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# HOLONOMIC QUANTUM FIELD THEORY OF BOSONS IN THE POINCARÉ DISK AND THE ZERO CURVATURE LIMIT

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We formulate a holonomic quantum field theory of bosons on a Poincaré disk of radius $R$ ($\mathbf{D}_R$) based on a monodromy preserving deformation of the laplacian operator defined on $\mathbf{D}_R$. First the isomonodromy problem for the laplacian is converted to an isomonodromy problem for a fuchsian system using the hyperbolic Laplace transform defined on $\mathbf{D}_R$. The deformation equations for the fuchsian system (Schlesinger equations) and an associated closed one-form $\omega$ are then discussed. Locally $\omega = \mathrm{d}\log\tau$ defines a $\tau$-function which is then identified with an $n$-point function of bosons defined on $\mathbf{D}_R$. At every step we discuss the limit $R \to \infty$ where the problem reduces to one on the euclidean plane. This facilitates a detailed comparison with the original results of Sato, Miwa and Jimbo on the euclidean plane. The two-point function is discussed in some detail and it is shown that it can be expressed in terms of a Painlevé transcendent of the sixth kind.

## 1. Introduction

Conformal field theory tells us a great deal about correlation functions of critical, or equivalently massless, 2D quantum field theories (see references in ref. [3]). On the other hand, progress in correlation functions for massive 2D quantum field theories has been restricted to a smaller class of models. The most general methods presently available are those coming from a series of papers [15] by Sato, Miwa and Jimbo (henceforth, SMJ I, II, etc.) called Holonomic Quantum Fields. The most important special case of their theory is the massive Ising field theory [10, 18] on $\mathbf{R}^2$ (see also ref. [13]). One direction to pursue to enlarge the class of solvable, massive 2D quantum field theories, is to construct holonomic quantum

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field theories on two-dimensional manifolds other than $\mathbf{R}^2$. Since the hyperbolic plane is the universal covering space for Riemann surfaces of genus greater than one, it is natural to first extend the SMJ analysis to massive Klein–Gordon and Dirac operators on the hyperbolic plane. For the case of the Klein–Gordon equation this was initiated in ref. [17] and for the Dirac case in ref. [14]. The model of 2D hyperbolic space used in these papers is the Poincaré upper half-plane. This model of the hyperbolic plane has the disadvantage that it is difficult to make a connection with the results of SMJ in $\mathbf{R}^2$ in the limit the curvature tends to zero. It is the purpose of this paper to establish a detailed connection between the euclidean results of SMJ and the hyperbolic results for the Klein–Gordon equation [17]. The corresponding results for the Dirac case will be considered in a future publication.

The hyperbolic plane can also be modeled by the Poincaré disk of radius $R$, $\mathbf{D}_R$. In this model the curvature of the manifold is $-(2/R)^2$. This then is a convenient model to analyze the limit of zero curvature, or equivalently, $R \to \infty$. In this paper we will show that the results associated with monodromy preserving deformation of the Klein–Gordon equation in SMJ III can be obtained as the limiting case of the corresponding results on $\mathbf{D}_R$. It is interesting to note that the analysis on $\mathbf{D}_R$ is in some sense nicer than the euclidean case, i.e. irregular singularities in $\mathbf{R}^2$ get replaced with regular singularities in $\mathbf{D}_R$, and the irregular singularities arise as a confluence of regular singularities as $R \to \infty$. This point was also observed by Atiyah [1] while working on the problem of monopoles.

A brief outline of the SMJ theory [15] is as follows (the review by Jimbo [5] is recommended): Consider the vector space $\mathrm{W}(L, A)$ of multivalued solutions $\phi$ of the (elliptic) Klein–Gordon equation on $\mathbf{R}^2 - \{a_1, \ldots, a_n\}$ satisfying the conditions: (i) any solution $\phi$ when analytically continued around any one of the points $a_j$ picks up a phase $\exp(2\pi i l_j)$, where $0 < l_j < 1$ are fixed, and (ii) such $\phi$ are of finite energy

$$\int_{\mathbf{R}^2} (4\,\bar{\partial}\phi\,\partial\bar{\phi} + m^2 \phi\bar{\phi})\,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{1.1}$$

An important result is that $\mathrm{W}(L, A)$ is finite dimensional and has dimension equal to $n$, the number of points removed from $\mathbf{R}^2$. The condition that the $l_j$ are fixed, and hence independent of $a_j$ and $\bar{a}_j$, is the condition of *isomonodromy*. Picking a basis of such wave functions, the Klein–Gordon system of equations satisfied by this basis of wave functions can be extended to a total system of linear differential equations in $z$ and $\bar{z}$. The condition of isomonodromy now results in a set of nonlinear equations, called the *deformation equations*, in the variables $a_j$ and $\bar{a}_j$ $(j = 1, \ldots, n)$. Associated with these deformation equations is a closed one-form $\omega$. It is shown in SMJ IV that this one-form $\omega$ is locally $\mathrm{d}\log\tau$ where $\tau(a_1, \ldots, a_n)$ is the vacuum expectation value of $n$ interacting bosonic quantum fields defined on

$\mathbf{R}^2$. Although the analysis is carried out in the Klein–Gordon variables $z$ and $\bar{z}$, it is pointed out in SMJ III that the (relativistic) Laplace transform of the extended linear system in $z$ and $\bar{z}$ results in a system of ordinary differential equations in the transform variable $u$. This transformed ODE has irregular singular points of rank one at $u = 0$ and $u = \infty$. The SMJ III deformation equations can then be viewed as an irregular singular point version of the classic Schlesinger equations [16]. That this is indeed the case was proved by Jimbo et al. [6] in their quite general theory of monodromy preserving deformation of ordinary differential equations which allows both regular and irregular singular points. Their theory defines a closed one-form $\omega$ associated to the deformation equations. When their theory is applied to the special case of the transformed ODE appearing in SMJ III, it is observed that the two $\omega$'s are equal.

In ref. [17] the corresponding problem of isomonodromy of the Klein–Gordon equation on $\mathbf{H}^2$ was studied. With the aid of the hyperbolic Laplace transform, the extended linear system was transformed into a fuchsian system in the transform variable. This is simpler than the euclidean case in the sense that the deformation equations resulting from the condition of isomonodromy are the Schlesinger equations. In their study of the Riemann–Hilbert problem [15], SMJ introduced a closed one-form $\omega$ expressible in terms of solutions of the Schlesinger equations. Furthermore, they showed that this "Riemann–Hilbert/Schlesinger" $\omega$ is the logarithmic derivative of the vacuum expectation of a product of "Riemann–Hilbert" fields $\Phi(a)$.

