Nuclear Physics B330 (1990) 681–704
North-Holland

ALGORITHMS FOR THE COMPUTATION OF POLYNOMIAL RELATIONSHIPS FOR THE HARD HEXAGON MODEL

Matthew P. RICHEY

Department of Mathematics, St. Olaf College, Northfield, MN 55057, USA

Craig A. TRACY*,**

Department of Mathematics,
and
Institute of Theoretical Dynamics, University of California, Davis, CA 95616, USA

Received 12 June 1989
(Revised 28 August 1989)

We calculate the polynomial relationships $P(\rho, z) = 0$ and $P(\rho, \kappa) = 0$ in both the ordered and disordered regimes for the hard hexagon model. We start with Baxter's exact solution which gives the physical quantities κ, ρ, and z parametrically as a function of a variable τ and exploit the modular properties of Baxter's solution. Using elementary Riemann surface theory, the computation can be reduced to algorithms involving only linear algebra. These algorithms are implemented using a computer algebra system. The method will be applicable to other exactly solvable models in which cusp expansions can be computed.

1. Introduction

Solvable models of lattice statistical mechanics are essentially equivalent to those for which the well-known Yang–Baxter equations [3, 10] admit a nontrivial solution. In many instances, these solutions involve elliptic functions; and hence the Boltzmann weights are parametrized in terms of a spectral parameter and a modulus x $(= e^{2\pi i\tau})$. The physical quantities derived from these solutions inherit these parameters; and in the case of the order parameters, are functions only of τ (the spectral parameter either cancels out or is set to a torsion point of the elliptic curve). This paper looks at the simplest IRF model that has these features; namely Baxter's solution of the hard hexagon model [1–3], and describes a computational method of deriving polynomial relationships between the physical quantities. The methods used are based on the modular properties of the functions giving the partition function per site κ, density ρ and activity z of this model. The computations

* Supported in part by the National Science Foundation, grant no. DMS 87-00867.
** This research was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation.

themselves use only linear algebra. However, in order to do the computations, it is essential that a computer algebra system be used. The work presented in this paper is a detailed account of the methods first announced in ref. [7].

In refs. [1, 2] Baxter (see also ref. [3]) derived exact expressions for κ, ρ, and z in the low activity (or disordered) regime, defined by $z < z_c$ where $z_c = (11 + 5\sqrt{5})/2$, and the high activity (or ordered) regime defined by $z > z_c$. In the disordered regime the local density is site independent whereas in the ordered regime the local density depends on which of three sublattices (denoted by 1, 2, or 3) it sits. A sublattice is defined by a possible close packed configuration of hexagons (we take boundary conditions so that the sublattice "1" is the close packed configuration ground state). Denoting the local density on the sublattice i ($i = 1,2,3$) by ρ_i, Baxter showed that in the ordered regime $\rho_2 = \rho_3 \neq \rho_1$. Following Pearce and Baxter [6], in the ordered regime we will work with $\Delta\rho = \sqrt{5}(\rho - \rho_c)$ where $\rho = \frac{1}{3}(\rho_1 + \rho_2 + \rho_3)$ and $\rho_c = (5 - \sqrt{5})/10$ is the critical density. All the formulas that we now quote with the exception of eq. (1.5) are due to Baxter [1–3]. Eq. (1.5) is due to Pearce and Baxter [6].

Disordered regime ($z < z_c$):

$$\kappa(\tau) = \prod_{m=1}^{\infty} \frac{(1-x^{5m})^2(1-x^{5m-1})^2(1-x^{5m-4})^2(1-x^{6m-3})^2(1-x^{6m-2})(1-x^{6m-4})}{(1-x^{5m-3})^3(1-x^{5m-2})^3(1-x^{6m-1})(1-x^{6m-5})(1-x^{6m})^2}, \tag{1.1}$$

$$\rho(\tau) = -x \prod_{m=1}^{\infty} \frac{(1-x^{6m-3})}{(1-x^{2m-1})(1-x^{5m-1})(1-x^{5m-4})(1-x^{30m-12})(1-x^{30m-18})}, \tag{1.2}$$

$$z(\tau) = -x \prod_{m=1}^{\infty} \left(\frac{(1-x^{5m-1})(1-x^{5m-4})}{(1-x^{5m-3})(1-x^{5m-2})} \right)^5. \tag{1.3}$$

Ordered regime ($z > z_c$):

$$\kappa(\tau) = x^{-1/3} \prod_{m=1}^{\infty} \frac{(1-x^{5m})^2(1-x^{5m-2})^2(1-x^{5m-3})^2(1-x^{3m-1})(1-x^{3m-2})}{(1-x^{5m-1})^3(1-x^{5m-4})^3(1-x^{3m})^2}, \tag{1.4}$$

$$\Delta\rho(\tau) = \frac{3-\sqrt{5}}{6} \prod_{m=1}^{\infty} \frac{(1-x^m)^2(1-\omega x^m + x^{2m})(1-\omega x^{3m} + x^{6m})}{(1-x^{3m})^2(1+\omega^{-1}x^m + x^{2m})^2}, \tag{1.5}$$

$$z(\tau) = x^{-1} \prod_{m=1}^{\infty} \left(\frac{(1-x^{5m-2})(1-x^{5m-3})}{(1-x^{5m-1})(1-x^{5m-4})} \right)^5, \tag{1.6}$$

where $\omega = (-1 + \sqrt{5})/2$.

It has been shown [9] that all of these expressions (or powers of) are modular functions with respect to some subgroup of $SL_2(\mathbb{L})$. More precisely, this was shown for all the above expressions except (1.5). The group theoretic significance of the Pearce–Baxter identity (1.5) is that although each ρ_i is modular with respect to the congruence subgroup Γ_1 [45], the density is modular with respect to the bigger subgroup Γ_1 [15]. If one examines the individual expressions (derived by Baxter) for the ρ_i $(i = 1, 2, 3)$ then it is not obvious that $\rho = \frac{1}{3}(\rho_1 + \rho_2 + \rho_3)$ is modular with respect to Γ_1 [15]; but using the Pearce–Baxter identity it is seen to be modular with respect to this subgroup using methods exactly the same as those used for the other expressions [9].

We derive below *polynomial relationships* which are known to exist between pairs of functions above (this follows from the fact that each of the above or their powers are modular functions). For example, in the disordered regime there is a polynomial, $P(x, y)$, such that

$$P(\rho(\tau), \kappa(\tau)) = 0$$

for all $\tau \in \mathrm{H} = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$. This equation defines implicitly the dependence of κ on ρ. In what follows, we will provide the details of the calculation of four such polynomials. In the ordered regime we will find

$$P_1(\Delta\rho, \kappa^2) = 0, \qquad P_2(\Delta\rho, z) = 0. \tag{1.7}$$

In the disordered regime we will find

$$P_3(\rho, \kappa) = 0, \qquad P_4(\rho, z) = 0. \tag{1.8}$$

The coefficients for P_1–P_4 are in tables 1–4. The simplest of these polynomials is P_2 which can be written in terms of ρ and z as

$$2 - 9\rho + 15\rho^2 - 11\rho^3 + 3\rho^4 + (1 - 12\rho + 45\rho^2 - 66\rho^3 + 33\rho^4)z + (\rho^3 - 3\rho^4)z^2 = 0. \tag{1.9}$$

From these polynomials it is a simple calculation to derive other interesting expressions such as the *isothermal compressibility*

$$\chi = \kappa\left(\frac{\partial\kappa}{\partial\rho}\right) = \frac{z}{\rho}\left(\frac{\partial\rho}{\partial z}\right),$$

where the derivatives may be obtained by implicitly differentiating one of the polynomials P_i.

