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LETTER TO THE EDITOR

Universality classes of some aperiodic Ising models

Craig A Tracy†

Department of Mathematics, University of California, Davis, CA 95616, USA

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Abstract. Numerical evidence is given to show that different aperiodic Ising models can have different universality classes.

With the discovery of quasicrystals (Schechtman *et al* 1984) various physical phenomena, where the underlying lattice, interaction constants, masses, etc, are aperiodic, have been of considerable interest. One such example is to consider an Ising model on the Penrose lattice (Godrèche *et al* 1986, Aoyama and Odagaki 1987). Though this model has not been solved exactly, the approximate renormalisation group studies suggest that in the ferromagnetic case the specific heat has a logarithmic singularity and hence the ferromagnetic Penrose Ising model is predicted to be in the same universality class as the Onsager solution.

A mathematically simpler problem is a two-dimensional ferromagnetic Ising model on a square lattice where the vertical bond strengths $E_2(j)$ are chosen to be an aperiodic function in the column variable j. The energy of interaction for such an Ising model is

$$E=-E_1\sum_j\sum_k \sigma_{j,k}\sigma_{j,k+1}-\sum_j\sum_k E_2(j)\sigma_{j,k}\sigma_{j+1,k}. \tag{1}$$

If this system has a logarithmic singularity in the specific heat, then it is physically reasonable to expect that the system defined by (1) can be obtained in the limit $n\to\infty$ from an nth-order layered Ising model (Fisher 1968, Au-Yang and McCoy 1974, Hamm 1977) defined by the energy of interaction

$$E=-E_1\sum_{j=1}^{nM+1}\sum_{k=N+1}^{N}\sigma_{j,k}\sigma_{j,k+1}-\sum_{j=0}^{M-1}\sum_{l=1}^{n}\sum_{k=-N+1}^{N}E_2(l)\sigma_{nj+l,k}\sigma_{nj+l+1,k}. \tag{2}$$

The advantage of working with (2) is that Au-Yang and McCoy (1974) have shown for any nth-order layered Ising model (in the thermodynamic limit M, $N\to\infty$) when $T\to T_c$ the specific heat diverges as

$$c/k_B=-A(n;\{E_2\})\ln|1-T/T_c|+O(1) \tag{3}$$

where an explicit expression for the amplitude $A(n;\{E_2\})$ was derived by Au-Yang and McCoy.

We now describe a layered Ising model coming from an inflation rule. Let $\mathscr{A}=\{a,b,\ldots\}$ be a finite alphabet and let $W(\mathscr{A})$ denote the words of finite length in $\mathscr{A}$, then an inflation rule is a mapping $T:\mathscr{A}\to W(\mathscr{A})$. We define the word $C_n=T^n a$ with c_n the number of letters in C_n and c_n^α the number of times the letter α occurs in C_n.

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The infinite sequence generated by T is $C_\infty = \lim_{n\to\infty} C_n$. For each letter $\alpha \in \mathscr{A}$ we let E_α denote a corresponding vertical bond strength and assume $E_\alpha \neq E_\beta$ for $\alpha \neq \beta$. The layering is defined by setting $E_2(j) = E_\alpha$ if the jth letter in the word C_n is α. This with (2) defines the c_n-order layered Ising model for inflation rule T. For example, the Fibonacci layering is defined by $\mathscr{A} = \{a, b\}$, $Ta = ab$, $Tb = a$ with $C_1 = ab$, $C_2 = aba$, $C_3 = abaab$, and in general $C_{n+1} = C_n C_{n-1}$. The characteristic polynomial associated with the Fibonacci rule T is $p_T(x) = x^2 - x - 1$ and the numbers c_n are the Fibonacci numbers F_n. It was proved (Tracy 1988) that for the Fibonacci layering the amplitude $A(c_n; \{E_2\})$ approaches a finite non-zero limit (an explicit expression for the limiting amplitude was also derived). This means, given the assumption mentioned above, that the Fibonacci Ising model, defined by (1) with $E_2(j) = E_\alpha$ when $j = \alpha$ in C_∞, $\alpha = a$ or b, has a logarithmic singularity in the specific heat.

A natural question is: to what extent is the Fibonacci Ising model typical of aperiodic Ising models? To examine this we look at two inflation rules on three letters due to Bombieri and Taylor (1986, 1987).

Rule I. $Ta = aac$, $Tb = ac$ and $Tc = b$ with $C_1 = aac$, $C_2 = aacaacb$, $C_3 = aacaacbaacaacbac$ and $C_{n+1} = C_n^2(C_{n-2}^{-1} C_{n-1})$ and associated characteristic polynomial $p_T(x) = x^3 - 2x^2 - x + 1$.

Rule II. $Ta = aaaaab$, $Tb = bbbbc$ and $Tc = a$ with $C_{n+1} = C_n^5(C_{n-1}^{-5} C_n)^4 C_{n-2}$ and characteristic polynomial $p_T(x) = x^3 - 9x^2 + 20x - 1$.