After some preliminary material on the Poincaré disk model of the hyperbolic plane in sect. 2, we formulate in sect. 3 the bosonic problem on $\mathbf{D}_R$. In sect. 4 we derive the extended system of linear equations. Also in sect. 4, we use the hyperbolic Laplace transform to convert this extended system to a fuchsian system of order $n$ with singular points at $a_j \in \mathbf{D}_R$ and at the conjugate points $R^2/\bar{a}_j$ ($j = 1, \dots, n$). In sect. 5 we show that the limit of this fuchsian system as $R \to \infty$ is precisely the transformed ODE of SMJ III. The closed one-form $\omega$ associated with the Schlesinger equations is obtained as a function of $R$. It is shown that as $R \to \infty$ this $\omega$ approaches the $\omega$ obtained in SMJ III for the euclidean Klein–Gordon equation. Therefore, the $\omega$ obtained on $\mathbf{D}_R$ is then $\mathrm{d} \log \tau$ where $\tau(a_1, \dots, a_n; R)$ is the $n$-point function of bosonic quantum fields defined on $\mathbf{D}_R$. We study in sect. 6 the $n = 2$ case in some detail, and show that the differential equation satisfied by $\tau$ is related to the Painlevé equation of the sixth kind ($\mathrm{P_{VI}}$). It is also shown that the differential equation satisfied by $\tau$ on $\mathbf{D}_R$ approaches the differential equation satisfied by the euclidean $\tau$-function as $R \to \infty$.

## 2. Poincaré model of hyperbolic geometry

In this section we present various properties of the Poincaré disk model of the hyperbolic plane that we will use in later sections. A reference for this section is

Helgason [2]. We keep the radius, $R$, of the disk explicit since we are interested in obtaining the results in the euclidean plane as $R \to \infty$. The metric on $\mathbf{D}_R$ is

$$\mathrm{d}s_R^2 := \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\left(1 - (x^2 + y^2)/R^2\right)^2}, \tag{2.1a}$$

and the invariant measure is

$$\mathrm{d}\mu_R := \frac{\mathrm{d}x \wedge \mathrm{d}y}{\left(1 - (x^2 + y^2)/R^2\right)^2}, \tag{2.1b}$$

where $x$ and $y$ are the cartesian coordinates in the euclidean plane in which $\mathbf{D}_R$ is embedded with the origin being at the center of the disk. The curvature is then $-(2/R)^2$. As $R \to \infty$ the disk $\mathbf{D}_R$ approaches the euclidean plane and the metric $\mathrm{d}s_R^2$ and invariant measure $\mathrm{d}\mu_R$ approach the standard euclidean metric and measure, respectively. The disk $\mathbf{D}_R$ can be mapped onto the upper half-plane by

$$w = -i\frac{z + iR}{z - iR}, \tag{2.2}$$

where $z = x + iy$ is the complex coordinate* on $\mathbf{D}_R$ and $w$ is the corresponding complex coordinate on the upper half-plane. The boundary of the disk $\mathbf{D}_R$ maps onto the real line. There is also a connection between $\mathbf{D}_R$ and a hyperboloid in three-dimensional Minkowski space. This is seen by letting

$$z = R\,\mathrm{e}^{i\theta} \tanh\tfrac{1}{2}r \tag{2.3}$$

which is also called the geodesic polar coordinates on $\mathbf{D}_R$ centered at the origin. The metric given by eq. (2.1a) then becomes

$$\mathrm{d}s^2 = \tfrac{1}{2}R^2(\mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\theta^2). \tag{2.4}$$

The above metric is the usual Minkowski metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2$ restricted to the hyperboloid $x^2 + y^2 - t^2 = (R/2)^2$.

The Poincaré disk $\mathbf{D}_R$ can be identified with the homogeneous space SU(1,1)/SO(2). An element $g$ of the isometry group SU(1,1) is given by a $2 \times 2$ matrix

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

* SMJ III use the convention $z = \frac{1}{2}(x + iy)$. This must be kept in mind, when in the latter sections, we identify certain formulae with those in SMJ III.

with the condition $a\bar{a} - b\bar{b} = 1$. The action of $g$ on $z \in \mathbf{D}_R$ is given by

$$z \to \frac{az + b}{\bar{b}z + \bar{a}} . \tag{2.5}$$

The corresponding Lie algebra su(1, 1) has three generators; one choice being

$$X_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad X_3 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{2.6}$$

with brackets

$$[X_1, X_2] = -2X_3, \qquad [X_2, X_3] = 2X_1, \qquad [X_3, X_1] = 2X_2 . \tag{2.7}$$

To each $X_i \in \mathrm{su}(1, 1)$ we associate a vector field $L_i$ acting on the $\mathrm{C}^\infty$-functions $f$: $\mathbf{D}_R \to \mathbf{C}$ by

$$(L_i f)(x, y) := \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tX_i)(x, y))\big|_{t=0} . \tag{2.8}$$

For the generators (2.1) we obtain

$$L_1 = \left(R - \frac{z^2}{R}\right)\partial + \left(R - \frac{\bar{z}^2}{R}\right)\bar{\partial}, \qquad L_2 = i\left(R + \frac{z^2}{R}\right)\partial - i\left(R + \frac{\bar{z}^2}{R}\right)\bar{\partial},$$

$$L_3 = 2i(z\partial - \bar{z}\bar{\partial}) , \tag{2.9}$$

where $\partial := \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar{\partial} := \frac{1}{2}(\partial_x + i\partial_y)$. Using the transformation (2.3) the vector fields given in eq. (2.9) become, respectively,

$$\bar{D} = \tfrac{1}{2}(L_1 + iL_2) = \mathrm{e}^{i\theta}\left(\partial_r + i\frac{\cosh r}{\sinh r}\partial_\theta\right),$$

$$D = \tfrac{1}{2}(L_1 - iL_2) = \mathrm{e}^{-i\theta}\left(\partial_r - i\frac{\cosh r}{\sinh r}\partial_\theta\right),$$

$$M = -\tfrac{1}{2}iL_3 = -i\partial_\theta . \tag{2.10}$$

To construct for a point $z \in \mathbf{D}_R$ geodesic polar coordinates centered at a point $z_0$, we first send $z_0$ to the origin by the isometry