After the appearance of ref. [7], work by Joyce [5] demonstrated another way of deriving the above results. Joyce's work exposes more of the intriguing relationship between modular functions and exactly solvable lattice models. Joyce uses the classical theory of modular functions as can be found in Klein and Fricke to derive the above polynomials. Our approach uses simple group theoretic notions and elementary results from Riemann surface theory coupled with the computational power provided by computer algebra systems. As a point of comparison, one can compare our derivation of the important equation (1.9) with that of Joyce's. We expect that the methods presented here will be more easily extended to other problems of a similar nature both in statistical physics and conformal field theory. In any case, we feel that these two approaches are sufficiently different to merit a long version of our Letter [7]. Once the polynomial relations have been computed, many interesting applications to statistical physics can be derived (some of these were briefly indicated in our Letter). However, this part of the analysis would be identical to that already done by Joyce so we refer the reader to Joyce [5] for these applications.

When exact solutions are not known, one method of extracting critical behavior is from series expansions. When extrapolating from series expansions it is important to know the class of functions being approximated (cf. ref. [4] and references therein). The results here show that the physical quantities (or powers of) such as κ, ρ, z, etc. satisfy polynomial relationships. We expect similar results for other solvable models coming from Baxter's methods (cf. ref. [3]) involving elliptic functions. This is to be expected since Baxter's corner transfer matrix methods are intimately related to modular functions. An important unsolved problem is to prove the *existence* of such polynomial relationships (and hopefully bounds on their degrees) *without* relying upon the explicit formulas. Perhaps such polynomial relationships exist for models "not solvable" by the methods of Baxter, and a proof of the solvable case can be generalized. For example, the hard square lattice gas is "not solvable" (cf. ref. [3]), but do κ, ρ, and z satisfy polynomial relations?

The paper is organized as follows. Sect. 2 contains a brief description of the aspects of modular function theory we will need. Included in this section are references in which more detailed information may be obtained. Sect. 3 consists of general techniques, theorems, and notation which are necessary for the computations of the four polynomials. Sect. 4 is divided into four parts, each part containing the algorithms for each computation. The details for the last three algorithms are somewhat brief. Those desiring more details may contact the authors. The appendix contains the cusp expansions necessary to do the computations.

2. Modular functions and cusp expansions

2.1. MODULAR FUNCTIONS

The key to deriving the polynomial relationships is the identification of the various physical expressions (or powers of) as modular functions with respect to certain modular subgroups. This has been done by Tracy et al. [9]. The modular subgroups we shall be interested in are $\Gamma_1[N]$ for $N = 5, 15, 30, 45$. Their definitions are

$$\Gamma_1[N] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

Matrices $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ act on points $\tau \in \mathrm{H}$ by

$$\gamma(\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}.$$

A meromorphic function $f(\tau)$, defined on H, is modular with respect to a modular subgroup $\Gamma^* \subset \mathrm{SL}_2(\mathbb{Z})$ if for any $\gamma \in \Gamma^*$ we have

$$f(\gamma(\tau)) = f(\tau),$$

for any $\tau \in \mathrm{H}$, and $f(\tau)$ is meromorphic at the cusps of Γ^* (we are assuming Γ^* has no parabolic points, cf. Schoenberg [8]). For the hard hexagon model, it is shown in ref. [9] that in the disordered regime

$$\begin{aligned} &\rho(\tau) \text{ is modular with respect to } \Gamma_1\,[30], \\ &\kappa(\tau) \text{ is modular with respect to } \Gamma_1\,[30], \\ &z(\tau) \text{ is modular with respect to } \Gamma_1\,[5]; \end{aligned} \tag{2.1}$$

and in the ordered regime

$$\begin{aligned} &\Delta\rho(\tau) \text{ is modular with respect to } \Gamma_1\,[15], \\ &\kappa^3(\tau) \text{ is modular with respect to } \Gamma_1\,[15], \\ &z(\tau) \text{ is modular with respect to } \Gamma_1\,[5]. \end{aligned} \tag{2.2}$$

Given a $\Gamma_1[N]$ and a fixed N, we may identify two points $z, w \in \mathrm{H}$ if there exists a $\gamma \in \Gamma_1[N]$ such that $\gamma(z) = w$. Using this equivalence relation on H, we can define a quotient space $\mathrm{H}/\Gamma_1[N] = \mathscr{F}_N$. If we include compactifying points then $\mathscr{F}_N$ is a compact Riemann surface (cf. ref. [7]). The compactifying points are either the point at infinity, $i\infty$, or rational points h/k on the real line. In either case these points are called *cusps*. A function modular with respect $\Gamma_1[N]$ is defined by its behavior on the compact Riemann surface $\mathscr{F}_N$. For $\Gamma_1[N]$, a fundamental domain is the closure of a connected, maximal set of inequivalent points. Often we will identify the fundamental set with $\mathscr{F}_N$.

A well-known result from modular function theory (cf. ref. [8]) which will be of particular importance to us is the following:

Theorem 2.1. If f and g are modular with respect to the same modular subgroup, then there exists a polynomial relationship between them. That is, there exists

$$P(x, y) = \sum_{i=0}^{N} \sum_{j=0}^{M} c_{ij} x^i y^i$$

such that $P(f(\tau), g(\tau)) = 0$ for all $\tau \in \mathrm{H}$.

In fact, the theorem says more. The *valence* of a meromorphic function f, defined on a compact Riemann surface $\mathscr{R}$, is defined in the following manner. Pick a $w \in \mathscr{R}$, then the size of the set

$$\{z \in \mathscr{R} : f(z) = w\}$$

does not depend on the choice of w. The valence of f, or val(f), is then just this size. Note that if val(f) $= \infty$, then f is constant. A particularly nice choice for w would be either 0 or ∞. In this case val(f) is just the number of zeros or poles of f. Knowing the valence of the functions f and g allows us to bound the degrees of the polynomial in the theorem above. In the following theorem, let deg(P, x) denote the degree of the polynomial P in the variable x.

Theorem 2.2. If val(f) $\leqslant M$ and val(g) $\leqslant N$ and P is irreducible, then deg(P, f) $\leqslant N$ and deg(P, g) $\leqslant M$.

Using theorem 2.1, we know that there must be polynomial relationships between any two of the functions listed in eqs. (2.1) and (2.2). This is because if N_1 divides N_2 then $\Gamma_1[N_2] \subset \Gamma_1[N_1]$. Hence if f is modular with respect to $\Gamma_1[N_1]$ then it must also be modular with respect to $\Gamma_1[N_2]$. For example, in the disordered regime, z is modular with respect to Γ_1 [5] while ρ is modular with respect to Γ_1 [30]. Thus both are modular with respect to Γ_1 [30] and thus by theorem 2.1 there must be a polynomial relationship between them.

We use theorem 2.2 to obtain bounds on the degrees of the polynomials we wish to compute. This is essential because otherwise we would not even know an upper limit on the number of unknown coefficients which we seek. To obtain these bounds we must know bounds on the valences of the functions. A glance at "definitions" (1.1)–(1.6) of z, κ, ρ, and $\Delta\rho$ makes clear that one needs more information in order to determine the valences. This additional information is obtained from the *cusp expansions*.

2.2. CUSP EXPANSIONS

A cusp expansion of a function f defined on a quotient space $\mathscr{F}_N = \mathrm{H}/\Gamma_1[N]$ is simply a definition of f in terms of a local variable defined about a particular cusp. The standard approach is to use a local uniformizing variable x_ξ defined near the cusp ξ. For details concerning the precise definition of a local variable we refer to Schoenberg [8].

This local uniformizing variable has the property that as τ approaches the cusp ξ, the local variable x approaches 0. For example, at the cusp $i\infty$, the local uniformizing variable is simply $x = \mathrm{e}^{2\pi i\tau}$. Thus eqs. (1.1)–(1.6) are simply cusp expansions in the local variable at $i\infty$. It is worth noting that a local variable defined about a cusp is in fact valid throughout the entire fundamental domain of $\Gamma_1[N]$.

