_{-\alpha}=\sum_{\sigma\in G_n} w_\alpha I_\alpha\otimes I_\alpha^{-1}.$$

where $w_\alpha=\omega^{\alpha_1\alpha_2}w_{\alpha-\alpha}$. If S is of the form (2.7) then using proposition 2.3, we conclude S is $\mathbf{Z}_n$-symmetric. □

3. Elliptic curves and their theta functions

In this section we collect the results we need from the theory of elliptic curves and theta functions. The reader is referred to either Krazer [13] or Mumford [15] for details. Wherever possible we follow the notation of Mumford.

Let H denote the upper half-plane, $\tau\in H$, and $\Lambda_\tau=\mathbf{Z}+\mathbf{Z}\tau$. We identify an elliptic curve with the complex torus $E_\tau=\mathbf{C}/\Lambda_\tau$. We define theta functions $\vartheta_{ab}(z,\tau)$ with rational characteristics $a,b\in(1/n)\mathbf{Z}$ by

$$\vartheta_{a,b}(z,\tau)=\sum_{m\in\mathbf{Z}}\exp\big(\pi i(m+a)^2\tau + 2\pi i(m+a)(z+b)\big). \tag{3.1}$$

We frequently suppress the dependence upon τ and write $\vartheta_{a,b}(z)$ for $\vartheta_{a,b}(z,\tau)$. Also we denote by ϑ_{ab} the null values $\vartheta_{a,b}(0,\tau)$, $\vartheta(z,\tau)$ for $\vartheta_{00}(z,\tau)$, and we sometimes use the abbreviated notation $\vartheta_\alpha(z)=\vartheta_{\alpha_1/n\,\alpha_2/n}(z,\tau)$ for $\alpha\in G_n$. For every holomorphic function $f(z)$ and real numbers a and b, let

$$\begin{aligned}(S_bf)(z)&=f(z+b),\\ (T_af)(z)&=\exp(\pi i a^2\tau+2\pi i a z)f(z+a\tau).\end{aligned} \tag{3.2}$$

In terms of these operators we have

$$\vartheta_{a,b}(z)=(S_bT_a\vartheta)(z), \tag{3.3a}$$

$$(S_{b_1}\vartheta_{a,b})(z)=\vartheta_{a,b+b_1}(z), \quad \text{for } a,b_1,b\in\frac{1}{n}\mathbf{Z}, \tag{3.3b}$$

$$(T_{a_1}\vartheta_{a,b})(z)=\exp(-2\pi i a_1 b)\vartheta_{a_1+a,b}(z), \quad \text{for } a,a_1,b\in\frac{1}{n}\mathbf{Z}, \tag{3.3c}$$

$$\vartheta_{a+p,b+q}(z)=\exp(2\pi i a q)\vartheta_{a,b}(z), \quad \forall p,q\in\mathbf{Z},\ a,b\in\frac{1}{n}\mathbf{Z}. \tag{3.3d}$$

The functions $\vartheta_{ab}(z)$, $a,b\in(1/n)\mathbf{Z}/\mathbf{Z}$, form a basis for the vector space V_n of entire functions invariant under S_n and T_n. Furthermore, an entire function $f(z)$ is in V_n if and only if

$$f(z)=\sum_{m\in\frac{1}{n}\mathbf{Z}} C_m\exp(\pi i m^2\tau+2\pi i m z),$$

such that $C_{m'}=C_m$ if $m'-m\in n\mathbf{Z}$ with the action of $S_{1/n}$ and $T_{1/n}$ on V_n as follows:

$$S_{1/n}\left(\sum_{m\in\frac{1}{n}\mathbf{Z}} C_m \exp(\pi i m^2\tau + 2\pi i m z)\right) = \sum_{m\in\frac{1}{n}\mathbf{Z}} C_m \exp\left(\frac{2\pi i}{n} m\right) \exp(\pi i m^2\tau + 2\pi i m z),$$

$$T_{1/n}\left(\sum_{m\in\frac{1}{n}\mathbf{Z}} C_m \exp(\pi i m^2\tau + 2\pi i m z)\right) = \sum_{m\in\frac{1}{n}\mathbf{Z}} C_{m-1/n} \exp(\pi i m^2\tau + 2\pi i m z).$$

If we use the coefficients $\{C_m | m\in(1/n)\mathbf{Z}/n\mathbf{Z}\}$ as coordinates, then on V_n we can define operators h and g in an analogous way as in (2.1) where now h and g are $n^2\times n^2$ matrices and $\omega=\exp(2\pi\sqrt{-1}/n^2)$. We then observe that $T_{1/n}$ and $S_{1/n}$ restricted to V_n can be identified as h^{-1} and g, respectively. These results are from Mumford [15, pages 8–10].

The geometrical significance of $\vartheta_{ab}(z)$ is that for $n\geq 2$ the complex torus E_τ can be embedded into $\mathbf{P}^{n^2-1}$ using these functions. Explicitly, let (a_i, b_i) be a set of coset representatives of $[(1/n)\mathbf{Z}/\mathbf{Z}]^2$ in $[(1/n)\mathbf{Z}]^2$, $0\leq i\leq n^2-1$. Write $\vartheta_i(z)=\vartheta_{a_i b_i}(z)$, then the map φ_n: $E_\tau\to\mathbf{P}^{n^2-1}$ defined by

$$z\mapsto(\ldots,\vartheta_i(nz,\tau),\ldots) \tag{3.4}$$

is a holomorphic embedding (see Mumford [15, pages 11–14]) of the elliptic curve into $\mathbf{P}^{n^2-1}$. The set $\varphi_n(E_\tau)$ is an algebraic subvariety and is defined by certain homogeneous polynomials.

To find these homogeneous polynomials it is easiest to proceed using the Riemann theta formulae. To every rational orthogonal $h\times h$ matrix T there is a corresponding Riemann theta identity (Mumford [15, page 211]). The classical choice of Riemann is

$$T=\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix} \tag{3.5}$$

and the various identities that result for the half-integer thetas are explicitly written out as equations R_1 thru R_{21} on page 20 of Mumford [15]. It will be these identities that are relevant for the $\mathbf{Z}_2$ Baxter model. If we define the $2n\times 2n$ matrix J_{2n} to be the matrix consisting of all 1's, then the choice

$$T=\frac{1}{n}(J_{2n}-nI_{2n}) \tag{3.6}$$

gives identities for the thetas $\vartheta_{a,b}(z)$, $a,b\in(1/n)\mathbf{Z}$. Such identities appear to have been first worked out by Krazer and Prym [14] over one-hundred years ago. Let η, $\rho^{(i)}$, and $\varepsilon\in\mathbf{Z}^2$ where we write $\eta=(\eta_1,\eta_2)$, etc. Define $\langle\varepsilon,\eta\rangle=\varepsilon_1\eta_2-\varepsilon_2\eta_1$ and recall $\omega=\exp[(2\pi i)/n]$ and the notation $\vartheta_\alpha(z)=\vartheta_{\alpha_1/n,\alpha_2/n}(z)$, $\alpha=(\alpha_1,\alpha_2)\in G_n$. Then the identities read (Krazer and Prym [14, page 274] and set $p=1$ in their Θ_2'')

$$n\prod_{i=1}^{2n}\vartheta_{\eta-\rho(i)}(nu_i)\,\omega^{-\eta_1\eta_2} = \sum_{\varepsilon_1,\varepsilon_2=0}^{n-1}\omega^{\langle\varepsilon,\eta\rangle-\varepsilon_1\varepsilon_2}\prod_{i=1}^{2n}\vartheta_{\varepsilon+\rho(i)}(nv_i), \tag{3.7}$$

where $\rho^{(1)}+\rho^{(2)}+\cdots+\rho^{(2n)}=(0,0)$, and the variables u_i are related to the v_i by $u=Tv$, $v=\begin{pmatrix}v_1\\ \vdots\\ v_{2n}\end{pmatrix}$, $u=\begin{pmatrix}u_1\\ \vdots\\ u_{2n}\end{pmatrix}$, and T is the $2n\times 2n$ matrix (3.6).