$$z \to \frac{w}{R} = \frac{(z - z_0)R}{R^2 - \bar{z}_0 z} \tag{2.11}$$

followed by the transformation (2.3) on $w$. It will be useful to express the $L_i$'s in terms of $\bar{D}$, $D$, $M$ and the point $z_0$. The result is

$$L_1 = \frac{1}{R^2 - z_0\bar{z}_0}\left[\left(R^2 - z_0^2\right) D_{z_0} + \left(R^2 - \bar{z}_0^2\right)\bar{D}_{z_0} + 2R(\bar{z}_0 - z_0)M_{z_0}\right],$$

$$L_2 = \frac{i}{R^2 - z_0\bar{z}_0}\left[\left(R^2 + z_0^2\right) D_{z_0} - \left(R^2 + \bar{z}_0^2\right)\bar{D}_{z_0} + 2R(\bar{z}_0 + z_0)M_{z_0}\right],$$

$$L_3 = \frac{2i}{R^2 - z_0\bar{z}_0}\left[Rz_0 D_{z_0} - R\bar{z}_0\bar{D}_{z_0} + \left(R^2 + z_0\bar{z}_0\right)M_{z_0}\right]. \tag{2.12}$$

The subscript $z_0$ on $D$, $\bar{D}$ and $M$ indicates that the geodesic polar coordinates in eq. (2.10) are centered at $z_0$.

The invariant differential operators on the homogeneous space SU(1, 1)/SO(2) are polynomials of the hyperbolic Laplace operator:

$$\Delta_{\mathrm{H}} = \left(1 - \frac{x^2+y^2}{R^2}\right)^2\left(\partial_x^2 + \partial_y^2\right) = 4\left(1 - \frac{z\bar{z}}{R^2}\right)^2 \partial\bar{\partial}$$

$$= \left(\frac{2}{R}\right)^2\left[\partial_r^2 + \frac{\cosh r}{\sinh r}\partial_r + \frac{1}{\sinh^2 r}\partial_\theta^2\right] = \frac{1}{R^2}\left(L_1^2 + L_2^2 - L_3^2\right). \tag{2.13}$$

The eigenfunctions of $\Delta_{\mathrm{H}}$ are now constructed by a geometrical procedure that is analogous to the euclidean case. Given a point $z \in \mathbf{D}_R$ we first construct a family of parallel geodesics emanating from a point $b$ on the boundary of $\mathbf{D}_R$. Then we construct the curve (called the horocycle) that is perpendicular to all these geodesics and passing through $z$. It is a circle tangential to the boundary at $b$. Let $w$ denote the point on the horocycle that is closest to the origin and let $\langle z, b\rangle$ denote the distance between $w$ and the origin (see fig. 1). If

$$P_s(z, b) := \mathrm{e}^{s\langle z, b\rangle}, \tag{2.14}$$

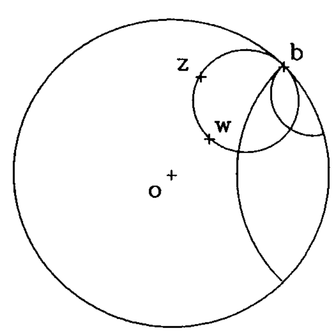


Fig. 1. Poincaré disk showing geodesics through $b$ and associated horocycle through $z$.

then it can be shown that

$$P_s(z,b) = \left[\frac{R^2 - z\bar{z}}{(z-b)(\bar{z}-\bar{b})}\right]^{Rs/2}, \tag{2.15}$$

where $b\bar{b} = R^2$ and

$$\Delta_{\mathrm{H}} P_s(z,b) = s(s-2/R) P_s(z,b). \tag{2.16}$$

It should be noted here that $P_s(z,b)$ is the same for all points $z$ on a horocycle and is parametrized by a point $b$ on the boundary. Also it is interesting to note that $P_s(z,b)$ is the Poisson kernel raised to the $Rs/2$ power. The eigenfunctions of $\Delta_{\mathrm{H}}$ are then of the form

$$f(z,\bar{z}) = \int_{\partial \mathbf{D}_R} P_s(z,b)\tilde{f}(b)\,\mathrm{d}b. \tag{2.17}$$

To be rigorous we must interpret $\tilde{f}(b)\mathrm{d}b$ as a hyperfunction, see refs. [2,8]. We now observe that

$$\lim_{R\to\infty} P_s(z,uR) = \exp\left[\tfrac{1}{2}s(zu^{-1} + \bar{z}u)\right], \tag{2.18}$$

which is the usual kernel, i.e. a plane wave, in the euclidean plane.

The following proposition tells us how the action of the vector fields $L_j$ translate into operators $H_j$ acting on the functions $\tilde{f}$ defined in eq. (2.17).

*Proposition 2.1.* Let $L_j$ $(j = 1,2,3)$ be the vector fields defined by (2.9), and suppose $f(z,\bar{z})$ and $\tilde{f}(b)$ are related by (2.17), then

$$(L_j f)(z,\bar{z}) = \int_{\partial \mathbf{D}_R} P_s(z,b)\left(H_j\tilde{f}\right)(b)\,\mathrm{d}b,$$

where

$$H_1 = \left(\frac{Rs}{2} - 2\right)\left(\frac{b}{R}\right) + \frac{Rs}{2}\left(\frac{\bar{b}}{R}\right) + \left(R - \frac{b^2}{R}\right)\partial_b,$$

$$H_2 = \left(\frac{Rs}{2} - 2\right)\left(\frac{-ib}{R}\right) + \frac{Rs}{2}\left(\frac{i\bar{b}}{R}\right) + i\left(R + \frac{b^2}{R}\right)\partial_b,$$

$$H_3 = 2i(1 + b\partial_b).$$

*Proof.* The proof is identical to the one in ref. [17]. QED

## 3. The space $\mathrm{W}(L, A)$ and local expansions

We are interested in spaces of multivalued solutions to the Klein–Gordon equation

$$\Delta_{\mathrm{H}} f(x, y)=s\left(s-\frac{2}{R}\right) f(x, y), \qquad s>\frac{2}{R}, \tag{3.1}$$

with isolated singularities $a_j \in \mathbf{D}_R$ $(j=1, \ldots, n)$ and specified monodromy at each point $a_j$. It is convenient to introduce the diagonal matrices

$$A=\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \qquad L=\begin{pmatrix} l_1 & & \\ & \ddots & \\ & & l_n \end{pmatrix}, \tag{3.2}$$

where $l_j$ $(j=1, \ldots, n)$ are real numbers between 0 and 1. Let $R_j(\theta)$ denote the rotation by $\theta$ about $a_j$.