In order to obtain cusp expansions of modular functions one must know their transformation properties. This is most easily done by writing the function in terms of generalized Dedekind eta functions (these functions also arise as determinants of Dirac operators on elliptic curves). The transformation properties of these are well known [8]. By writing a function in terms of the generalized eta functions one can also recognize a subgroup under which a function is modular. The process of identifying infinite product expansions as products of generalized Dedekind eta functions and then doing the transformations to local variables is tedious but mechanical. For a complete description of this process, see ref. [9].

To obtain a complete description of a function modular with respect to $\Gamma_1[N]$, one first derives cusp expansions at a complete inequivalent set of cusps for $\mathscr{F}_N$. For the functions listed in eqs. (1.1)–(1.6), cusp expansions are given in the appendix. From the cusp expansion at $i\infty$ given by eqs. (1.1)–(1.6), it is clear that functions of this type can only have zeros and poles at the cusps. Hence once all the cusp expansions are done one simply counts the zeros (including multiplicities) at all the cusps to obtain the valence. From the cusp expansions of ρ in the disordered regime, we see that in $\mathscr{F}_{30}, \mathrm{val}(\rho) = 8$. For valence of z in the disordered regime, we first must convert the cusp expansion of z in $\mathscr{F}_5$ to cusp expansions in $\mathscr{F}_{30}$. For each cusp in $\mathscr{F}_5$, one finds a complete set of inequivalent cusps in $\mathscr{F}_{30}$ (but equivalent under Γ_1 [5]). Then the local variable about the cusp in $\mathscr{F}_5$ is easily converted to a local variable about the new cusps in $\mathscr{F}_{30}$. Once again, the reader is referred to ref. [8] for details. In this case we discover that $\mathrm{val}(z) = 24$ in $\mathscr{F}_{30}$. Thus we know that the polynomial relationship between ρ and z must have the form

$$P(\rho, z) = \sum_{i=0}^{24} \sum_{j=0}^{8} c_{ij}\rho^i z^j = 0\,.$$

At this stage if one had access to unlimited computing power it would be straightforward to find the $25 \times 9 = 225$ coefficients c_{ij}. Since the polynomial is only unique up to an overall multiplicative factor, arbitrarily set one of the coefficients equal to, say, 1. At any cusp, derive Laurent series expansions with at least 224 terms in the local variable x. Substitute these series into the unknown polynomial $P(\rho, z) = 0$. The result of this substitution would be a series whose coefficients depend linearly on the c_{ij}. Since this series must be identically zero, all of its coefficients must also be identically zero. Equating the first 224 to zero would yield a system which determines the remaining 224 unknowns. However, under limits on computing power, this approach is troubled because of round-off error.

The approach in this paper will be to use the information given by the cusp expansions to reduce dramatically the number of equations that must be solved at any one time. For example, for the polynomial above, we will see that it is never necessary to solve more than four equations in four of the unknown coefficients at any one time. This allows us to do the computations in exact rational form.

3. General techniques

In this section we will outline some of the general techniques and notation to be used in the subsequent calculations. First of all, we will be quite interested in coefficients of power series in the local variable x. Let f be any modular function on H and ξ any cusp. About that cusp we will have a local variable x. Let

$$\mathrm{co}\,f_\xi(n, f) = \text{coefficient of } x^n \text{ in the Laurent series of } f \text{ at cusp } \xi .$$

Suppose that we have two functions f and g, both modular with respect to the same modular subgroup and that $\mathrm{val}(f) = M$ and $\mathrm{val}(g) = N$ on some compact Riemann surface. Therefore, the resulting irreducible polynomial takes the form

$$P(f, g) = \sum_{i=0}^{N} \sum_{j=0}^{M} c_{ij} f^i g^j = 0 .$$

A means by which we may reduce the number of equations that need to be considered at any one time in order to determine the c_{ij} is illustrated in the following scenario. Suppose there is a cusp ξ at which f and g have the power series expansions:

$$f(x) = a_0 + a_1 x + \dots , \qquad g(x) = x^m (b_0 + b_1 x + \dots) = x^m \tilde{g}(x) . \tag{3.1}$$

Let Q_j and P_k be defined by

$$Q_j = \sum_{i=0}^{N} c_{ij} f^i , \qquad P_k = \sum_{j=0}^{k} g^j Q_j$$

so that $P = P_M$. For convenience, let $P_{-1} \equiv 0$. Thus we have for $i = 0, \dots, m-1$

and $k \leqslant N$,

$$\begin{aligned} 0 &= \operatorname{co} f_\xi(mk+i, P) = \operatorname{co} f_\xi(mk+i, P_k) \\ &= \operatorname{co} f_\xi(mk+i, g^k Q_k) + \operatorname{co} f_\xi(mk+i, P_{k-1}) \\ &= \operatorname{co} f_\xi(i, \tilde{g}^k Q_k) + \operatorname{co} f_\xi(mk+i, P_{k-1}), \end{aligned}$$

which we may write as

$$\operatorname{co} f_\xi(i, \tilde{g}^k Q_k) = -\operatorname{co} f_\xi(mk+i, P_{k-1}).$$

The left-hand side only involves c_{ik} and the right-hand side only involves c_{in} with $n < k$. Thus we have m linear equations for c_{ik} defined recursively in terms of c_{in} with $n < k$.

These equations may be written in matrix form as

$$G^k A \begin{pmatrix} c_{0,k} \\ \vdots \\ c_{N,k} \end{pmatrix} = -B_k,$$

where G is a lower triangular $m \times m$ matrix whose nonzero entries are given by $(G)_{ij} = b_{i-j}$, A is an $m \times (N+1)$ matrix whose entries are only dependent on the a_0 through a_{m-1}, B_k is an $m \times 1$ column whose ith entry is $\operatorname{co} f_\xi(mk+i, P_{k-1})$. What is particularly nice about the matrices G and A is that each is independent of k. The key is either to get several of these type equations from different cusps with the same series structure for f and g or to have enough of the coefficients c_{ik} already determined so that only m of them remain. In our case, the entries of G, A, and B_k are either rational numbers or in the field $\mathbb{Q}(\sqrt{5})$.

Another important consideration is to ensure that the bounds on the degrees of the polynomial are as sharp as possible. Determining the degrees from the valences is fine as long as we are sure that we are working with the largest possible modular subgroup for the functions in question (equivalently, the smallest possible common fundamental domain). If one does not identify this largest modular subgroup, then the bounds on the degrees will not be optimal and one would conclude that the degrees of the polynomial P are higher than they actually should be. Of course what would happen upon calculation is that all the higher degree terms would have zero coefficients. However, the larger number of unknown coefficients might make the method of this paper computationally infeasible. Thus we would like to be able to recognize when we have not found the optimal modular subgroup. It is often possible to do so by looking at the cusp expansions.

Theorem 3.1. Suppose that f and g are modular functions with respect to a common modular subgroup Γ^*. Let a set of inequivalent cusps be given by $\mathscr{C} = \mathscr{C}_1 \cup \ldots \cup \mathscr{C}_k$ where $\mathscr{C}_i \cap \mathscr{C}_j = \varnothing$ for $i \neq j$ and $|\mathscr{C}_i| = |\mathscr{C}_j| = S$ for all i, j. Let $N = \mathrm{val}(f)$, $M = \mathrm{val}(g)$ on H/Γ^*. For each cusp, let us use the same symbol x to denote a local variable at that cusp. Suppose that for either f or g, that its cusp expansion in x only depends on the set $\mathscr{C}_i$ containing that cusp. Then, if $P(f, g) = 0$ is the irreducible polynomial relationship between f and g, we have

$$\deg(P, f) \leqslant M/S, \qquad \deg(P, g) \leqslant N/S.$$

The importance of this theorem is that it allows one not to worry about finding the optimal modular subgroup for a pair of functions. Instead any common subgroup will work. One simply uses that subgroup to calculate all the cusp expansions for the two functions at the inequivalent cursps. Then if all the expansions do not group up as in the theorem, the subgroup is optimal. If they do, one simply remembers that the degrees of the polynomial can be reduced by some factor.