We set $nv_{n+1}=\cdots=nv_{2n}=x$; $v_1=\cdots=v_n=0$; $\rho_1=\cdots=\rho_n=(0,0)$; $\rho_{n+1}=\cdots=\rho_{2n-1}=\sigma\in G_n$, $\rho_{2n}=-(n-1)\sigma$; and $\eta=(0,0)$ in (3.7) to obtain

$$n(\vartheta_{00}(x))^n(\vartheta_{-\sigma})^n = \sum_{\varepsilon_1,\varepsilon_2=0}^{n-1}\omega^{-\varepsilon_1(\varepsilon_2+\sigma_2)}(\vartheta_\varepsilon)^n(\vartheta_{\sigma+\varepsilon}(x))^n, \tag{3.8}$$

where we used (3.3d), i.e.

$$\vartheta_{\frac{n-1}{n}\sigma_1,\frac{n-1}{n}\sigma_2}=\omega^{-\sigma_1\sigma_2}\vartheta_{\frac{-\sigma_1}{n},\frac{-\sigma_2}{n}}=\omega^{-\sigma_1\sigma_2}\vartheta_{-\sigma}.$$

The geometrical significance of (3.8) is seen by setting $x_{ij} \equiv \vartheta_{i/n,j/n}(x,\tau)$ in (3.8) to obtain the n^2 homogeneous polynomials

$$n x_{00}^n \vartheta_{-\sigma}^n = \sum_{\varepsilon_1,\varepsilon_2=0}^{n-1} \omega^{-\varepsilon_1(\varepsilon_2+\sigma_2)}(\vartheta_\varepsilon)^n x_{\sigma+\varepsilon}^n, \quad \sigma \in G_n, \tag{3.9}$$

which in $\mathbb{P}^{n^2-1}$ define an algebraic variety $\mathscr{E}_n$. For $n=2$ the equations reduce to

$$\vartheta_{00}^2\vartheta_{00}^2(x) = \vartheta_{01}^2\vartheta_{01}^2(x) + \vartheta_{10}^2\vartheta_{10}^2(x), \tag{E$_1$}$$

$$\vartheta_{01}^2\vartheta_{00}^2(x) = \vartheta_{00}^2\vartheta_{01}^2(x) - \vartheta_{10}^2\vartheta_{11}^2(x), \tag{E$_1'$}$$

$$\vartheta_{10}^2\vartheta_{00}^2(x) = \vartheta_{00}^2\vartheta_{10}^2(x) + \vartheta_{01}^2\vartheta_{11}^2(x), \tag{E$_2'$}$$

$$\vartheta_{00}^2\vartheta_{11}^2(x) = \vartheta_{10}^2\vartheta_{01}^2(x) - \vartheta_{01}^2\vartheta_{10}^2(x), \tag{E$_2$}$$

for $\sigma = (0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ respectively. For $n=2$ the above four equations are the famous Jacobi equations (Mumford [15, pages 23 and 57], Krazer [13, page 331]). Only two of the above equations are independent (one normally chooses E_1 and E_2). Furthermore the intersection of E_1 and E_2 defines a curve $\mathscr{E}_2$ which coincides with the image $\varphi_2(E_\tau)$ from the Lefschetz embedding (Mumford [15, page 23]).

For arbitrary $n \geq 2$ the variety $\mathscr{E}_n$ is at least of dimension 1 in $\mathbb{P}^{n^2-1}$ since as x varies over E_τ the theta functions satisfy (3.9). The problem is whether this specialization of the Riemann theta formula gives a complete set of homogeneous polynomials characterizing the embedded curve. The problem is that there are too many theta nulls in (3.9) and they satisfy many identities (set $x=0$ in (3.8), for example; for $n=2$ this reduces to a single identity $\vartheta_{00}^4 = \vartheta_{01}^4 + \vartheta_{10}^4$). These questions are taken up again in section 6 for the special case $n=3$.

4. Yang–Baxter equations

Let S be a completely $\mathbb{Z}_n$ symmetric matrix with representation (2.7). Following Baxter [1, 3] (for $n=2$) and Belavin [4] (for arbitrary $n \in N$, $n \geq 2$) we set

$$w_\alpha(z) = \frac{\vartheta_\alpha(z+\eta)}{\vartheta_\alpha(\eta)}, \tag{4.1}$$

where $z, \eta \in E_\tau$, $\alpha \in G_n$ and η is chosen so that $\vartheta_\alpha(\eta) \neq 0$ for all $\alpha \in G_n$. Observe that $w_{\alpha+n} = w_\alpha$ for all $\alpha \in G_n$ (use (3.3d)). If we define

$$S^{12}(z) = \sum_{\alpha \in G_n} w_\alpha(z) I_\alpha \otimes I_\alpha^{-1} \otimes I, \tag{4.2a}$$

$$S^{13}(z) = \sum_{\alpha \in G_n} w_\alpha(z) I_\alpha \otimes I \otimes I_\alpha^{-1}, \tag{4.2b}$$

and

$$S^{23}(z) = \sum_{\alpha \in G_n} w_\alpha(z) I \otimes I_\alpha \otimes I_\alpha^{-1} \tag{4.2c}$$

(I is the $n \times n$ identity matrix) then the Yang–Baxter equation for the $\mathbb{Z}_n$ Baxter model is

$$S^{12}(z_1)S^{13}(z_1+z_2)S^{23}(z_2) = S^{23}(z_2)S^{13}(z_1+z_2)S^{12}(z_1) \tag{4.3}$$

for all $z_1, z_2 \in E_\tau$.

Proposition 4.1. The Yang–Baxter equation (4.3) is equivalent to the identity

$$F_{ab}(z_1, z_2) = 0, \quad \text{for all } z_1, z_2 \in E_\tau \text{ and all } a, b \in G_n, \tag{4.4}$$

where

$$F_{ab}(z_1, z_2) = \sum_{c \in G_n} w_c(z_1) w_{a-c}(z_1 + z_2) w_{b+c}(z_2) [\omega^{\langle a-c, b\rangle} - \omega^{\langle c, b\rangle}]. \tag{4.5}$$

Proof. Using (4.2) and the multiplication rule for the I_a's we have

$$S^{12}(z_1) S^{13}(z_1 + z_2) S^{23}(z_2) = \sum_{c,d,e} w_c(z_1) w_d(z_1 + z_2) w_e(z_2) \times \omega^{-d_2 e_1 - e_1 e_2 + c_2 e_1 - c_1 c_2 - d_1 d_2 - c_2 d_1}$$
$$\times I_{c+d} \otimes I_{-c+e} \otimes I_{-d-e}.$$

Let $a = c + d$, $b = -c + e$, and summing over a, b and c the above expression becomes

$$\sum_{a,b} \omega^{-a_1 a_2 - b_1 b_2 - a_2 b_1} \left[\sum_c \omega^{\langle b, c\rangle} w_c(z_1) w_{a-c}(z_1 + z_2) w_{b+c}(z_2) \right] \cdot I_a \otimes I_b \otimes I_{-a-b}.$$

A similar calculation gives

$$S^{23}(z_2) S^{13}(z_1 + z_2) S^{12}(z_1) = \sum_{a,b} \omega^{-a_1 a_2 - b_1 b_2 - a_1 b_2} \left[\sum_c \omega^{-\langle b, c\rangle} w_c(z_1) w_{a-c}(z_1 + z_2) w_{b+c}(z_2) \right]$$
$$\cdot I_a \otimes I_b \otimes I_{-a-b}.$$

Using the fact $\{I_a\}$ is a basis we see that (4.4) and (4.5) now follow. □

Observe that (4.4) is immediate for $b = (0, 0)$. Thus we assume $b \neq (0, 0)$ for the remainder of this section. Let $z_2 \in E_\tau$ be fixed and for simplicity we write $F_{ab}(z) = F_{ab}(z, z_2)$.