*Definition 3.1.* $\mathrm{W}(L, A)$ will denote the complex linear space of multivalued solutions to (3.1) on $\mathbf{D}_R-\{a_1, \ldots, a_n\}$ with isolated singularities at $a_j$ $(j=1, \ldots, n)$ satisfying

$$f\left(R_j(2 \pi)(x, y)\right)=\exp(2 \pi i l_j) f(x, y) \tag{3.3a}$$

for $(x, y)$ near $a_j$ and the finite energy condition

$$I(f, f)=\int_{\mathbf{D}_R}\left(4\left(1-\frac{x^2+y^2}{R^2}\right)^2 \bar{\partial} f \partial \bar{f}+s\left(s-\frac{2}{R}\right) f \bar{f}\right) \mathrm{d} \mu_R<\infty . \tag{3.3b}$$

Because of the monodromy condition (3.3a), the functions $f\bar{f}$ and $\bar{\partial} f \partial \bar{f}$ are single-valued functions on $\mathbf{D}_R$ and the integral in eq. (3.3b) is well defined. Observe, at least formally, that as $R \to \infty$, eq. (3.1) tends to the euclidean Klein–Gordon equation (with the identification of $s=m$) and the finite-energy condition (3.3b) approaches the euclidean finite-energy condition (1.1).

We develop a local representation of a function $f \in \mathrm{W}(L, A)$ (or local expansion) near a point $a_j$ by expanding $f$ in terms of the eigenfunctions of $L_3$ which have the appropriate monodromy given by eq. (3.3a). To achieve this we use the representation of the hyperbolic laplacian in geodesic polar coordinates (see eq. (2.13)) and separate the variables. This along with the finite-energy condition (3.3b) results in the following expansion for $f$ valid for $z$ near $a_j$:

$$\begin{aligned} f(r_j, \theta_j)= & \sum_{k=l_j-1, l_j, l_j+1, \ldots} c_j(k,+) f_1(r_j, \theta_j ; k) \\ & +\sum_{k=l_j-1, l_j-2, l_j-3, \ldots} c_j(k,-) f_2(r_j, \theta_j ; k), \end{aligned} \tag{3.4}$$

where $(r_j, \theta_j)$ are the geodesic polar coordinates (2.3) centered at $a_j$,

$$f_1(r_j, \theta_j; k) = \mathrm{e}^{ik\theta_j} P^{-k}_{Rs/2-1}(\cosh r_j),$$
$$f_2(r_j, \theta_j; k) = \mathrm{e}^{ik\theta_j} P^{k}_{Rs/2-1}(\cosh r_j), \tag{3.5}$$

and $P^{\mu}_{\nu}(z)$ is the Legendre function of the first kind. The local expansion of $f$ near infinity is

$$f(r_\infty, \theta_\infty) = \sum_{k \equiv l_\infty [\mathbf{Z}]} c_\infty(k) f_\infty(r_\infty, \theta_\infty; k), \tag{3.6}$$

where $(r_\infty, \theta_\infty)$ are the geodesic polar coordinates centered at the origin of the disk,

$$f_\infty(r_\infty, \theta_\infty; k) = \mathrm{e}^{ik\theta_\infty} Q^{k}_{Rs/2-1}(\cosh r_\infty), \qquad r_\infty > R_\infty, \tag{3.7}$$

$Q^{\mu}_{\nu}(z)$ is the Legendre function of the second kind,

$$l_\infty \equiv l_1 + \dots + l_n [\mathbf{Z}], \quad 0 < l_\infty < 1,$$

and $R_\infty$ is chosen so that all the points $a_j$ $(j = 1, \dots, n)$ are inside the circle defined by $R_\infty$.

We now consider the limit of the eigenfunctions in eqs. (3.5) and (3.7) as $R \to \infty$. We focus on the Legendre function of the first kind, $P^{-k}_{Rs/2-1}(\cosh r)$. Since $r$ is dimensionless, we write $r := 2\rho/R$ where $\rho$ has dimensions of length. Using the integral representation of the Legendre function we find that

$$\begin{aligned}
&\lim_{R\to\infty} \left(\frac{Rs}{2}\right)^k P^{-k}_{Rs/2-1}\left(\cosh \frac{\rho}{R}\right) \\
&\quad = \lim_{R\to\infty} \frac{((Rs/2)\sinh(\rho/R))^k}{2^{\mu}\sqrt{\pi}\,\Gamma(k+\frac{1}{2})} \int_{-1}^{1} \frac{(1-t^2)^{k-1/2}}{(\cosh(\rho/R) + t\sinh(\rho/R))^{k+1-Rs/2}} \mathrm{d}t \\
&\quad = \frac{(\rho s/4)^k}{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})} \int_{-1}^{1} \mathrm{e}^{t\rho s/2} (1-t^2)^{k-1/2} \mathrm{d}t \\
&\quad = I_k(\rho s/2),
\end{aligned} \tag{3.8}$$

where $I_k(z)$ is the modified Bessel function. This then shows that the eigenfunctions

$$\left(\frac{Rs}{2}\right)^k P^{-k}_{Rs/2-1}\left(\cosh \frac{\rho}{R}\right) \mathrm{e}^{ik\theta}$$

on $\mathbf{D}_R$ go into the eigenfunctions $I_k(\rho s/2)e^{ik\theta}$ of the euclidean laplacian as $R\to\infty$. A similar limit holds for the Legendre function of the second kind.

Using the recursion relations for the Legendre functions it is straightforward to prove:

*Proposition 3.2.*

$$\begin{aligned}
Df_1(r,\theta;k) &= f_1(r,\theta;k-1),\\
Df_2(r,\theta;k) &= (k+\tfrac{1}{2}Rs-1)(\tfrac{1}{2}Rs-k)f_2(r,\theta;k-1),\\
\bar{D}f_1(r,\theta;k) &= (-k+\tfrac{1}{2}Rs-1)(\tfrac{1}{2}Rs+k)f_1(r,\theta;k+1),\\
\bar{D}f_2(r,\theta;k) &= f_2(r,\theta;k+1),\\
Df_\infty(r,\theta;k) &= (k+\tfrac{1}{2}Rs-1)(\tfrac{1}{2}Rs-k)f_\infty(r,\theta;k-1),\\
\bar{D}f_\infty(r,\theta;k) &= f_\infty(r,\theta;k+1),\\
Mf_i(r,\theta;k) &= kf(r,\theta;k),\quad i=1,2,\infty.
\end{aligned}\tag{3.9}$$

Following exactly the arguments in SMJ III the following theorem can be established.