There are other techniques that make the calculation of the coefficients c_{ij} easier. One, for instance, is to exploit any symmetry in the cusp expansions. For example, if there is a pair of cusps at which the expansion of f does not change, but the expansion of g goes to $-g$, then the polynomial $P(f, g)$ must be even in g. To see this, suppose the pair of cusps with this property is ξ and ξ'. In the local variable x at ξ, write

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \qquad g(x) = b_0 + b_1 x + b_2 x^2 = \ldots.$$

In the local variable x' at ξ' we can write

$$f(x') = a_0 + a_1 x' + a_2 x'^2 + \ldots, \qquad g(x') = -\left(b_0 + b_1 x' + b_2 x'^2 + \ldots\right),$$

where the coefficients a_i and b_i are the same in both cases. For any value $\tau \in \mathrm{H}$, there is a value τ' such that $x(\tau) = x'(\tau')$. At these values we have

$$f(\tau) = f(\tau'), \qquad g(\tau) = -g(\tau').$$

Since the polynomial $P(f, g)$ must be zero we get for any value in H,

$$P(f(\tau), g(\tau)) = 0 = P(f(\tau'), g(\tau')),$$

or

$$P(f(\tau), g(\tau)) = P(f(\tau), -g(\tau))$$

and hence $P(f, g)$ is even in g.

4. Algorithms

In this section we will present details of the algorithms used to calculate the four polynomials in eqs. (1.7) and (1.8). We will drop the subscripts 1, 2, 3, and 4 from the polynomials since in each subsection we will be referring to only one polynomial at a time. For the first polynomial, that is subsect. 4.1, we will include a large amount of detail so that the reader may see the type of calculations which must be performed. In the subsequent subsections we will simply sketch the outlines of the algorithms.

4.1. $P(\Delta\rho, \kappa^3) = 0$ FOR $\Delta\rho$ AND κ IN THE ORDERED REGIME

In this subsection we will calculate the polynomial $P(\Delta\rho, \kappa^3) = 0$ for $\Delta\rho$ and κ in the ordered regime. First we see from the cusp expansions that $\mathrm{val}(\Delta\rho) = 4$ and $\mathrm{val}(\kappa^3) = 22$. At any of the cusp pairs $(0, \frac{1}{4}), (\frac{1}{6}, \frac{1}{9})$, etc., we see that in terms of the appropriate local variable x that $\kappa^3 \to -\kappa^3$ while $\Delta\rho$ remains unchanged. Thus P must be even in κ^3. Define a new variable $\tilde{\kappa} = (\kappa^3)^2$ and write

$$P(\Delta\rho, \tilde{\kappa}) = \sum_{i=0}^{22} \sum_{j=0}^{2} c_{ij} \Delta\rho^i \tilde{\kappa}^j = 0 .$$

The coefficients c_{ij} are displayed in table 1.

Step 1. Consider the cusps $\frac{1}{6}$ and $\frac{2}{3}$. At each of these cusps $\tilde{\kappa}$ has a pole of order 10 and $\Delta\rho$ has a pole of order 1. This implies that $c_{22,2} = c_{21,2} = \cdots = c_{3,2} = 0$ and $c_{22,1} = c_{21,1} = \cdots = c_{13,1} = 0$.

Step 2. We now look at the cusp $i\infty$. At this cusp $\tilde{\kappa}$ has a pole of order 2 and $\Delta\rho$ is nonzero. This gives us equations in $c_{i,2}$. We already know that $c_{i,2} = 0$ for $i > 2$, and that one coefficient of P, say $c_{0,2}$, is arbitrary. The other two coefficients, $c_{1,2}$ and $c_{2,2}$, are determined by the two equations above. Therefore we have completely determined the coefficients $c_{i,2}$ for $i = 0, \dots, 22$.

Step 3. We now consider the cusp 0. Here we have the expansions:

$$\tilde{\kappa} = k_0 + 0x + 0x^2 + k_3 x^3 + \dots , \qquad \Delta\rho = x^2\left(1 - x + 0x^2 + r_3 x^3 + \dots\right).$$

The fact that $\tilde{\kappa}$ has no x^1 or x^2 coefficient will be important. Define Q_i and P_n by

$$Q_i = c_{i,0} + c_{i,1}\tilde{\kappa} + c_{i,2}\tilde{\kappa}^2 , \qquad P_n = \sum_{i=0}^{n} \Delta\rho^i Q_i .$$

We see that

$$\mathrm{co} f_0(2k, P) = \mathrm{co} f_0(2k, P_k) = \mathrm{co} f_0(0, Q_k) + \mathrm{co} f_0(2k, P_{k-1}) = 0 . \quad (4.1)$$

This gives one equation in the coefficients $c_{k,0}$ and $c_{k,1}$ in terms of c_{ij} with $i < k$

TABLE 1

The coefficients c_{ij} are represented as an ordered pair (a_{ij}, b_{ij}): $c_{ij} = a_{ij} + \sqrt{5}\, b_{ij}$. To simplify the entries, κ^6 is rescaled so that the leading coefficient is 1 at the cusps $(0, \frac{1}{4})$.

To obtain κ as normalized in eq. (1.1) let $c_{ij} \to F^j c_{ij}$, where

$$F = \left[\frac{3\sqrt{3}}{10} \frac{\sin^2(2\pi/5)}{\sin^3(\pi/5)} \right]^{-6} = \frac{5^7}{3^9}(1525 - 682\sqrt{5}).$$

i \ j	0	1	2
0	2, 0	−4, 0	2, 0
1	−18, −6	36, 12	−18, −6
2	−27, 117	−36, −144	63, 27
3	−840, −120	−1860, 1020	0, 0
4	9855, −2745	−6075, 4365	0, 0
5	45324, −24480	73368, −41400	0, 0
6	−54621, 32793	194538, −66534	0, 0
7	−1880064, 827424	−1459188, 622908	0, 0
8	−7997310, 3593970	−2478465, 1136475	0, 0
9	−2914480, 1284520	6991720, −3144320	0, 0
10	127074558, −56812470	5635290, −2513190	0, 0
11	737266608, −329728464	−16357680, 7313760	0, 0
12	2459142582, −1099753902	7329150, −3277530	0, 0
13	5794100820, −2591204940	0, 0	0, 0
14	10280792160, −4597708140	0, 0	0, 0
15	14087220816, −6299997360	0, 0	0, 0
16	15015313971, −6715052343	0, 0	0, 0
17	12388779054, −5540430474	0, 0	0, 0
18	7781944320, −3480191290	0, 0	0, 0
19	3604574760, −1612014840	0, 0	0, 0
20	1162500012, −519885810	0, 0	0, 0
21	233397882, −104378706	0, 0	0, 0
22	21986883, −9832833	0, 0	0, 0

and $c_{k,2}$, namely,

$$c_{k,0} + k_0 c_{k,1} = -k_0^2 c_{k,2} - \operatorname{co} f_0(2k, P_{k-1}). \tag{4.2}$$

Now consider the x^{2k+1} coefficient. We see that

$$\begin{aligned}\operatorname{co} f_0(2k+1, P) &= \operatorname{co} f_0(2k+1, P_k)\\ &= \operatorname{co} f_0(2k+1, \Delta\rho^k Q_k) + \operatorname{co} f_0(2k+1, P_{k-1}) = 0.\end{aligned} \tag{4.3}$$

However, Q_k has no x^1 coefficient. Thus

$$\operatorname{co} f_0(2k+1, \Delta\rho^k Q_k) = \operatorname{co} f_0(2k+1, \Delta\rho^k) \operatorname{co} f_0(0, Q_k)$$

and hence (4.3) becomes

$$k \operatorname{co} f_0(0, Q_k) = -\operatorname{co} f_0(2k+1, P_{k-1}). \tag{4.4}$$

We may eliminate $\operatorname{co} f_0(0, Q_k)$ from eq. (4.4) using (4.1) and replace $k-1$ by k to obtain

$$(k+1) \operatorname{co} f_0(2k+2, P_k) + \operatorname{co} f_0(2k+3, P_k) = 0. \tag{4.5}$$