Proposition 4.2. For $a, b \in G_n$, $z \in E_\tau$, the function $F_{ab}(z)$ vanishes at

(i) $z = 0$,

(ii) $z = \frac{1}{n}(b_1 \tau + b_2)$.

Proof. (i) Setting $z = 0$ in (4.5), using the fact that $w_c(0) = 1$, and changing the summation index ($c \mapsto a - b - c$) in the term with the factor $\omega^{\langle a-c, b\rangle}$ shows that $F_{ab}(0) = 0$.

(ii) Using (3.3) we first observe

$$w_c\left(z_2 + \frac{b_1}{n}\tau + \frac{b_2}{n}\right) = \frac{\vartheta_{c+b}(z_2 + \eta)}{\vartheta_c(\eta)} \exp\left(-\frac{2\pi i}{n^2} b_1 (c_2 + b_2)\right) \exp\left(-i\pi \frac{b_1^2}{n^2}\tau - 2\pi i \frac{b_1}{n}(z_2 + \eta)\right).$$

Then

$$w_c\left(\frac{b_1}{n}\tau+\frac{b_2}{n}\right)w_{a-c}\left(z_2+\frac{b_1}{n}\tau+\frac{b_2}{n}\right)=\lambda\frac{\vartheta_{b+c}(\eta)}{\vartheta_c(\eta)}\frac{\vartheta_{a+b-c}(z_2+\eta)}{\vartheta_{a-c}(\eta)},$$

where

$$\lambda=\exp\left(-\frac{2\pi i}{n^2}b_1(a_2+2b_2)-2i\pi\frac{b_1^2}{n^2}\tau-4\pi i\frac{b_1}{n}\eta-2\pi i\frac{b_1}{n}z_2\right).$$

Hence

$$F_{ab}\left(\frac{b_1}{n}\tau+\frac{b_2}{n}\right)=\lambda\sum_{c\in G_n}\frac{\vartheta_{a+b-c}(z_2+\eta)\vartheta_{b+c}(z_2+\eta)}{\vartheta_c(\eta)\vartheta_{a-c}(\eta)}\left[\omega^{\langle a-c,b\rangle}-\omega^{\langle c,b\rangle}\right].$$

Letting, $c\mapsto a-c$ in the term with the factor $\omega^{\langle a-c,b\rangle}$, we conclude that $F_{ab}\left(\frac{b_1}{n}\tau+\frac{b_2}{n}\right)=0$. □

Let $f_{ab}(z)$, $a,b\in(1/n)\mathbf{Z}$, be an entire function of z, not identically zero, with the transformation properties

$$f_{ab}(z+1)=\exp(2\pi i a)f_{ab}(z) \tag{4.6}$$

and

$$f_{ab}(z+\tau)=\exp(-2\pi i b)\exp(-2\pi i(z_2+\tau)-4\pi i(z+\eta))f_{ab}(z).$$

We have

Proposition 4.3. The function $f_{ab}(z)$ has exactly two zeros in the fundamental region for $\mathbf{C}/\Lambda_\tau$ and their sum is equal to

$$-a\tau-(b+z_2+2\eta)\pmod{\Lambda_\tau}. \tag{4.7}$$

Proof. This is a standard contour argument, i.e. look at

$$\frac{1}{2\pi i}\int_\Gamma\frac{d}{dz}\log f_{ab}(z)\,dz=\frac{1}{2\pi i}\int_0^1\frac{d}{dx}\log\frac{f_{ab}(x)}{f_{ab}(x+\tau)}=2$$

to conclude there are exactly two zeros, and look at

$$\frac{1}{2\pi i}\int_\Gamma z\,d\log f_{ab}(z)=\frac{1}{2\pi i}\int_0^1\left\{x\,d\log\frac{f_{ab}(x)}{f_{ab}(x+\tau)}-\tau\,d\log f_{ab}(x+\tau)\right\}$$
$$+\frac{1}{2\pi i}\int_0^\tau\left\{x\,d\log\frac{f_{ab}(x+1)}{f_{ab}(x)}+d\log f_{ab}(x+1)\right\}$$

to conclude the sum is given by (4.7). The contour Γ is the boundary of the fundamental paralleogram.

Putting all this together, we have

Theorem 4.4. If S^{12}, S^{13}, and S^{23} are defined by (4.2), then the Yang–Baxter equation (4.3) is satisfied for all $z_1, z_2 \in E_\tau$.

Proof. Observe that $F_{ab}(z)$ is an entire function of z with transformation properties (4.6). In view of propositions 4.2 and 4.3 we must conclude $F_{ab}(z)$ is identically zero. The transformation properties (4.6) follow from

$$w_a(z+1) = \frac{(S_1\vartheta_a)(z+\eta)}{\vartheta_a(\eta)} = \exp\left(2\pi i \frac{a_1}{n}\right) w_a(z),$$

$$w_a(z+\tau) = \exp(-i\pi\tau - 2\pi i(z+\eta)) \frac{(T_1\vartheta_a)(z+\eta)}{\vartheta_a(\eta)}$$

$$= \exp(-i\pi\tau - 2\pi i(z+\eta)) \exp\left(-2\pi i \frac{a_2}{n}\right) w_a(z),$$

where $a \in G_n$. □

The proof of theorem 4.4 is short though not exactly instructive. It is the author's belief that $F_{ab}(z_1, z_2) = 0$ can be derived as a special case of the generalized Riemann theta identity (see Mumford [15, page 212]) for some choice of T. It would be very nice if the Yang–Baxter equation came from the identity (3.7); however, the author has been unable to show this. To substantiate this conjecture I point out that for $n = 2$, the Yang–Baxter equations are precisely a special case of the Riemann quartic identity. Explicitly, for $n = 2$, the YB equation is

$$\begin{aligned}\left[(-1)^{\langle a,b\rangle} - 1\right]\Big[& w_{00}(z_1) w_a(z_1+z_2) w_b(z_2) + (-1)^{b_1} w_{01}(z_1) w_{a_1,a_2+1}(z_1+z_2) w_{b_1,b_2+1}(z_2) \\ & + (-1)^{b_2} w_{10}(z_1) w_{a_1+1,a_2}(z_1+z_2) w_{b_1+1,b_2}(z_2) \\ & + (-1)^{b_1+b_2} w_{11}(z_1) w_{a_1+1,a_2+1}(z_1+z_2) w_{b_1+1,b_2+1}(z_2)\Big] = 0. \end{aligned} \tag{4.8}$$

For $(-1)^{\langle a,b\rangle} \neq 1$ we obtain six equations (see Baxter [2, page 210]). We multiply each of these six equations by $\vartheta_{00}(\eta)\vartheta_{01}(\eta)\vartheta_{10}(\eta)\vartheta_{11}(\eta)$ (we are using the standard notation $\vartheta_{11}(z) = \vartheta_{\frac{1}{2}\frac{1}{2}}(z)$, etc. for the half-thetas) and obtain six quartic identities for the half-thetas. These equations are precisely a special case of the Riemann quartic theta formulae. For example, for $a = (0,1)$ and $b = (1,0)$, (4.8) becomes

$$\begin{aligned} & \vartheta_{00}(z_1+\eta)\vartheta_{01}(z_1+z_2+\eta)\vartheta_{10}(z_2+\eta)\vartheta_{11}(\eta) - \vartheta_{01}(z_1+\eta)\vartheta_{00}(z_1+z_2+\eta)\vartheta_{11}(z_2+\eta) \\ & \quad \times \vartheta_{10}(\eta) + \vartheta_{10}(z_1+\eta)\vartheta_{11}(z_1+z_2+\eta)\vartheta_{00}(z_2+\eta)\vartheta_{01}(\eta) \\ & \qquad - \vartheta_{11}(z_1+\eta)\vartheta_{10}(z_1+z_2+\eta)\vartheta_{01}(z_2+\eta)\vartheta_{00}(\eta) = 0. \end{aligned} \tag{4.9}$$