*Theorem 3.3.* The dimension of $\mathrm{W}(L,A)$ is $n$. There exists a basis $\{f_\mu\}_{\mu=1}^n$ of $\mathrm{W}(L,A)$ such that if the local expansion (3.4) of $f_\mu$ at $a_\nu$ is written as

$$\begin{aligned}
f_\mu(r_\nu,\theta_\nu) = &\sum_{k=l_\nu-1,l_\nu,l_\nu+1,\dots} c_{\mu\nu}(k,+)f_1(r_\nu,\theta_\nu;k)\\
&+\sum_{k=l_\nu-1,l_\nu-2,l_\nu-3,\dots} c_{\mu\nu}(k,-)f_2(r_\nu,\theta_\nu;k),
\end{aligned}$$

then $c_{\mu\nu}(l_\nu-1,+)=\delta_{\mu\nu}$.

The basis in this theorem is called the canonical basis. We now introduce the following matrices which will be useful for later purposes.

$$\begin{aligned}
&[C_\pm(k)]_{\mu\nu} = c_{\mu\nu}(l_\nu-1\pm k,\pm),\qquad G=-(\sin\pi L)^{-1}C_-^{-1}(0)C_+(0),\\
&\tilde{A} = G^{-1}\bar{A}G,\qquad \Lambda^+ = C_+^{-1}(0)C_+(1),\\
&\Lambda^- = (\sin\pi L)^{-1}C_-^{-1}(0)C_-(1)\sin\pi L,\qquad \alpha = A/R,\\
&F^+ = [\alpha,\Lambda^+]+(I-\alpha\bar{\alpha})(I-L),\qquad F^- = [\bar{\alpha},\Lambda^-]-(I-\alpha\bar{\alpha})(I-L).
\end{aligned}\tag{3.10}$$

We shall write $p(x, y)$ for a polynomial in $x$ and $y$ with constant coefficients. Now define a space of multivalued functions $\mathrm{W}^1(L, A)$ by

$$\mathrm{W}^1(L, A) = \{ p(L_1, L_2)w \,|\, w \in \mathrm{W}(L, A), \deg p \leqslant 1 \} . \tag{3.11}$$

The following gives a useful characterization of $\mathrm{W}^1(L, A)$ in terms of local expansions.

*Theorem 3.4.* The multivalued function $f$ is in $\mathrm{W}^1(L, A)$ if and only if it satisfies the Klein–Gordon equation (3.1) on $\mathbf{D}_R - \{a_1, \ldots, a_n\}$ and

(i) $$f\big(R_j(2\pi)(x, y)\big) = \exp(2\pi i l_j) f(x, y), \qquad (x, y) \text{ near } a_j .$$

(ii) The local expansion at $a_j$ is of the form

$$f(r_j, \theta_j) = \sum_{k = l_j - 2, l_j - 1, l_j, \ldots} c_j(k, +) f_1(r_j, \theta_j; k) + \sum_{k = l_j, l_j - 1, l_j - 2, \ldots} c_j(k, -) f_2(r_j, \theta_j; k) ,$$

(iii) There exists $R_\infty > 0$ such that for geodesic polar coordinate $r_\infty$ greater than $R_\infty$

$$f(r_\infty, \theta_\infty) = \sum_{k \equiv l_\infty [\mathbf{Z}]} c_\infty(k) f_\infty(r_\infty, \theta_\infty; k) .$$

*Proof.* The proof is similar to the one in ref. [17]. We note that the existence of the inverse of $A - \tilde{A}$ must be assumed. QED

## 4. Extended system of equations

If $\{f_\mu\}_{\mu=1}^n$ is any basis of $\mathrm{W}(L, A)$, then a basis for $\mathrm{W}^1(L, A)$ is $\{L_1 f_\mu, L_2 f_\mu, f_\mu\}_{\mu=1}^n$. Using eq. (2.12), proposition 3.2, and theorem 3.4 it is easy to check that $L_3 f_\mu$ is in $\mathrm{W}^1(L, A)$. Hence there exist coefficients $B_{\mu\lambda}^{(1)}$, $B_{\mu\lambda}^{(2)}$ and $B_{\mu\lambda}^{(3)}$, independent of $z$ and $\bar{z}$, such that

$$L_3 f_\mu = \sum_{\lambda=1}^{n} \left( B_{\mu\lambda}^{(1)} L_1 f_\lambda + B_{\mu\lambda}^{(2)} L_2 f_\lambda + B_{\mu\lambda}^{(3)} f_\lambda \right), \tag{4.1}$$

or in matrix notation,

$$L_3 f = B^{(1)} L_1 f + B^{(2)} L_2 f + B^{(3)} f . \tag{4.2}$$

A local expansion of both sides of eq. (4.1) at $a_\nu$ $(\nu = 1, \ldots, n)$ (see (3.4)) and equating coefficients of $f_1(r_\nu, \theta_\nu; l_\nu - 2)$ and $f_2(r_\nu, \theta_\nu; l_\nu)$ leads to the equations

$$2iRC_+(0)A = B^{(1)}C_+(0)(R^2I - A^2) + iB^{(2)}C_+(0)(R^2I + A^2),$$
$$2iRC_-(0)\bar{A} = -B^{(1)}C_-(0)(R^2I - \bar{A}^2) + iB^{(2)}C_-(0)(R^2I + \bar{A}^2). \tag{4.3}$$

We define

$$B^{\pm} = B^{(2)} \pm iB^{(1)}, \qquad \tilde{\alpha} = R\tilde{A}^{-1}, \qquad \beta^{(2)} = 2(B^+)^{-1},$$
$$\beta^{(1)} = \tfrac{1}{2}\beta^{(2)}B^-, \qquad \beta^{(3)} = \tfrac{1}{2}i\beta^{(2)}B^{(3)}. \tag{4.4}$$

Solving for $\beta^{(1)}$ and $\beta^{(2)}$ in the canonical basis, we get

$$\beta^{(1)} = (\alpha - \tilde{\alpha})\tilde{\alpha}(\alpha - \tilde{\alpha})^{-1}\alpha = (\alpha - \tilde{\alpha})\alpha(\alpha - \tilde{\alpha})^{-1}\tilde{\alpha},$$
$$\beta^{(2)} = (\alpha^2 - \tilde{\alpha}^2)(\alpha - \tilde{\alpha})^{-1} = \tilde{\alpha} + (\alpha - \tilde{\alpha})\alpha(\alpha - \tilde{\alpha})^{-1} = \alpha + (\alpha - \tilde{\alpha})\tilde{\alpha}(\alpha - \tilde{\alpha})^{-1}. \tag{4.5}$$