This will give us a second equation in $c_{k,0}$ and $c_{k,1}$. All we need to isolate is the portion of eq. (4.5) which only depends on k. This is straightforward. Once this is done we have the two equations

$$c_{k,0} + k_0 c_{k,1} = -k_0^2 c_{k,2} - \operatorname{co} f_0(2k, P_{k-1}), \qquad k_3 c_{k,1} = -2k_0 k_3 c_{k,2} + L_k,$$

where

$$L_k = (k+1) \operatorname{co} f_0(2k+2, \Delta\rho^k) \operatorname{co} f_0(2k, P_{k-1}) + \operatorname{co} f_0(2k+3, \Delta\rho^k) \operatorname{co} f_0(2k, P_{k-1})$$
$$-(k+1) \operatorname{co} f_0(2k+2, P_{k-1}) - \operatorname{co} f_0(2k+3, P_{k-1}).$$

At the kth iteration, the right-hand side is completely determined. Thus we may recursively calculate c_{ki} $(i = 1, 2)$ for $k = 0, \ldots, 22$. We already know that $c_{k,1} = 0$ for $k \geqslant 13$. So if desired, we may use only one of the above equations once $k \geqslant 13$ in order to speed up the algorithm. In order to carry out the recursive calculation, we must know the Laurent series expansions of $\tilde{\kappa}$ and $\Delta\rho$ out to x^{44}.

4.2. $P(\Delta\rho, z) = 0$ FOR $\Delta\rho$ AND z IN THE ORDERED REGIME

In this subsection we will calculate the polynomial $P(\Delta\rho, z) = 0$ for $\Delta\rho$ and z in the ordered regime. The valences of these functions are $\operatorname{val}(\Delta\rho) = 4$ and $\operatorname{val}(z) = 8$. However, we see that $\Delta\rho$ and z have the same cusp expansions at 8 cusp pairs. Thus we may use theorem 3.1 and halve the degrees of P. This gives us

$$P(\Delta\rho, z) = \sum_{i=0}^{4} \sum_{j=0}^{2} c_{ij} \Delta\rho^i z^j = 0.$$

The coefficients c_{ij} appear in table 2. A change of variables from $\Delta\rho$ to ρ gives (1.9).

Step 1. Consider the cusp 0. Here we see that $\Delta\rho$ has a zero of order 2 while $z = z_0 + z_3 x^3 + z_6 x^6 + \ldots$. This gives us two equations in the three unknown coefficients $c_{0,0}$, $c_{0,1}$, and $c_{0,2}$. One arises immediately from the x^0 coefficient of $P(\Delta\rho, z)$ and the other from the fact that z has no x^1 or x^2 terms. Since one

TABLE 2
The coefficients c_{ij} of $P(\Delta\rho, z) = 0$ in the ordered regime.

i \ j	0	1	2
0	$11 + 5\sqrt{5}$	-4	$-11 + 5\sqrt{5}$
1	$-33 - 15\sqrt{5}$	12	$33 - 15\sqrt{5}$
2	$39 + 15\sqrt{5}$	54	$-39 + 15\sqrt{5}$
3	$-12 - 10\sqrt{5}$	-132	$12 - 10\sqrt{5}$
4	6	66	6

coefficient is arbitrary, we may set $c_{0,0}$ equal to $11 + 5\sqrt{5}$ and determine $c_{0,1}$ and $c_{0,2}$. Thus we have determined $c_{0,k}$ for $k = 0, 1, 2$.

Step 2. Now proceed to the cusps $i\infty$ and $\frac{1}{5}$. At $\frac{1}{5}$ we see that $\Delta\rho$ is nonzero while z has a zero of order 3. In this case we obtain three equations in c_{mk}, $m = 0, \ldots, 4$ which are defined in terms of c_{jk}, with $j > m$.

To get a fourth such equation, we move to $i\infty$ at which we see again that $\Delta\rho$ is nonzero while now z has a simple zero. Recall that from step 1, we already know $c_{0,k}$. Therefore we have four equations in the four unknowns $c_{i,k}$, $(i = 1, 2, 3, 4)$, in terms of c_{ij} for $j < k$ and $c_{0,k}$. In order to implement this recursive calculation, we need the cusp expansions of $\Delta\rho$ and z out to x^{14}.

4.3. $P(\rho, \kappa) = 0$ FOR ρ AND κ IN THE DISORDERED REGIME

In this subsection we will calculate the polynomial $P(\rho, \kappa) = 0$ for ρ and κ in the disordered regime. We see that the valences are $\mathrm{val}(\rho) = 8$ and $\mathrm{val}(\kappa) = 22$. At the cusp pairs $(i\infty, \frac{1}{30}), (\frac{4}{5}, \frac{1}{5})$, etc., the cusp expansion of κ goes to $-\kappa$ while ρ remains unchanged. Thus $P(\rho, \kappa)$ must be even in κ. We will let $\tilde{\kappa} = \kappa^2$ and write

$$P(\rho, \tilde{\kappa}) = \sum_{i=0}^{22} \sum_{j=0}^{4} c_{ij} \rho^i \tilde{\kappa}^j = 0 .$$

The coefficients c_{ij} appear in table 3.

Step 1. Consider the cusps $\frac{5}{6}$ and $\frac{1}{12}$. At each of these cusps ρ has a pole of order one while $\tilde{\kappa}$ has a pole of order 10. Since $P(\rho, \tilde{\kappa})$ can have no poles, we may conclude that $c_{22,4} = \ldots = c_{3,4} = c_{22,3} = \ldots = c_{13,3} = 0$.

Step 2. At the cusp $\frac{7}{30}$, we see that ρ is nonzero while $\tilde{\kappa}$ has a pole of order 2. By considering the x^{-8} and x^{-7} coefficients of the polynomial $P(\rho, \tilde{\kappa})$, we get two equations in $c_{i,4}$. Since one coefficient in P is arbitrary, we may set $c_{0,4}$ equal to 1, and solve for $c_{1,4}$ and $c_{2,4}$. Therefore, after completing steps 1 and 2, we have completely determined $c_{i,4}$ for $i = 0, \ldots, 22$.

TABLE 3
The coefficients c_{ij} of $P(\rho, \tilde{\kappa}) = 0$ in the disordered regime.

$i \backslash j$	0	1	2	3	4
0	186624	−190944	3877	442	1
1	−4105728	4574016	−98398	−6168	−2
2	43110144	−51321600	1072713	37230	1
3	−287400960	360237888	−6981832	−130984	0
4	1365154560	−1780447824	31291462	303237	0
5	−4914556416	6606372960	−104811948	−491616	0
6	13924576512	−19147033584	276579010	576476	0
7	−31827603456	44499329856	−595159240	−494868	0
8	59676756480	−84438496656	1066656183	309375	0
9	−92830510080	132446333088	−1610038466	−137500	0
10	120679663104	−173147673168	2055774797	41250	0
11	−131650541568	189573384960	−2221200636	−7500	0
12	120679663104	−174162517632	2026754828	625	0
13	−92830510080	134143224768	−1556002700	0	0
14	59676756480	−86293880064	999566100	0	0
15	−31827603456	46038428352	−532982500	0	0
16	13924576512	−20144497392	233167375	0	0
17	−4914556416	7110122976	−82293750	0	0
18	1365154560	−1975050000	22859375	0	0
19	−287400960	415800000	−4812500	0	0
20	43110144	−62370000	721875	0	0
21	−4105728	5940000	−68750	0	0
22	186624	−270000	3125	0	0

Step 3. We move to the cusps $i\infty$, $\frac{3}{5}$, $\frac{7}{10}$, and $\frac{2}{15}$. At the cusp we see that $\tilde{\kappa}$ is nonzero while ρ has a zero of order 1. This gives us four equations (one at each cusp) in the four unknowns $c_{n,0}$ $c_{n,1}$, $c_{n,2}$, and $c_{n,3}$. Each equation is defined in terms of c_{ij} with $i<n$ and $c_{n,4}$. Therefore we may recursively calculate the remaining coefficients. In this case it is necessary to have the series expansions of $\tilde{\kappa}$ and ρ out to the x^{22} term.