Comparing this with equation R_{20} of Mumford [15, page 20] and letting $x \mapsto z_1 + \eta$, $y \to z_1 + z_2 + \eta$, $u \to z_2 + \eta$, $v \to \eta$, we see (4.9) and R_{20} are identical. The other five equations resulting from (4.8) can be obtained by similar reductions of equations R_{19}, R_{20}, and R_{21} of Mumford. Observe that the RHS of the Riemann theta formulae vanishes in the above special cases since in each case $\vartheta_{11}(z)$ is evaluated at $z = 0$. I leave it as an exercise to show that for $n = 2$ the YB equations are also a special case of the Riemann theta formulae based upon the choice $T = \frac{1}{2}(J_4 - 2I_4)$.

5. Polynomial equation for the Boltzmann coordinates

By specializing the Riemann quartic theta formulae, all the identities reduce to the two independent equations E_1 and E_2 (such equations as R_{19}–R_{21} in Mumford which contain the Yang–Baxter equations for $n=2$ reduce to the trivial statement zero equals zero). As mentioned in section 3, equations E_1 and E_2 define the embedded elliptic curve in $\mathbb{P}^3$. Baxter, in his analysis of the $n=2$ case, was led to the two polynomial equations

$$\frac{cd}{ab}=c_1, \tag{B$_1$}$$

$$\frac{a^2+b^2-c^2-d^2}{2ab}=c_2, \tag{B$_2$}$$

where c_1 and c_2 are constants. In terms of the notation of section 2, $a=w_{00}+w_{01}$, $b=w_{00}-w_{01}$, $c=w_{10}+w_{11}$, and $d=w_{10}-w_{11}$.

Theorem 5.1. Assume $n=2$ and that w_α, $\alpha\in G_2$, are given by (4.1). Then the Jacobi equations (E_1) and (E_2) imply the Baxter equations (B_1) and (B_2).

Proof.

(i) E_1 and $E_2\Rightarrow B_1$:
We write $a_{ij}=\vartheta_{ij}(\eta)$, $i,j\in G_2$,

$$\frac{cd}{ab}=\frac{w_{10}^2-w_{11}^2}{w_{00}^2-w_{01}^2}=\left[\frac{a_{00}a_{01}}{a_{10}a_{11}}\right]^2\left[\frac{a_{11}^2\vartheta_{10}^2(x+\eta)-a_{10}^2\vartheta_{11}^2(x+\eta)}{a_{01}^2\vartheta_{00}^2(x+\eta)-a_{00}^2\vartheta_{01}^2(x+\eta)}\right].$$

Using E_1 and E_2 with $x\to x+\eta$, we express $\vartheta_{11}^2(x+\eta)$ and $\vartheta_{00}^2(x+\eta)$ in terms of $\vartheta_{01}^2(x+\eta)$ and $\vartheta_{10}^2(x+\eta)$ (and the theta nulls). The $(a_{ij})^2$ also satisfy E_1 and E_2. Using E_1 and E_2 a second time, we conclude the second factor in the above equation is 1 (this could be obtained more quickly if we simply used the addition formulae). Thus

$$\frac{cd}{ab}=\left[\frac{\vartheta_{00}(\eta)\vartheta_{01}(\eta)}{\vartheta_{10}(\eta)\vartheta_{11}(\eta)}\right]^2$$

giving the constant c_1 in terms of the fixed η and τ.

(ii) To show E_1 and $E_2\Rightarrow B_2$, we first calculate

$$\frac{c_2+c_1+1}{c_2-c_1-1}=\frac{w_{00}^2-w_{11}^2}{w_{01}^2-w_{10}^2}.$$

A similar argument shows that this expression equals

$$-\left(\frac{a_{01}a_{10}}{a_{00}a_{11}}\right)^2$$

and hence (B_2) follows. □

For arbitrary $n\geq 2$ the Boltzmann coordinates w_α will satisfy polynomial equations. To obtain these equations we need only to obtain the homogeneous polynomials giving the embedded elliptic curve coming from the Lefschetz embedding. In these equations, recall $x_{ij}=\vartheta_{i/n,j/n}(x,\tau)$, we simply let $x\to x+\eta$ and use (4.1) to obtain the polynomials in w_α. As shown above, the $n=2$ case of (3.9) reduce to the Jacobi equations E_1 and E_2, which in turn imply the Baxter equations B_1 and B_2. In this sense (3.9) under the transformation $x\to x+\eta$, $x_{ij}\to\vartheta_{i/n,j/n}(\eta)w_{ij}$ give the natural generalization for the $\mathbb{Z}_n$ Baxter model.

Remarks.

(1) The $\vartheta_\varepsilon(\eta)$ also satisfy (3.9). Presumably the theta nulls may be eliminated in the polynomials for w_α with only the constants $\vartheta_\varepsilon(\eta)$ appearing.

(2) As mentioned above, we have not proved that (3.9) gives enough equations to give the embedded elliptic curve.

(3) For $n=2$ and each point $x\in\mathbb{P}^3$, $x=(x_0,x_1,x_2,x_3)$, $x_j\neq 0$, it is clear from (B_1) and (B_2) that thru this point passes an embedded elliptic curve. For $n>2$ this is not the case. It would be of interest to characterize the algebraic surface in $\mathbb{P}^{n^2-1}$ on which the embedded curves lie. It would be important to be able to produce an explicit basis for the associated polynomial ideal *before* the parametrization in terms of theta func-

tions. In other examples, this typically requires long calculations.

6. The case $n = 3$

The ideas and calculations of this section follow Bianchi [5], Krazer [13], and Klein and Fricke [11]. A 20th century reference is Hulek [9]. Let $x \in E_\tau$ and consider the map from E_τ to $\mathbb{P}^2$ given by

$$\begin{aligned} x_0 &= \vartheta_{00}(x)\vartheta_{0\frac{1}{3}}(x)\vartheta_{0\frac{2}{3}}(x), \\ x_1 &= \vartheta_{\frac{1}{3}0}(x)\vartheta_{\frac{1}{3}\frac{1}{3}}(x)\vartheta_{\frac{1}{3}\frac{2}{3}}(x), \\ x_2 &= \vartheta_{\frac{2}{3}0}(x)\vartheta_{\frac{2}{3}\frac{1}{3}}(x)\vartheta_{\frac{2}{3}\frac{2}{3}}(x). \end{aligned} \tag{6.1}$$

In the language of theta functions, x_0, x_1, and x_2 are third order theta functions of characteristic $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (see Krazer [13, pg. 36]). Such third order thetas form a vector space of dimension 3 and $\{x_0, x_1, x_2\}$ is a basis. The functions x_i can also be thought of as sections of the line bundle L^3 [8], but we will not need this formulation here. What we do need is that the map (6.1) is an embedding of E_τ into $\mathbb{P}^2$ (see, for example, [8, pg. 317]).