Equating the coefficients of $f_1(r_\nu, \theta_\nu; l_\nu - 1)$ results in

$$\beta^{(3)} = \left[-F^+\alpha + (\alpha - \tilde{\alpha})\tilde{\alpha}(\alpha - \tilde{\alpha})^{-1}F^+\right](I - \alpha\bar{\alpha})^{-1}, \tag{4.6}$$

where $F^+$ is defined in eq. (3.10)*. Equating the coefficients of $f_2(r_\nu, \theta_\nu; l_\nu - 1)$ results in the following identity

$$G\beta^{(3)}G^{-1}(I - \alpha\bar{\alpha}) = -\Lambda^- - 2\alpha(I - L) - G\beta^{(1)}G^{-1}\Lambda^-\bar{\alpha}^2 - 2G\beta^{(1)}G^{-1}\bar{\alpha}(I - L)$$
$$+ G\beta^{(2)}G^{-1}\Lambda^-\bar{\alpha} + G\beta^{(2)}G^{-1}(I + \alpha\bar{\alpha})(I - L), \tag{4.7}$$

which is one of an infinite sequence of identities that now arise from eq. (4.1).

We now use the hyperbolic Laplace transform (2.17) to transform $f(z, \bar{z})$ into $\tilde{f}(b)$. We note here that for $f(z, \bar{z})$ to have the prescribed monodromy around the $a_j$'s, the function $\tilde{f}(b)$ is expected to have branch points and therefore a deformation of the boundary of $\mathbf{D}_R$ is expected in the actual situation. We now use proposition 2.1 to transform the extended system (4.2) into a system of ordinary differential equations:

$$\Delta(b)\frac{\mathrm{d}\tilde{f}}{\mathrm{d}b} = \left[-\beta^{(1)}\frac{R^2 s}{2b} + \beta^{(2)} + \beta^{(3)} + \left(\frac{Rs}{2} - 2\right)\frac{bI}{R}\right]\tilde{f}, \tag{4.8}$$

* Note the misprint in going from eq. (A.3) to (4.5) of ref. [17].

where $\Delta(b)=(b^2/R)I-b\beta^{(2)}+R\beta^{(1)}$. It can then be shown that

$$\Delta^{-1}(b)=R^{-1}\left[\left(\frac{b}{R}I-\alpha\right)^{-1}-\left(\frac{b}{R}I-\tilde{\alpha}\right)^{-1}\right](\alpha-\tilde{\alpha})^{-1}. \tag{4.9}$$

Thus the system (4.8) has the form

$$\mathrm{d}\tilde{f}/\mathrm{d}b=A(b)\tilde{f}, \tag{4.10}$$

where

$$A(b)=R^{-1}\left[\left(\frac{b}{R}I-\alpha\right)^{-1}-\left(\frac{b}{R}I-\tilde{\alpha}\right)^{-1}\right](\alpha-\tilde{\alpha})^{-1}$$
$$\times\left[-\beta^{(1)}\frac{R^2s}{2b}+\beta^{(2)}+\beta^{(3)}+\left(\frac{Rs}{2}-2\right)\frac{b}{R}I\right]. \tag{4.11}$$

We now reduce $A(b)$ to the standard form

$$A(b)=\frac{A_0}{b}+\sum_{j=1}^{n}\frac{A_j}{b-a_j}+\frac{A_j'}{b-R^2/\bar{a}_j}, \tag{4.12}$$

where $A_0$, $A_j$ and $A_j'$ are the $n\times n$ matrices

$$A_0=-\tfrac{1}{2}RsI,$$
$$A_j=E_j(\alpha-\tilde{\alpha})^{-1}\left[-\beta^{(1)}\frac{R^2s}{2a_j}+\beta^{(2)}+\beta^{(3)}+\left(\frac{Rs}{2}-2\right)\frac{a_j}{R}I\right],$$
$$A_j'=-\tilde{E}_j(\alpha-\tilde{\alpha})^{-1}\left[-\beta^{(1)}\frac{s\bar{a}_j}{2}+\beta^{(2)}+\beta^{(3)}+\left(\frac{Rs}{2}-2\right)\frac{R}{\bar{a}_j}I\right], \tag{4.13}$$

with

$$(E_j)_{kl}=\delta_{jk}\delta_{jl},\qquad \tilde{E}_j=G^{-1}E_jG.$$

It follows from eq. (4.13) that

$$A_0+\sum_{j=1}^{n}(A_j+A_j')=(\tfrac{1}{2}Rs-2)I, \tag{4.14}$$

which implies that the system (4.10) has a simple pole at infinity. Hence the system (4.10) is fuchsian of order $n$ with $2n$ singularities at $a_j$, $R^2/\bar{a}_j$. The residue of the singularities at zero and infinity are proportional to the identity, and can therefore be moved to any other point, say $b_0$, by a transformation

$$\tilde{f} \to \tilde{f} b^{-Rs/2}(b-b_0)^{Rs-2}.$$

We note that the matrices $A_j$ and $A'_j$ are rank one due to the rank one nature of $E_j$. We now proceed to compute the traces of $A_j$ and $A'_j$. For convenience, we let $U=(\alpha-\tilde{\alpha})^{-1}$ and $U'=GUG^{-1}=(G\alpha G^{-1}-\bar{\alpha}^{-1})^{-1}$. To compute the trace of $A_j$ we note that it is of the form $E_j M$ and that

$$\mathrm{Tr}(E_j M)=\sum_{k,l}\delta_{jk}\delta_{jl}M_{lk}=M_{jj}.$$

Then using the identities*

$$U\beta^{(1)}=\alpha U\alpha-\alpha, \qquad U\beta^{(2)}=\alpha U+U\alpha-I,$$

$$U\beta^{(3)}=(\alpha UF^{+}-UF^{+}\alpha-F^{+})(I-\alpha\bar{\alpha})^{-1}, \tag{4.15}$$

we find that

$$\mathrm{Tr}(A_j)=\tfrac{1}{2}Rs+l_j-2. \tag{4.16}$$

To compute the trace of $A'_j$ we note that it is of the form $\tilde{E}_j M$ and that

$$\mathrm{Tr}\big(\tilde{E}_j M\big)=\mathrm{Tr}\big(G^{-1}E_j GM\big)=\mathrm{Tr}\big(E_j GMG^{-1}\big)=(GMG^{-1})_{jj}.$$

Then using the identities*

$$GU\beta^{(1)}G^{-1}=\bar{\alpha}^{-1}U'\bar{\alpha}^{-1}+\bar{\alpha}^{-1},$$

$$GU\beta^{(2)}G^{-1}=\bar{\alpha}^{-1}U'+U'\bar{\alpha}^{-1}+I,$$

$$GU\beta^{(3)}G^{-1}=(\bar{\alpha}^{-1}F^{-}\bar{\alpha}+\bar{\alpha}^{-1}U'\bar{\alpha}^{-1}F^{-}\bar{\alpha}-U'\bar{\alpha}^{-1}F^{-})(I-\alpha\bar{\alpha})^{-1}, \tag{4.17}$$

we find that

$$\mathrm{Tr}(A'_j)=\tfrac{1}{2}Rs-l_j. \tag{4.18}$$

Collecting these results together we have

* These identities are derived in an analogous way that the identities in the appendix of ref. [17] were derived.