4.4. $P(\rho, z) = 0$ FOR ρ AND z IN THE DISORDERED REGIME

In this subsection we will consider the polynomial $P(\rho, z) = 0$, for ρ and z in the disordered regime. This is certainly the most involved algorithm of the four. We see that val$(\rho) = 8$ and val$(z) = 24$ and we note that the cusp pairs, $(i\infty, \frac{11}{30})$, $(\frac{4}{5}, \frac{1}{5})$, etc., and the cusp expansions of ρ, z do not change. Thus we may use theorem 3.1 and halve the degrees of the polynomial. Hence P looks like

$$P(\rho, z) = \sum_{i=0}^{12} \sum_{j=0}^{4} c_{ij}\rho^i z^j = 0 .$$

The coefficients c_{ij} appear in table 4.

TABLE 4
The coefficients c_{ij} of $P(\rho, z) = 0$ in the disordered regime.

i \ j	0	1	2	3	4
0	0	1	0	0	0
1	−1	−18	0	0	0
2	11	141	−1	0	0
3	−55	−635	15	0	0
4	165	1840	−90	0	0
5	−330	−3641	264	1	0
6	462	5095	−319	−13	0
7	−462	−5148	−264	66	0
8	330	3795	1485	−165	0
9	−165	−2035	−2255	220	0
10	55	770	1760	−165	0
11	−11	−187	−714	77	−1
12	1	22	119	−22	1

Step 1. Consider the cusp $i\infty$ at which both ρ and z have a simple zero. This implies that $c_{0,0} = 0$ and that

$$c_{1,0} + c_{0,1} = 0. \tag{4.6}$$

Step 2. We move to the cusps $\frac{3}{5}$, $\frac{3}{10}$, and $\frac{2}{15}$. At all three cusps, ρ has a simple zero, while at $\frac{3}{5}$, z has a pole of order 6; at $\frac{3}{10}$, z has a pole of order 3; and at $\frac{2}{15}$, z has a pole of order 2. Using this information we can conclude from looking at the poles of $P(\rho, z)$ that

$$c_{0,4} = c_{1,4} = \ldots = c_{10,4} = 0, \qquad c_{0,3} = c_{1,3} = \ldots = c_{4,3} = 0, \qquad c_{0,2} = c_{1,2} = 0.$$

Looking at poles of lower order, we obtain the equations

$$\mathrm{co}\, f_{3/5}(-13, P) = c_{11,4} + c_{5,3} = 0,$$

$$\mathrm{co}\, f_{3/10}(-4, P) = c_{5,3} + c_{2,2} = 0,$$

$$\mathrm{co}\, f_{2/15}(-2, P) = c_{2,2} + c_{0,1} = 0, \tag{4.7}$$

$$\mathrm{co}\, f_{3/5}(-12, P) = c_{12,4} + c_{6,3} = -\mathrm{co}\, f_{3/5}(-12, c_{11,4}\rho^{11}z^4 + c_{5,3}\rho^5 z^3). \tag{4.8}$$

These will be useful shortly.

For steps 3 and 4 it is necessary to define a new function $\tilde{\rho}$ by

$$\tilde{\rho} = \rho/(1-\rho).$$

Note that $\tilde{\rho}$ has a pole where ρ has leading coefficient 1. In addition it is clear that $\tilde{\rho}$ is still modular with respect to the same modular subgroup as ρ and it also has the same valence.

Step 3. At the cusp $\frac{7}{30}$ we see that z and $\tilde{\rho}$ each has a pole of order 1. This implies that $c_{12,4} = 1$ and $c_{11,4} = 1$, and the coefficients c_{ij} for $j = 4$ are determined. Since $c_{11,4} = -1$, we have from eqs. (4.7) and (4.8) that $c_{5,3} = 1$, $c_{6,3} = -13$, $c_{2,2} = -1$, and $c_{0,1} = 1$ and from eq. (4.6) that $c_{1,0} = -1$. In particular, note that the coefficients c_{ij} for $i \leqslant 6$ are determined.

Step 4. We consider the cusps $\frac{4}{5}$, $\frac{1}{10}$, and $\frac{1}{15}$. Again using $\tilde{\rho}$, we see that we have a situation much like step 2, only with zeros and poles interchanged. Using a similar argument, we see that $c^*_{12,0} = c^*_{11,0} = \dots = c^*_{2,0} = 0$. Here c^*_{ij} indicates the coefficient of the corresponding polynomial in $\tilde{\rho}$. Since $c_{0,0} = 0$, we have that $c^*_{0,0} = 0$. Therefore only $c^*_{1,0}$ remains undetermined. It is straightforward to determine this and to hence conclude that

$$c_{i,0} = (-1)^{i+1}\binom{11}{i-1}.$$

We have now completely determined c_{ij} for $j = 0$ and $j = 4$.

Step 5. We will now find a recursive set of equations which will determine c_{ij} for $j = 1, 2, 3$ in terms of c_{kj} for $k > i$ and c_{ij} for $j = 0, 4$. We will start with $i = 12$ and proceed in steps of -1.

First we observe that at the cusp $i\infty$, both ρ and z have power series whose coefficients are elements of $\mathbb{Q}$. This implies that the coefficients c_{ij} of the polynomial $P(\rho, z) = 0$ are also in $\mathbb{Q}$. This is an important fact, for we will have equations of the form:

$$\sum_{j=0}^{4} a_j c_{ij} = 0,$$

where the numbers a_j are in $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$, an extension of $\mathbb{Q}$. Define conjugation in $\mathbb{Q}(\sqrt{5})$ by $\overline{a + \sqrt{5}\,b} = a - \sqrt{5}\,b$. Note that conjugation leaves elements of $\mathbb{Q}$ unchanged. Upon conjugation of the above equation, we have a second equation in c_{ij}, namely

$$\sum_{j=0}^{4} \bar{a}_j c_{ij} = 0 .$$

Consider the cusps $\frac{1}{7}$ and $\frac{1}{6}$. At $\frac{1}{7}$ we have the power series:

$$\rho = \rho_0 x^{-1}(1 + r_1 x + \dots), \qquad z = z_0\left(1 + z_6 x^6 + \dots\right),$$

where $\rho_0 = (\sqrt{5}\,)^{-1}$, $z_0 = \zeta_2^5 = (11 + 5\sqrt{5}\,)/2$, and r_i, $z_i \in \mathbb{Q}(\sqrt{5}\,)$. As usual, we may obtain one equation, and in this case a second by conjugation, for c_{nk}, $k = 0,\dots,4$, which is recursive in n. This procedure is somewhat complicated by the fact that z has no x^i coefficient for $i = 1,\dots,5$.

To obtain at least one more equation, we now look at the cusp $\frac{1}{6}$. Here we have the expansions

$$\rho = \rho_0 x^{-1}(1 + r_1 x + \dots), \qquad z = \bar{z}_0(1 + \bar{z}_6 x + \dots),$$

where ρ_0, z_0, r_i, and z_i are the same as before. In fact, we get the previous expansion of z from this one by replacing x by x^6 and conjugating. This fact is needed in determining the recursive set of equations. The equations we obtain are, for $n = 12,\dots,7$,

$$\operatorname{co} f_{1/7}(0, Z_n) = 0 ,$$

$$\rho_0^n \operatorname{co} f_{1/6}(1, Z_n) = \begin{cases} -\operatorname{co} f_{1/6}(-n+1, R_{n+1}), & n \neq 12, \\ 0, & n = 12, \end{cases}$$

where $Z_n = c_{n,0} + c_{n,1} z + \dots + c_{n,4} z^4$ and $R_n = \rho^{12} Z_{12} + \dots + \rho^n Z_n$. We proceed recursively from $n = 12$ to $n = 7$. At each stage the right-hand side is completely determined. Thus we have three (actually four) equations in the unknown coefficients $c_{n,j}$, $j = 1,2,3$. Once $n \leqslant 6$, only $c_{n,j}$ for $j = 1$ and 2 are undetermined. Thus only one equation and its conjugate are needed. A good choice would be

$$\rho_0^n \operatorname{co} f_{1/7}(0, Z_n) = -\operatorname{co} f_{1/7}(-n, R_{n+1}) .$$

For this polynomial we need the cusp expansions of ρ and z out to x^{12}.