Recalling the results from section 3 we can easily verify that

$$\begin{aligned} S_{\frac{1}{3}}x_0 &= x_0, \quad T_{\frac{1}{3}}x_0 = \lambda x_1; \\ S_{\frac{1}{3}}x_1 &= \omega x_1, \quad T_{\frac{1}{3}}x_1 = \lambda x_2; \\ S_{\frac{1}{3}}x_2 &= \omega^2 x_2, \quad T_{\frac{1}{3}}x_2 = \lambda x_0, \end{aligned} \tag{6.2}$$

where $\omega \equiv \exp(\frac{2}{3}\pi i)$ and $\lambda \equiv \omega^{-1}\exp(-i\pi\frac{2}{9}\tau - 4\pi i x/3)$. Thus on $\mathbb{P}^2$ the Heisenberg elements $S_{\frac{1}{3}}$ and $T_{\frac{1}{3}}$ can be identified with the 3×3 matrices g and h^{-1}, respectively, of (2.1) for the case $n = 3$.

Now the Heisenberg group H_3 maps E_τ to itself such that the nine torsion points of order 3 are permuted amongst themselves (in geometrical terms, these nine points are the nine inflection points of the elliptic curve). Since this action extends to $\mathbb{P}^2$ we must have that in $\mathbb{P}^2$ the embedded curve is invariant under H_3 (that is, a point on the curve is taken to a point on the curve). Now it is easy to see that no quadratic equation in x_0, x_1 and x_2 can satisfy this requirement. The cubic homogeneous polynomials that are invariant under $S_{\frac{1}{3}}$ are x_0^3, x_1^3, x_2^3 and $x_0x_1x_2$. Writing

$$f(x_0, x_1, x_2) = Ax_0^3 + Bx_1^3 + Cx_2^3 + Dx_0x_1x_2,$$

we see that $A = B = C$ follows from invariance under $T_{\frac{1}{3}}$. It is customary to write $D = 6aA$ to obtain the elliptic normal curve $\mathscr{C}_3$ (Hesse form)

$$x_0^3 + x_1^3 + x_2^3 + 6ax_0x_1x_2 = 0 \tag{6.3}$$

that the theta functions given by (6.1) must satisfy. The parameter a will be a function of τ and in fact is an elliptic modular function of level 3. The classic reference on modular functions is Klein and Fricke [11]; a modern reference is Schoeneberg [18]. Recalling (3.1) and letting v_α denote the normalized Boltzmann coordinates $\vartheta_\alpha(\eta)w_\alpha$, we see that we have the single equation

$$\begin{aligned} &v_{00}^3v_{01}^3v_{02}^3 + v_{10}^3v_{11}^3v_{12}^3 + v_{20}^3v_{21}^3v_{22}^3 \\ &\quad + 6av_{00}v_{01}v_{02}v_{10}v_{11}v_{12}v_{20}v_{21}v_{22} = 0. \end{aligned} \tag{6.4}$$

Of course, the v_α will satisfy more independent equations. These additional equations arise by considering different bases $\{x_0', x_1', x_2'\}$ for the space of third order thetas of characteristic $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It is easy to check that for α, $\beta \in G_3$, $\beta \neq (0,0)$, α, 2α that

$$\begin{aligned} x_0' &= \vartheta_{00}(x)\vartheta_\alpha(x)\vartheta_{2\alpha}(x), \\ x_1' &= \vartheta_\beta(x)\vartheta_{\alpha+\beta}(x)\vartheta_{\beta+2\alpha}(x), \\ x_2' &= \vartheta_{2\beta}(x)\vartheta_{\alpha+2\beta}(x)\vartheta_{2\beta+2\alpha}(x), \end{aligned} \tag{6.5}$$

is a basis for third order thetas of characteristic $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (recall convention: $\vartheta_\alpha(x) \equiv \vartheta_{\alpha_1/3\alpha_2/3}(x)$, $\alpha \in G_3$). The choice $\alpha = (0,1)$ and $\beta = (1,0)$ corresponds to (6.1).

We now examine (6.5) in the case $\alpha \neq (0,0)$; thus 0, α, and 2α are distinct elements in G_3.

Define

$$S' = T_{\alpha_1/3} S_{\alpha_2/3}, \quad T' = T_{\beta_1/3} S_{\beta_2/3},$$

so that when $\alpha = (0,1)$ and $\beta = (1,0)$ we have $S' = S_{\frac{1}{3}}$ and $T' = T_{\frac{1}{3}}$. Using the transformation properties of the theta functions we have

$$S'x_0' = \lambda x_0',$$
$$S'x_1' = \lambda \omega^{\langle \beta, \alpha \rangle} x_1',$$
$$S'x_2' = \lambda \omega^{2\langle \beta, \alpha \rangle} x_2',$$

where λ is a common exponential factor (never vanishing) and

$$T'x_0' = \lambda' \omega^{-\beta_1\beta_2} x_1',$$
$$T'x_1' = \lambda' \omega^{-2\beta_1\beta_2} x_2',$$
$$T'x_2' = \lambda' x_0'.$$

The elements S' and T' generate H_3. We see as before that the invariant polynomial must be of the form

$$f'(x_0', x_1', x_2') = (x_0')^3 + (x_1')^3 + (x_2')^3 + 6a'x_0'x_1'x_2'. \tag{6.6}$$

There are four subgroups of G_3 that are isomorphic to $\mathbf{Z}_3$; $A_0 = \{(0,0),(0,1),(0,2)\}$, $A_1 = \{(0,0),(1,0),(2,0)\}$, $A_2 = \{(0,0),(1,1),(2,2)\}$ and $A_3 = \{(0,0),(1,2),(2,1)\}$. Observe that if $\alpha \in A_l$ then so is 2α. Hence the condition $\beta \neq (0,0)$, α, 2α can be given as if $\alpha \in A_l$, then $\beta \notin A_l$, $\beta \neq (0,0)$. Note further that choosing different (nonzero) elements of a fixed subgroup A_l for α leads to the same basis (6.5). Hence there are four distinct choices of α and six choices for β. For a fixed α the six different choices of β correspond to either a different ordering of $\{x_0', x_1', x_2'\}$ or certain coordinates are multiplied by either ω or ω^2. Hence for fixed α, we get the transformations

$$a' = \omega a \quad \text{and} \quad a' = \omega^2 a. \tag{6.7}$$

As far as distinct equations for the normalized Boltzmann coordinates v_α we get four distinct equations corresponding to the four distinct choices of α and any representative choice of β.

But, of course, these change of bases are related by linear transformations on (x_0, x_1, x_2). Thus there must exist a theta identity connecting (6.1) and (6.5). This is the meaning of eq. (98) on page 395 of Krazer [13]. In the notation here this equation reads

$$\begin{aligned} x_0 &= k(x_0' + x_1' + x_2'), \\ \omega^2 x_1 &= k(x_0' + \omega^2 x_1' + \omega x_2'), \\ \omega x_2 &= k(x_0' + \omega x_1' + \omega^2 x_2'), \end{aligned} \tag{6.8}$$

where we denote by a prime, the choice $\alpha = (1,0)$, $\beta = (0,1)$ in (6.5). The constant k is expressible in terms of the theta nulls, but its exact form need not concern us here. Using the fact that $\omega^2 + \omega + 1 = 0$, we see that substituting (6.8) into (6.3) the a is related to the a' of (6.6) by

$$a' = \frac{1-a}{1+2a}, \quad \text{for } \alpha = (1,0),\ \beta = (0,1) \tag{6.9}$$

(a useful identity in these calculations is $(A+B+C)\ (A+\omega B+\omega^2 C)\ (A+\omega^2 B+\omega C) = A^3+B^3+C^3-3ABC$). Eq. (6.8) can be viewed as a theta identity. It is in fact derived from the Riemann theta formulae (3.7) for the case $n=3$. The details of this can be found in Krazer's book or complete details in Krazer [12]. There are two more equations we get by setting $\alpha = (1,1)$, $\beta = (0,1)$ and $\alpha = (1,2)$, $\beta = (0,1)$ in (6.5). Again there must exist a linear transformation and this will lead to a transformation on the corresponding a'. We have (use formula I_1 of Krazer [12])

$$a \to \frac{1-\omega a}{1+2\omega a}, \quad \text{for } \alpha = (1,1),\ \beta = (0,1) \tag{6.10}$$

and

$$a \to \frac{1-\omega^2 a}{1+2\omega^2 a}, \quad \text{for } \alpha = (1,2),\ \beta = (0,1). \tag{6.11}$$

The transformations S: $a \to \omega a$ and T: $a \to (1-$

$a)/(1+2a)$ generate the tetrahedron group. The group generated by S and T is isomorphic to $\Gamma/\Gamma(3)$ (see Schoeneberg [18]).