With an inflation T (Bombieri and Taylor 1986, 1987) we associate a one-dimensional lattice with lattice points

$$x_k = \sum_\alpha l_\alpha u_\alpha(k)$$

where $u_\alpha(k)$ is the number of times the letter α occurs in the first k letters of C_∞ and l_α is the length of tile α (see Bombieri and Taylor (1986, 1987) for a discussion on making the lattice points extend to infinity in both directions). By a one-dimensional quasicrystal we mean an aperiodic sequence x_k of lattice points whose Fourier transform is of the form

$$\hat{\rho}(\xi) = \sum_{\lambda \in \Lambda} a_\lambda \delta(\xi - \lambda) + g(\xi) \tag{4}$$

where δ denotes the Dirac delta function, $g(\xi)$ is some function (possibly 0) and Λ is a set which is not contained in the integral multiples of any single frequency. Bombieri and Taylor (1986, 1987) show that any inflation rule T produces a quasicrystal only if the characteristic polynomial $p_T(x)$ has just one root greater than one in absolute value. For both the Fibonacci inflation rule and inflation rule I, the characteristic polynomial has only one root of absolute value greater than one whereas inflation rule II has two such roots.

For an inflation rule T and word $C_n = T^n a$ define $z_\alpha = \tanh(\beta_c E_\alpha)$, $y_\alpha = z_\alpha^2$, $z_1 = \tanh(\beta_c E_1)$ with β_c the inverse critical temperature. Then the amplitude in (3) is

$$A(c_n; \{E_2\}) = \frac{\beta_c^2}{4\pi}(z_1^{-1} - z_1)\left(2E_1 + \sum_{\alpha \in \mathscr{A}} \frac{c_n^\alpha}{c_n} E_\alpha (z_\alpha^{-1} - z_\alpha)\right)^2 \frac{1}{I_n} \tag{5}$$

where

$$I_n^2 = \frac{1}{c_n^2} \sum_{m=0}^{c_n - 1} \sum_{l=1}^{c_n} \prod_{\alpha \in \mathscr{A}} y_\alpha^{\Delta N_\alpha(l,m,n)} \tag{6}$$

where

$$\Delta N_\alpha(l, m, n) = N_\alpha(l, m, n) - \bar{N}_\alpha(m) \qquad (7)$$

$N_\alpha(l, m, n)$ = the number of times letter α appears between l and $l+m$ (including l and $l+m$) in layering C_n (8)

$\bar{N}_\alpha(m) = (m+1)c_n^\alpha/c_n$ = the average number of times letter α appears in a segment of length $(m+1)$ in a C_n layering. (9)

The numerator in (5) clearly has a limit as $n \to \infty$. The question is: does $\lim_{n\to\infty} I_n$ exist?

The numerical evaluation of (6) suggests that as $n \to \infty$ the quantity I_n converges to a finite non-zero limit for inflation rule I whereas it diverges to $+\infty$ for inflation rule II. Further evidence of the difference of inflation rules I and II can be obtained by examining

$$\mathrm{var}(N_\alpha) = \frac{1}{c_n} \sum_{l=1}^{c_n} (\Delta N_\alpha(l, m, n))^2. \qquad (10)$$

For inflation rule I, $\mathrm{var}(N_\alpha)$ is bounded in m whereas for inflation rule II, equation (10) more or less increases with increasing m. For the Fibonacci inflation rule, $\mathrm{var}(N_\alpha) = \{N_\alpha\}(1-\{N_\alpha\})$ with $\{x\}$ the fractional part of x and the range of $N_\alpha(l, m, n)$ is $[N_\alpha]$ and $[N_\alpha]+1$ where $[x]$ is the greatest integer function (Tracy 1988). From explicit computation for $1 \leq n \leq 7$, it is conjectured that for all inflation levels n the range of $N_\alpha(l, m, n)$ for inflation rule I is $[N_\alpha]$, $[N_\alpha] \pm 1$, $[N_\alpha]+2$. From low values of n it is clear that possible values of $N_\alpha(l, m, n)$ for inflation rule II increase with increasing m. Thus inflation rule I is similar to the Fibonacci rule whereas inflation rule II is qualitatively different.

Thus on the basis of the numerical work we conjecture: (i) the singularity of the aperiodic Ising model (1) defined by inflation rule I has a logarithmic singularity in the specific heat, (ii) the amplitude (5) diverges to zero for inflation rule II which is interpreted to mean that the aperiodic Ising model (1) defined by inflation rule II does not have a logarithmic singularity in the specific heat.

It is interesting to note that the logarithmic singularity in the specific heat appears only in the cases where the inflation rule T generates a quasicrystal. We conjecture that it is a general phenomenon that the ferromagnetic aperiodic Ising model (1) has a logarithmic singularity in the specific heat only when the characteristic polynomial $p_T(x)$ associated with the inflation rule T has only one root greater than one in absolute value. Certainly, the clarification of the universality classes of various aperiodic Ising models deserves further study.

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