*Theorem 4.1.* Let $\{f_\mu\}_{\mu=1}^{n}$ be a basis for $W(L,A)$ and $f$ the column vector $\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$. If

$$f(z,\bar{z}) = \int_{\Gamma_R^n} P_s(z,b)\tilde{f}(b)\,\mathrm{d}b\,,$$

then $\tilde{f}(b)$ satisfies the fuchsian system

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}b} = \left[\frac{A_0}{b} + \sum_{j=1}^{n} \frac{A_j}{b-a_j} + \frac{A'_j}{b-R^2/\bar{a}_j}\right]\tilde{f},$$

where $A_0$, $A_j$ and $A'_j$ are $n\times n$ matrices given by eq. (4.13) and they satisfy (4.14). Further $A_j$ and $A'_j$ are rank one and

$$\begin{aligned} \mathrm{Tr}(A_j) &= \tfrac{1}{2}Rs + l_j - 2, \\ \mathrm{Tr}(A'_j) &= \tfrac{1}{2}Rs - l_j, \end{aligned} \qquad j = 1,\dots,n.$$

*Proof.* The only part that remains to be proven is that the contour $\Gamma_R^n$ as in fig. 2 gives the correct monodromy around all the points $a_j$ $(j = 1,\dots,n)$.

First of all, one comment is in order. The singularities at 0 and $\infty$ in the differential equation for $\tilde{f}$ can be moved to any other point because $A_0$ and $A_\infty$ (given by eq. (4.14)) are proportional to the identity. In particular they can be moved to one of the pairs $a_j$ and $R^2/\bar{a}_j$. This is the reason why there is no branch cut connecting 0 and $\infty$ in the contour $\Gamma_R^n$.

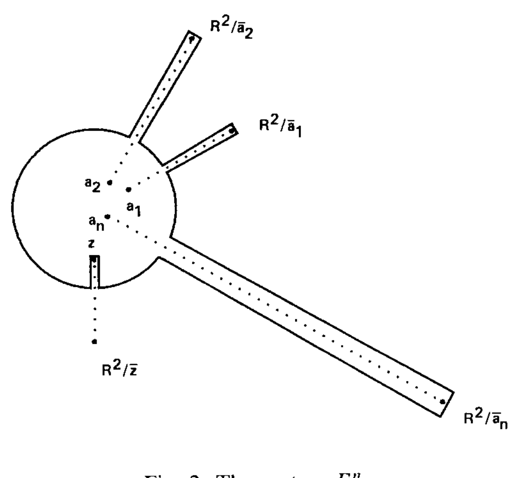


Fig. 2. The contour $\Gamma_R^n$.

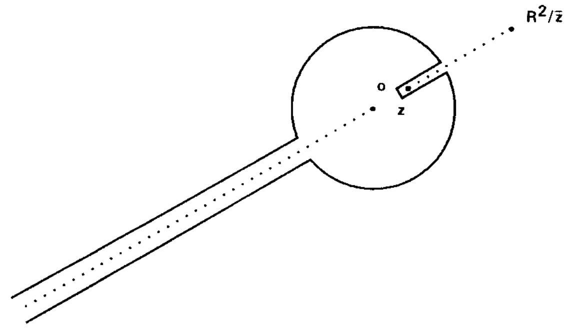


Fig. 3. The contour $\Gamma_R^1$.

We first analyze the case $n = 1$. The proof for the general case will then follow naturally. Without loss of generality we can choose $a_1 = 0$. For this case the space $W(L, A)$ is one-dimensional and it is spanned by

$$f(z,\bar{z}) = e^{i(l_1-1)\theta} Q^{l_1-1}_{Rs/2-1}(\cosh r)\,. \tag{4.19}$$

We now show that this result comes out of the fuchsian system with the choice of contour $\Gamma_R^1$ (fig. 3).

From theorem 4.1, the fuchsian system reduces for $n = 1$ to

$$\frac{d\tilde{f}}{db} = \frac{l_1-2}{b}\tilde{f}. \tag{4.20}$$

This can be integrated to give

$$\tilde{f}(b) = f_0 b^{l_1-2}, \tag{4.21}$$

where $f_0$ is the integration constant. Substitution of eq. (4.21) into (2.17) gives

$$f(z,\bar{z}) = \int_{\Gamma_R^1}\left[\frac{R^2 - z\bar{z}}{(z-b)(\bar{z}-\bar{b})}\right]^{Rs/2} f_0 b^{l_1-2}\,db\,, \tag{4.22}$$

where $\Gamma_R^1$ is the contour for the case of $n = 1$ and is shown in fig. 3. To perform the integration, we write $z = R\,e^{i\theta}\tanh r/2$ and $b = uR$. Then the integral (4.22) reduces to

$$f(z,\bar{z}) = f_0 R^{l_1-1}\int_{\Gamma^1_{R=1}}\left[\cosh r - \tfrac{1}{2}\sinh r\left(u\,e^{-i\theta} + (1/u)e^{i\theta}\right)\right]^{-Rs/2} u^{l_1-2}\,du\,. \tag{4.23}$$

A contour integration then shows that (4.23) is a multiple of (4.19). Note that the correct phase factor of $\exp i(l_1-1)\theta$ arises from the factor $b^{l_1-2}\,\mathrm{d}b$ in eq. (4.22).