We now turn to the case $\alpha=(0,0)$ in (6.5). Writing y_i' for this case, (6.5) becomes (and changing the β label to α'.)

$$y_0' = \vartheta_{00}^3(x), \quad y_1' = \vartheta_{\alpha}^3(x), \quad y_2' = \vartheta_{2\alpha}^3(x), \tag{6.12}$$

where there are four distinct choices for α; namely, $\alpha \neq (0,0)$ and one representative from each subgroup A_l, $l=0,1,2,3$. We will choose $\alpha=(0,1)$, $(1,0)$, $(1,1)$, and $(1,2)$. We now fix $\beta=(0,1)$ in the basis (6.5). Since (6.5) defines a basis there exists a linear transformation, depending upon α, that takes (6.12) to (6.5). Krazer gives these maps in eq. (107), page 397 of [13]. It reads

$$\begin{aligned} X_1 &= k_1(Y_1+Y_2+Y_3), \\ X_2 &= k_2(Y_1+\omega Y_2+\omega^2 Y_3), \\ X_3 &= k_2(Y_1+\omega^2 Y_2+\omega Y_3), \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} X_1 &= \vartheta_{00}(x)\vartheta_{\alpha}(x)\vartheta_{-\alpha}(x), \\ X_2 &= \vartheta_{\beta}(x)\vartheta_{\beta+\alpha}(x)\vartheta_{\beta-\alpha}(x)\omega^{\beta_1\beta_2} \\ X_3 &= \vartheta_{-\beta}(x)\vartheta_{-\beta+\alpha}(x)\vartheta_{-\beta-\alpha}(x)\omega^{\beta_1\beta_2}, \end{aligned} \tag{6.14a}$$

$$\begin{aligned} Y_1 &= \vartheta_{00}^3(x), \\ Y_2 &= \vartheta_{\alpha}^3(x)\omega^{\alpha_1\alpha_2}, \\ Y_3 &= \vartheta_{-\alpha}^3(x)\omega^{\alpha_1\alpha_2}, \end{aligned} \tag{6.14b}$$

with

$$k_2 = -2ak_1 \tag{6.14c}$$

and k_1 is expressible in terms of theta nulls depending on the theta characteristic α. Again (6.13) follows from the Riemann theta formulae (3.7) though here more work is required to see this [12, 13]. Choosing $\alpha=(0,1)$ then X_1, X_2 and X_3 are essentially x_0, x_1 and x_2 ($X_1 \to x_0$, $X_2 \to \omega^2 x_1$, $X_3 \to \omega x_2$) and $Y_1 \to y_0$, $Y_2 \to y_1$, and $Y_3 \to y_2$. Working in this basis we compute

$$x_0^3 + x_1^3 + x_2^3 + 6ax_0x_1x_2 \tag{6.15}$$

using (6.13). The polynomial that results is (up to a nonzero constant)

$$(y_0+y_1+y_2)^3 + 6by_0y_1y_2,$$

where

$$b = -\frac{36a^3}{1+8a^3} \quad (\alpha=(0,1)). \tag{6.16}$$

Now setting $\alpha=(1,0)$, (6.13) gives the map from (6.12) to (6.5). The resulting b' is related to the corresponding a' exactly by (6.16) since the polynomial in x_0', x_1', x_2' is of the form (6.15). We hence need only know how this a' is related to a. From the above work we see that $a' = Ta$ so that

$$b \to -\frac{4(1-a)^3}{1-2a+4a^2} \quad (\alpha=(1,0)). \tag{6.17a}$$

Similarly,

$$b \to -\frac{4(1-\omega a)^3}{1-2\omega a+4\omega^2 a^2} \quad (\alpha=(1,1)), \tag{6.17b}$$

$$b \to -\frac{4(1-\omega^2 a)^3}{1-2\omega^2 a+4\omega a^2} \quad (\alpha=(1,2)). \tag{6.17c}$$

This produces an additional four polynomials and in all eight polynomials.

We now collect these eight homogeneous polynomials:

$$
\begin{aligned}
f_1 &\equiv x_{00}^3 x_{01}^3 x_{02}^3 + x_{10}^3 x_{11}^3 x_{12}^3 + x_{20}^3 x_{21}^3 x_{22}^3 \\
&\quad + 6a x_{00} x_{01} x_{02} x_{10} x_{11} x_{12} x_{20} x_{21} x_{22}, \\
f_2 &\equiv x_{00}^3 x_{10}^3 x_{20}^3 + x_{01}^3 x_{11}^3 x_{21}^3 + x_{02}^3 x_{12}^3 x_{22}^3 \\
&\quad + 6\frac{1-a}{1+2a} x_{00} x_{01} x_{02} x_{10} x_{11} x_{12} x_{20} x_{21} x_{22}, \\
f_3 &\equiv x_{00}^3 x_{11}^3 x_{22}^3 + x_{01}^3 x_{12}^3 x_{20}^3 + x_{02}^3 x_{10}^3 x_{21}^3 \\
&\quad + 6\frac{1-\omega a}{1+2\omega a} \omega^2 x_{00} x_{01} x_{02} x_{10} x_{11} x_{12} x_{20} x_{21} x_{22}, \\
f_4 &\equiv x_{00}^3 x_{12}^3 x_{21}^3 + x_{01}^3 x_{10}^3 x_{22}^3 + x_{02}^3 x_{11}^3 x_{20}^3 \\
&\quad + 6\frac{1-\omega^2 a}{1+2\omega^2 a} \omega x_{00} x_{01} x_{02} x_{10} x_{11} x_{12} x_{20} x_{21} x_{22}, \\
f_5 &\equiv \left(x_{00}^3 + x_{01}^3 + x_{02}^3\right)^3 \\
&\quad - 216\frac{a^3}{1+8a^3} x_{00}^3 x_{01}^3 x_{02}^3, \\
f_6 &\equiv \left(x_{00}^3 + x_{10}^3 + x_{20}^3\right)^3 \\
&\quad - 24\frac{(1-a)^3}{1-2a+4a^2} x_{00}^3 x_{10}^3 x_{20}^3, \\
f_7 &\equiv \left(x_{00}^3 + x_{11}^3 + x_{22}^3\right)^3 \\
&\quad - 24\frac{(1-\omega a)^3}{1-2\omega a+4\omega^2 a^2} x_{00}^3 x_{11}^3 x_{22}^3, \\
f_8 &\equiv \left(x_{00}^3 + x_{12}^3 + x_{21}^3\right)^3 \\
&\quad - 24\frac{(1-\omega^2 a)^3}{1-2\omega^2 a+4\omega a^2} x_{00}^3 x_{12}^3 x_{21}^3.
\end{aligned}
\tag{6.18}
$$