C.A.T. would like to thank Prof. Avner Friedman and Prof. Willard Miller, Jr. for the invitation to visit the IMA at the University of Minnesota, where part of this

work was done. Also, he would like to thank Dr. George A. Baker, Jr. for helpful comments.

Appendix

CUSP EXPANSIONS

In tables A.1–A.5 cusp expansions are given. To simplify the tables we will use the following notation:

$$[j] = \prod_{m=1}^{\infty} (1 - x^{jm}), \qquad \{j, k\} = \prod_{m=1}^{\infty} (1 - x^{jm-k}),$$

$$[j, k] = \prod_{m=1}^{\infty} (1 - x^{jm-k})(1 - x^{jm+k-j}),$$

TABLE A.1

Cusp expansion of $\Delta\rho(\tau) = \sqrt{5}\,(\rho - \rho_c)$, ordered regime, at inequivalent cusps of Γ_1 [15]

Cusp	Local expansion
$0, \frac{1}{4}$	$x^2\left(\frac{[15]}{[5]}\right)^2 \frac{[5,1][15,3]}{[15,6]^2}$
$\frac{1}{6}, \frac{1}{9}$	$-\frac{1}{3}x^{-1}\left(\frac{[5]}{[15]}\right)^2 \frac{[15,6][5,1]}{[5,2]^2}$
$\frac{1}{2}, \frac{1}{7}$	$\left(\frac{[15]}{[5]}\right)^2 \frac{[5,2][15,6]}{[15,3]^2}$
$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{3}x^{-1}\left(\frac{[5]}{[15]}\right)^2 \frac{[15,3][5,2]}{[5,1]^2}$
$i\infty, \frac{4}{15}$	$\frac{3-\sqrt{5}}{6}\left(\frac{[1]}{[3]}\right)^2 \frac{[1;\omega_t][3;\omega_t]}{[1;\omega_5^2]^2}$
$\frac{1}{5}, \frac{4}{5}$	$\frac{1-\sqrt{5}}{2}\left(\frac{[3]}{[1]}\right)^2 \frac{[1;\omega_5^2][3;\omega_5]}{[3;\omega_5^2]^2}$
$\frac{2}{5}, \frac{3}{5}$	$\frac{1+\sqrt{5}}{2}\left(\frac{[3]}{[1]}\right)^2 \frac{[1;\omega_5][3;\omega_5^2]}{[3;\omega_5]^2}$
$\frac{2}{15}, \frac{7}{15}$	$\frac{3+\sqrt{5}}{6}\left(\frac{[1]}{[3]}\right)^2 \frac{[3;\omega_5^2][1;\omega_5^2]}{[1;\omega_5]^2}$

$$[j; a] = \prod_{m=1}^{\infty} (1 - a x^{jm})(1 - a^{-1} x^{jm}),$$

$$[j, k; a] = \prod_{m=1}^{\infty} (1 - a x^{jm-j})(1 - a^{-1} x^{jm+k-j}).$$

In each of the tables, let $\omega_n = \exp(2\pi i/n)$. To convert a local expansion in Γ_1 [5] to Γ_1 [15] or Γ_1 [30] use the conversion tables A.6 and A.7. For example, to find the cusp expansion z in the ordered regime at the cusp $\frac{1}{15}$, use the expansion of z given at the equivalent cusp of Γ_1 [5], namely, $i\infty$, and replace x by x^2.

TABLE A.2
Expansion of $\kappa^3(\tau)$, ordered regime, at inequivalent cusps of Γ_1 [15]. Use the lower sign for second cusp. $c_1 = \sin^2(\pi/5)/\sin^3(2\pi/5)$ and $c_2 = \sin^2(2\pi/5)/\sin^3(\pi/5)$

Cusp	$\kappa^3(\tau)$
$i\infty, \frac{4}{15}$	$\pm x^{-1} \frac{[5]^6}{[3]^6} \frac{[5,2]^6}{[5,1]^9} [3,1]^3$
$\frac{1}{5}, \frac{4}{5}$	$\pm i 3^{9/2} \frac{[15]^6}{[1]^6} \frac{[15,6]^6}{[15,3]^9} [1; \omega_3]^3$
$0, \frac{1}{4}$	$\pm \left(\frac{3\sqrt{3}}{10} c_2 \right)^3 \frac{[3]^6}{[5]^6} \frac{[3; \omega_5^2]^6}{[3; \omega_5]^9} [5; \omega_3]^3$
$\frac{1}{6}, \frac{1}{9}$	$\pm i \left(\frac{c_2}{10} \right)^3 x^{-5} \frac{[1]^6}{[15]^6} \frac{[1; \omega_5^2]^6}{[1; \omega_5]^9} [15,5]^3$
$\frac{1}{7}, \frac{1}{2}$	$\pm \left(\frac{3\sqrt{3}}{10} c_1 \right)^3 \frac{[3]^6}{[5]^6} \frac{[3; \omega_5]^6}{[3; \omega_5^2]^9} [5; \omega_3]^3$
$\frac{2}{3}, \frac{1}{3}$	$\pm i \left(\frac{c_1}{10} \right)^3 x^{-5} \frac{[1]^6}{[15]^6} \frac{[1; \omega_5]^6}{[1; \omega_5^2]^9} [15,5]^3$
$\frac{3}{5}, \frac{2}{5}$	$\pm i 3^{9/2} x^9 \frac{[15]^6}{[1]^6} \frac{[15,3]^6}{[15;6]^9} [1, \omega_3]^3$
$\frac{2}{15}, \frac{7}{15}$	$\pm x^2 \frac{[15]^6}{[3]^6} \frac{[5,1]^6}{[5,2]^9} [3,1]^3$

TABLE A.3
Local expansions of $\rho(\tau)$, disordered regime, at inequivalent cusps of Γ_1 [30]. $\rho_c = (5-\sqrt{5})/10$, $\rho_{NP} = (5+\sqrt{5})/10$, and $\rho_0 = 1/\sqrt{5}$

Cusp	Local expansion
$i\infty, \frac{11}{30}, \frac{3}{5}, \frac{2}{5}$	$-x\dfrac{\{6,3\}}{\{2,1\}[5,1][30,12]}$
$\frac{1}{5}, \frac{4}{5}, \frac{7}{30}, \frac{13}{30}$	$\dfrac{\{6,3\}}{\{2,1\}[30,6][5,2]}$
$\frac{1}{10}, \frac{9}{10}, \frac{1}{15}, \frac{4}{15}$	$\dfrac{\{2,1\}}{\{6,3\}[10,2][15,3]}$
$0, \frac{1}{11}, \frac{1}{12}, \frac{5}{12}$	$-\rho_0 x^{-1}\dfrac{\{30,15\}}{\{10,5\}[6;\omega_5][1;\omega_5^2]}$
$\frac{1}{2}, \frac{1}{8}, \frac{1}{3}, \frac{2}{3}$	$\rho_c\dfrac{\{10,5\}}{\{30,15\}[3;\omega_5^2][2;\omega_5^2]}$
$\frac{1}{4}, \frac{1}{14}, \frac{1}{9}, \frac{2}{9}$	$\rho_{NP}\dfrac{\{10,5\}}{\{30,15\}[3;\omega_5][2;\omega_5]}$
$\frac{1}{7}, \frac{1}{13}, \frac{1}{6}, \frac{5}{6}$	$\rho_0 x^{-1}\dfrac{\{30,15\}}{\{10,5\}[6;\omega_5^2][1;\omega_5]}$
$\frac{3}{10}, \frac{7}{10}, \frac{2}{15}, \frac{7}{15}$	$x\dfrac{\{2,1\}}{\{6,3\}[10,4][15,6]}$