The modular function a of level 3 can be expressed in terms of theta nulls. There are many different looking expressions for a (there are many theta null identities!). The one in Krazer [13] is

$$
a = a(\tau) = -\frac{1}{2}\frac{\vartheta_{00}^3 + \vartheta_{0\frac{1}{3}}^3 + \vartheta_{0\frac{2}{3}}^3}{\vartheta_{\frac{1}{3}0}^3 + \vartheta_{\frac{1}{3}\frac{1}{3}}^3 + \vartheta_{\frac{1}{3}\frac{2}{3}}^3}.
$$

Note there is a misprint in the middle of eq. (94) on page 394 of Krazer [13]. In Krazer's notation replace

$$
\vartheta^3\{\beta\} + \tau\vartheta^3\{\beta+\alpha\} + \tau^2\vartheta^3\{\beta-\alpha\}
$$

by

$$
\vartheta^3\{\beta\} + \tau^{\langle\alpha,\beta\rangle}\vartheta^3\{\beta+\alpha\} + \tau^{2\langle\alpha,\beta\rangle}\vartheta^3\{\beta-\alpha\},
$$

where $\langle\alpha,\beta\rangle \equiv \alpha_1\beta_2 - \alpha_2\beta_1$. Without this correction the small q expansion of the two expressions occurring in eq. (94) of Krazer do not agree. Also the expression for $a(\tau)$ now agrees with eq. III of Krazer [12]. Note that we have substituted $\vartheta_{0\frac{1}{3}}^3 = \vartheta_{0-\frac{1}{3}}^3 = \vartheta_{0\frac{2}{3}}^3$ in Krazer to make $a(\tau)$ look more symmetric. These equations express the τ dependence in terms of the single function $a(\tau)$ compared with the nine theta nulls appearing in (3.7). The price is that they are harder to generalize and their degree is higher than in (3.7).

As a comparison we recall the $n=2$ case

$$
\begin{aligned}
f_1 &\equiv x_{00}^2 - k'(\tau)x_{01}^2 - k(\tau)x_{10}^2, \\
f_2 &\equiv x_{11}^2 - k(\tau)x_{01}^2 + k'(\tau)x_{10}^2,
\end{aligned}
$$

where

$$
k^2(\tau) + k'^2(\tau) = 1, \quad k(\tau) \equiv \frac{\vartheta_{01}^2}{\vartheta_{00}^2}.
$$

Thus the function $a(\tau)$ for $n=3$ plays an analogous role to $k(\tau)$ in the $n=2$ case. Also from (6.18) we can answer (for $n=3$) the question posed at the end of section 5. Namely, by using say f_1 to solve for a, we can eliminate $a(\tau)$ in $f_2, \dots, f_8$. These remaining polynomials define a two-dimensional surface in $\mathbb{P}^8$ on which the renormalized Boltzmann coordinates v_α must lie in order for there to exist commuting transfer matrices.

7. The cases $n > 3$

In this section we indicate the method of reducing the number of theta nulls appearing in the set of homogeneous equations for the normalized Boltzmann coordinates. The method is a generalization of the ideas presented in the previous case $n = 3$, though the details will be considerably more involved. For this reason we present only a sketch of the steps involved. The problem naturally divides into two cases, n even and n odd. For n odd the generalization of (6.1) is immediate:

$$x_j = \prod_{k=0}^{n-1} \vartheta_{j/n,k/n}(x,\tau), \quad j = 0,1,\ldots,n-1,\ n \equiv 1(0). \tag{7.1}$$

For n even the map from E_τ to $\mathbb{P}^{n-1}$ has to be slightly modified as was first shown by Hurwitz [10] in the case $n = 4$. We restrict the discussion here to n odd; and though not necessary at first, we further restrict the discussion to n prime. For details of the general case of elliptic normal curves see volume 2 of Klein and Fricke [11].

We have for all $j = 0,1,\ldots,n-1$

$$S_{1/n}x_j = \omega^j x_j, \quad T_{1/n}x_j = \lambda x_{j+1}, \quad x_{j+n} = x_j, \tag{7.2}$$

where $\omega \equiv \exp(2\pi i/n)$ and λ is a nowhere vanishing function independent of j. Thus H_n lifts to $\mathbb{P}^{n-1}$ as before, and we have the crucial fact that the embedded curve $\mathscr{C}_n$ in $\mathbb{P}^{n-1}$ is invariant under this action. For $n = 3$ we determined a single polynomial in x_0, x_1, and x_2 that defined $\mathscr{C}_3$. Now we seek a collection of homogeneous polynomials in $x_0,\ldots,x_{n-1}$ that defines the algebraic curve $\mathscr{C}_n$. That these polynomials can be chosen to be quadrics goes back to Bianchi [5] who considered both $n = 3$ and $n = 5$ in considerable detail.

Though a more modern proof can be devised (see Hulek [9]), in the spirit of the Riemann theta formulae we follow Krazer [13, pges 399–404] to show the existence of a set of quadrics $\{Q_i\}$ defining $\mathscr{C}_n$ follows essentially from the Riemann quartic identity.

The first step is to realize that (7.1) can be written as a single theta function and then to use the Riemann theta identity for this theta. Start with the identity

$$\vartheta_{ab}(x,\tau)\vartheta_{a,b+1/n}(x,\tau)\cdots\vartheta_{a,b+(n-1)/n}(x,\tau) = c\vartheta_{a,nb+\frac{1}{2}(n-1)}(nx,n\tau), \tag{7.3}$$

where c is independent of x, a, and b. A quick proof of (7.3) follows the standard Liouville argument (look at the ratio and observe the n zeros in E_τ for both the numerator and denominator agree. To show c is independent of a and b look at the transformation of the ratio under $x \to x + 1/n$ and $x \to x + 1/n\tau$).

Now it is a classical result of the Riemann quartic identity (details are in Krazer) that

$$\begin{aligned}&\vartheta_{11}(t+u,\tau)\vartheta_{11}(t-u,\tau)\vartheta_{11}(v+w,\tau)\vartheta_{11}(v-w,\tau)\\ &\quad+\vartheta_{11}(t+v,\tau)\vartheta_{11}(t-v,\tau)\vartheta_{11}(w+u,\tau)\vartheta_{11}(w-u,\tau)\\ &\quad+\vartheta_{11}(t+w,\tau)\vartheta_{11}(t-w,\tau)\vartheta_{11}(u+v,\tau)\vartheta_{11}(u-v,\tau)=0\end{aligned} \tag{7.4}$$

for all $t,u,v,w \in E_\tau$ (here $\vartheta_{11}(z,\tau)$ is the half-theta $\vartheta_{\frac{1}{2}\frac{1}{2}}(z,\tau)$). It is somewhat more convenient to write

(7.4) in terms of $\vartheta(z,\tau)$ (I have written both out since (7.4) is more readily recognizable):

$$\begin{aligned}
&e^{2\pi i v}\vartheta(t+u+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(t-u+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(v+w+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(v-w+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\\
&\quad+e^{2\pi i w}\vartheta(t+v+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(t-v+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(w+u+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(w-u+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\\
&\quad+e^{2\pi i u}\vartheta(t+w+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(t-w+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(u+v+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)\vartheta(u-v+\tfrac{1}{2}\tau+\tfrac{1}{2},\tau)=0,
\end{aligned}\tag{7.5}$$

where common nowhere vanishing factors have been removed. Now let

$$\begin{aligned}
t&=\tfrac{1}{2}nx+\frac{\alpha_1}{n}\tau^*-\tfrac{1}{4}(\tau+1),\\
u&=\tfrac{1}{2}nx+\frac{\alpha_2}{n}\tau^*-\tfrac{1}{4}(\tau+1),\\
v&=\tfrac{1}{2}nx+\frac{\alpha_3}{n}\tau^*-\tfrac{1}{4}(\tau+1),\\
w&=\tfrac{1}{2}nx+\frac{\alpha_4}{n}\tau^*-\tfrac{1}{4}(\tau+1),
\end{aligned}\tag{7.6}$$

in (7.5), remove common nowhere vanishing factors, let $\tau\to n\tau$, and then set $\tau^*=\tau$. The resulting identity reads

$$\begin{aligned}
&e^{2\pi i\alpha_3\tau/n}\vartheta\Big(nx+\frac{\alpha_1+\alpha_2}{n}\tau,n\tau\Big)\vartheta\Big(\frac{\alpha_1-\alpha_2}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\vartheta\Big(nx+\frac{\alpha_3+\alpha_4}{n}\tau,n\tau\Big)\\
&\times\vartheta\Big(\frac{\alpha_3-\alpha_4}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\\
&+e^{2\pi i\alpha_4\tau/n}\vartheta\Big(nx+\frac{\alpha_1+\alpha_3}{n}\tau,n\tau\Big)\vartheta\Big(\frac{\alpha_1-\alpha_3}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\vartheta\Big(nx+\frac{\alpha_2+\alpha_4}{n}\tau,n\tau\Big)\\
&\times\vartheta\Big(\frac{\alpha_4-\alpha_2}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\\
&+e^{2\pi i\alpha_2\tau/n}\vartheta\Big(nx+\frac{\alpha_1+\alpha_4}{n}\tau,n\tau\Big)\vartheta\Big(\frac{\alpha_1-\alpha_4}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)\vartheta\Big(nx+\frac{\alpha_2+\alpha_3}{n}\tau,n\tau\Big)\\
&\times\vartheta\Big(\frac{\alpha_2-\alpha_3}{n}\tau+\tfrac{1}{2}n\tau+\tfrac{1}{2},n\tau\Big)=0.
\end{aligned}\tag{7.7}$$

Since n is odd, $\vartheta_{a,nb+\frac{1}{2}(n-1)}(z,\tau)=\exp(\pi i(n-1)a)\vartheta_{a,nb}(z,\tau)$. Hence

$$x_\alpha=\vartheta_{\alpha/n,0}(nx,n\tau),\quad \alpha=0,1,\ldots,n-1,$$

where we used (7.3). Comparing with (7.7) we conclude that

$$c_1x_{\alpha_1+\alpha_2}x_{\alpha_3+\alpha_4}+c_2x_{\alpha_1+\alpha_3}x_{\alpha_2+\alpha_4}+c_3x_{\alpha_1+\alpha_4}x_{\alpha_2+\alpha_3}=0,\tag{7.8}$$

where c_1, c_2, and c_3 are independent of x but depend upon α_1, α_2, α_3, and α_4. Eq. (7.8) is the desired set $\{Q_i\}$ of quadrics. Observe that the α_j must be distinct for (7.8) to be nontrivial (thus (7.8) does not say anything about the case $n=3$).

Once the existence of the quadrics is established, their form can more readily be deduced using the action of the Heisenberg group H_n. For example, choosing $\alpha_1=2$, $\alpha_2=3$, $\alpha_3=1$, and $\alpha_4=4$, one choice of (7.8) for $n=5$ is

$$Q_0(x_0,\dots,x_4)=x_0^2+ax_2x_3+bx_1x_4. \tag{7.9a}$$

Observe that Q_0 is invariant under $S_{1/5}$ and the action of $T_{1/5}$ generates an additional four polynomials:

$$\begin{aligned}Q_1(x_0,\dots,x_4)&=x_1^2+ax_3x_4+bx_2x_0,\\ Q_2(x_0,\dots,x_4)&=x_2^2+ax_4x_0+bx_3x_1,\\ Q_3(x_0,\dots,x_4)&=x_3^2+ax_0x_1+bx_4x_2,\\ Q_4(x_0,\dots,x_4)&=x_4^2+ax_1x_2+bx_0x_3.\end{aligned} \tag{7.9b}$$

Furthermore, as shown by Bianchi [5],

$$b=-\frac{1}{a}. \tag{7.10}$$

To see this, first recall that $\vartheta_{00}(x,\tau)$ is zero at $x=\frac{1}{2}(\tau+1)$. Thus $x_0(x)$ vanishes at this point, and evaluating Q_2 and Q_4 at $x=\frac{1}{2}(\tau+1)$ gives

$$a=-\frac{x_4^2(\frac{1}{2}\tau+\frac{1}{2})}{x_1(\frac{1}{2}\tau+\frac{1}{2})x_2(\frac{1}{2}\tau+\frac{1}{2})},\quad b=-\frac{x_2^2(\frac{1}{2}\tau+\tau)}{x_1(\frac{1}{2}\tau+\frac{1}{2})x_3(\frac{1}{2}\tau+\frac{1}{2})}. \tag{7.11}$$

To simplify (7.11) we write $x_4(\frac{1}{2}\tau+\frac{1}{2})=x_4(-\frac{1}{2}\tau-\frac{1}{2}+\tau+1)$ and $x_2(\frac{1}{2}\tau+\frac{1}{2})=x_2(-\frac{1}{2}\tau-\frac{1}{2}+\tau+1)$, use the facts $x_4(-u)=\omega x_1(u)$, $x_2(-u)=\omega^3x_3(u)$, and $x_j(u+\tau+1)=\lambda_5(u)x_j(u)$ where $\lambda_5(u)\equiv\exp(5(-\pi\tau-2\pi iu))$ to show that (7.11) implies (7.10). In (7.9) there are three independent quadrics; in fact,

$$x_1Q_3=x_2Q_0+ax_0Q_1-ax_3Q_2,\quad x_1Q_4=-ax_4Q_0+ax_2Q_1+x_0Q_2.$$

Thus the set $\{Q_i\}_{i=0}^2$ for $n=5$ defines the elliptic normal curve $\mathscr{C}_5$. In general there will be $n-2$ independent quadrics.

To obtain the additional homogeneous polynomials for the $x_{ij}=\vartheta_{i/n,j/n}(x,\tau)$ we first consider the generalization of (6.5):

$$x'_j=\prod_{k=0}^{n-1}\vartheta_{j\beta+k\alpha}(x,\tau),\quad j=0,1,\dots,n-1. \tag{7.12}$$

Since n is prime there are $n+1$ subgroups A_k of G_n that are isomorphic to $\mathbb{Z}_n$ (see Hulek [9]). They are generated by (0,1) and (1, l) for $l=0,1,\dots,n-1$. The choice $\alpha=(0,1)$ and $\beta=(1,0)$ in (7.12) corresponds to (7.1). We fix $\beta=(1,0)$ and choose α to be a representative from each of the subgroups A_k, $k\geq 1$. This change of basis from $x\to x'$ will in each case correspond to the generalization of (6.8) and its variants in the $n=3$ case. For $n=5$, Bianchi has shown that the quadrics $\{Q_i\}$ retain their form under this basis change and the a now transforms according to the icosahedral group. Thus for $n=5$ there is a single modular function, $a=a(\tau)$, of level 5 that appears in the quadrics Q_i. The final step is to define the basis

(use the fact that n is prime)

$$y_j' = \vartheta_{j\alpha}^n(x, \tau), \quad j = 0, 1, \ldots, n-1,$$

where again α is a representative from the subgroups A_k. Now we must generalize (6.13) to obtain the generalization of (6.16). This step is involved and will be left for future work.

Added remarks

After this paper was written, M. Jimbo brought to my attention the paper "Factorized S-matrices and generalized Baxter models" by A. Bovier (J. Math. Phys. 24 631–641 (1983)) in which a proof with complete details of the Belavin conjecture is given. The proof given in this paper in section 4, though not identical, is quite similar to that in Bovier.

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