TABLE A.4
Cusp expansion of $\zeta^5(\tau)$ at inequivalent cusps of Γ_1 [5]. In the ordered regime $z=\zeta^{-5}$, in the disordered regime $z=-\zeta^5$. $\zeta_1=(\sqrt{5}-1)/2$ and $\zeta_2=(\sqrt{5}+1)/2$

Cusp	Local expansion
$i\infty$	$x\left(\dfrac{[5,1]}{[5,2]}\right)^5$
0	$\zeta_1^5\left(\dfrac{[1;\omega_5]}{[1;\omega_5^2]}\right)^5$
$\frac{2}{5}$	$-x^{-1}\left(\dfrac{[5,2]}{[5,1]}\right)^5$
$\frac{1}{2}$	$-\zeta_2^5\left(\dfrac{[1;\omega_5^2]}{[1;\omega_5]}\right)^5$

TABLE A.5
Cusp expansions of $\kappa(\tau)$, disordered regime, at inequivalent cusps of Γ_1 [30]. Use lower sign for second cusp. c_1 and c_2 are defined in table A.2.

Cusp	Local expansion
$i\infty, \frac{11}{30}$	$\pm \dfrac{[5]^2[5,1]^2[6,3][6,2]}{[5,2]^3[6,1][6]^2}$
$\frac{4}{5}, \frac{1}{5}$	$\pm 12i\sqrt{3}\,x^6 \dfrac{[30]^2[30,6]^2[1;-1][1;\omega_3]}{[30,12]^3[1;\omega_6][1]^2}$
$\frac{1}{10}, \frac{9}{10}$	$\pm 3i\sqrt{3}\,x^3 \dfrac{[15]^2[15,3]^2[2,1][2;\omega_3]}{[15,6]^3[2,1;\omega_3][2]^2}$
$\frac{1}{15}, \frac{4}{15}$	$\pm 4x^2 \dfrac{[10]^2[10,2]^2[3;-1[3,1]}{[10,4]^3[3,1;-1][3]^2}$
$\frac{3}{5}, \frac{2}{5}$	$\pm 12i\sqrt{3} \dfrac{[30]^2[30,12]^2[1;-1][1;\omega_3]}{[30,6]^3[1;\omega_6][1]^2}$
$\frac{7}{10}, \frac{3}{10}$	$\pm 3i\sqrt{3} \dfrac{[15]^2[15,6]^2[2,1][2;\omega_3]}{[15,3]^3[2,1;\omega_3][2]^2}$
$\frac{7}{15}, \frac{2}{15}$	$\pm 4 \dfrac{[10]^2[10,4]^2[3;-1][3,1]}{[10,2]^3[3,1;-1][3]^2}$
$\frac{7}{30}, \frac{13}{30}$	$\pm x^{-1} \dfrac{[5]^2[5,2]^2[6,3][6,2]}{[5,1]^3[6,1][6]^2}$
$0, \frac{1}{11}$	$\pm \dfrac{6\sqrt{3}}{5} c_1 \dfrac{[6]^2[6;\omega_5]^2[5;-1][5;\omega_3]}{[6;\omega_5^2]^3[5;\omega_6][5]^2}$
$\frac{1}{14}, \frac{1}{4}$	$\pm \dfrac{3\sqrt{3}}{10} c_1 \dfrac{[3]^2[3;\omega_5]^2[10,5][10;\omega_3]}{[3;\omega_5^2]^3[10,5;\omega_3][10]^2}$
$\frac{1}{9}, \frac{2}{9}$	$\pm \dfrac{2ic_1}{5} \dfrac{[2]^2[2;\omega_5]^2[15;-1][15,5]}{[2;\omega_5^2]^3[15,5;-1][15]^2}$
$\frac{5}{6}, \frac{1}{6}$	$\pm \dfrac{i}{10} c_1 x^{-5} \dfrac{[1]^2[1;\omega_5^2]^2[30,15][30,10]}{[1;\omega_5]^3[30,5][30]^2}$
$\frac{1}{2}, \frac{1}{8}$	$\pm \dfrac{3\sqrt{3}}{10} c_2 \dfrac{[3]^2[3;\omega_5]^2[10,5][10;\omega_3]}{[3;\omega_5]^3[10,5;\omega_3][10]^2}$
$\frac{1}{13}, \frac{1}{7}$	$\pm \dfrac{6\sqrt{3}}{5} c_2 \dfrac{[6]^2[6;\omega_5^2]^2[5;-1][5;\omega_3]}{[6;\omega_5]^3[5;\omega_6][5]^2}$
$\frac{2}{3}, \frac{1}{3}$	$\pm \dfrac{2ic_2}{5} \dfrac{[2]^2[2;\omega_5^2][15;-1][15,5]}{[2;\omega_5]^3[15,5;-1][15]^2}$
$\frac{1}{12}, \frac{5}{12}$	$\pm \dfrac{i}{10} c_2 x^{-5} \dfrac{[1]^2[1;\omega_5^2]^2[30,15][30,10]}{[1;\omega_5]^3[30,5][15]^2}$

TABLE A.6
Conversion table for local variable "x" in Γ_1 [5] to local variable in Γ_1 [30]. The cusps are grouped in Γ_1 [5]-equivalent sets.

Cusp	New variable
$i\infty, \frac{11}{30}$	x
$\frac{1}{5}, \frac{4}{5}$	x^6
$\frac{1}{10}, \frac{9}{10}$	x^3
$\frac{1}{15}, \frac{4}{15}$	x^2
$\frac{3}{5}, \frac{2}{5}$	x^6
$\frac{3}{10}, \frac{7}{10}$	x^3
$\frac{2}{15}, \frac{7}{15}$	x^2
$\frac{7}{30}, \frac{13}{30}$	x
$0, \frac{1}{11}$	x^6
$\frac{1}{4}, \frac{1}{14}$	x^3
$\frac{1}{9}, \frac{2}{9}$	x^2
$\frac{1}{6}, \frac{5}{6}$	x
$\frac{1}{2}, \frac{1}{8}$	x^3
$\frac{1}{7}, \frac{1}{13}$	x^6
$\frac{1}{3}, \frac{2}{3}$	x^2
$\frac{1}{12}, \frac{5}{12}$	x

TABLE A.7
Conversion table for local variable "x" in Γ_1 [5] to local variable in Γ_1 [15]. The cusps are grouped in Γ_1 [5]-equivalent sets.

Cusp	New variable
$0, \frac{1}{4}$	x^3
$\frac{1}{6}, \frac{1}{9}$	x
$\frac{1}{2}, \frac{1}{7}$	x^3
$\frac{1}{3}, \frac{2}{3}$	x
$i\infty, \frac{4}{15}$	x
$\frac{1}{5}, \frac{4}{5}$	x^3
$\frac{2}{3}, \frac{3}{5}$	x^3
$\frac{2}{15}, \frac{7}{15}$	x

References

[1] R.J. Baxter, J. Phys. A13 (1980) L61
[2] R.J. Baxter, J. Stat. Phys. 13 (1981) 427
[3] R.J. Baxter, Exactly solved models in statistical mechanics (Academic Press, London, 1982)
[4] D.L. Hunter and G.A. Baker, Jr., Phys. Rev. B19 (1979) 3808
[5] G.S. Joyce, Phil. Trans. R. Soc. London A. 325 (1988) 643
[6] P.A. Pearce and R.J. Baxter, J. Phys. A17 (1984) 2095
[7] M.P. Richey and C.A. Tracy, J. Phys. A20 (1987) L1121
[8] B. Schoenberg, Elliptic modular functions (Springer, Berlin, 1974)
[9] C.A. Tracy, L. Grove and M.F. Newman, J. Stat. Phys. 48 (1986) 477
[10] C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312