Now we comment on the general case. We need to prove that the contour chosen gives the correct monodromy around every $a_j$, thus we investigate the situation when $z$ is close to $a_j$. We first make a transformation

$$\tilde{f} \to \tilde{f} b^{-Rs/2}\left(b-a_j\right)^{Rs/2}\left(b-R^2/\bar{a}_j\right)^{Rs/2-2},$$

which brings the singularities at 0 and $\infty$ to $a_j$ and $R^2/\bar{a}_j$ respectively. Next we send the point $a_j$ to 0 by an isometry of the type given by eq. (2.11). The pole at $b=0$ in the fuchsian system is then $(A_0+A_j)/b$. We now diagonalize $A_0+A_j$ by a similarity transformation on the fuchsian system, and using eqs. (4.13) and (4.16) we find that the pole at $b=0$ is $(l_j-2)/b$ and the branch cut connects $b=0$ and $b=\infty$. This part of the contour integration is then similar to eqs. (4.20) and (4.22) and the associated contour $\Gamma_R^1$ which implies the correct monodromy around $a_j$. QED

We now investigate the $R\to\infty$ limit of the fuchsian system in theorem 4.1. We first set $b=Ru$ where $u\bar{u}=1$. We write the $a_j$ in geodesic polar coordinates as

$$a_j = R\,\mathrm{e}^{i\theta}\tanh\frac{\rho_j}{R} = \rho_j\,\mathrm{e}^{i\theta_j} + \mathrm{O}(R^{-2}) := a_j^{\mathrm{E}} + \mathrm{O}(R^{-2}), \tag{4.24}$$

where $a_j^{\mathrm{E}}$ is the corresponding point in $\mathbf{R}^2$. Next, we establish a connection between the coefficients $c_{\mu\nu}(k,\pm)$ in theorem 3.3 and the limiting coefficients in the euclidean case. In eq. (3.8) the connection between the Legendre functions and the Bessel functions was established. This along with the local expansion for $f(z,\bar{z})$ in theorem 3.3 imply that if we set

$$c_{\mu\nu}(k,\pm) = \left(\tfrac{1}{2}Rs-1\right)^{\pm k}\tilde{c}_{\mu\nu}(k,\pm), \tag{4.25a}$$

then

$$\lim_{R\to\infty}\tilde{c}_{\mu\nu}(k,\pm) = \tilde{c}^{\mathrm{E}}_{\mu\nu}(k,\pm), \tag{4.25b}$$

where $\tilde{c}^{\mathrm{E}}_{\mu\nu}(k,\pm)$ are the local expansion coefficients of the limiting basis in $\mathbf{R}^2$. But if we start with a canonical basis, i.e. $c_{\mu\nu}(l_\nu-1,+)=\delta_{\mu\nu}$, then the limiting basis in $\mathbf{R}^2$ is not canonical. Therefore, we change the basis on the hyperbolic plane by multiplying all coefficients by $(Rs/2-1)^{1-l_\nu}$. We now write down the limiting behavior of the hyperbolic local expansion coefficient matrices $C_\pm(0)$ and $C_\pm(1)$ which will be of use later:

$$C_+(0)\sim I, \qquad C_-(0)\sim C_-^{\mathrm{E}}(0)\left(\tfrac{1}{2}Rs-1\right)^{2(I-L)},$$
$$C_+(1)\sim C_+^{\mathrm{E}}(1)\left(\tfrac{1}{2}Rs-1\right), \qquad C_-(1)\sim C_-^{\mathrm{E}}(1)\left(\tfrac{1}{2}Rs-1\right)^{(3I-2L)}. \tag{4.26}$$

Finally, we establish a connection between $\tilde{f}$ and the corresponding $\tilde{f}^{\mathrm{E}}$ in the $R \to \infty$ limit. If we denote by $f^{\mathrm{E}}(z, \bar{z})$ a multivalued wave functions in $\mathbf{R}^2$, then it is related to $\tilde{f}^{\mathrm{E}}$ by [SJM III]

$$f^{\mathrm{E}}(z,\bar{z}) = \int_{\Gamma_\infty} \exp\left[\frac{s}{2}(zu^{-1} + \bar{z}u)\right] \tilde{f}^{\mathrm{E}}(u)\frac{\mathrm{d}u}{u}. \tag{4.27}$$

The contour $\Gamma_\infty$ is essentially the same as the one in fig. 3 and is connected to the contour $\Gamma_R^n$ in theorem 4.1 in the following way. First scale the contour $\Gamma_R^n$ by the change of variable $b = uR$. This then moves the singularities in the fuchsian system in theorem 4.1 to $a_j/R$ and $R/\bar{a}_j$. As $R \to \infty$, using eq. (4.24) we see that all the singularities at $a_j/R$ ($j = 1, \dots, n$) accumulate at the origin and all the singularities at $R/\bar{a}_j$ ($j = 1, \dots, n$) accumulate at $\infty$ giving rise to the contour $\Gamma_\infty$ as shown in fig. 3. Note that there is no need for a superscript $n$ in the contour $\Gamma_\infty$ because the singularities for all $n$ are only at 0 and $\infty$. Comparing eq. (4.27) with (2.17) and (2.18) we establish the following connection between $\tilde{f}$ and $\tilde{f}^{\mathrm{E}}$. First define $\check{f}(u) = Ru\tilde{f}(Ru)$, then

$$\check{f}(u) \sim \tilde{F}^{\mathrm{E}}(u). \tag{4.28}$$

The fuchsian system (4.10) for $\tilde{f}$ then results in a system of equations for $\check{f}(u)$, namely,

$$\mathrm{d}\check{f}(u)/\mathrm{d}u = \left[u^{-1}I + RA(Ru)\right]\check{f}(u). \tag{4.29}$$

We now let $R \to \infty$ in the above equation. Using eqs. (3.10) and (4.24) we get

$$\alpha = A^{\mathrm{E}}/R + \mathrm{O}(R^{-3}), \tag{4.30}$$

where $(A^{\mathrm{E}})_{ij} = a_i^{\mathrm{E}}\delta_{ij}$. From eqs. (4.4) and (4.24) we find that

$$\tilde{\alpha} = R(A^{\mathrm{E}})^{-1} + \mathrm{O}(R^{-1}), \tag{4.31}$$

where $\tilde{A}^{\mathrm{E}} = G^{-1}\bar{A}^{\mathrm{E}}G$. We note that eqs. (3.10) and (4.26) imply

$$G = -(\sin\pi L)^{-1}\left(\tfrac{1}{2}Rs - 1\right)^{2(L-I)}\left(C_-^{\mathrm{E}}(0)\right)^{-1} := \left(\tfrac{1}{2}Rs - 1\right)^{2(L-I)}G^{\mathrm{E}},$$

and therefore $\tilde{A}^{\mathrm{E}} = (G^{\mathrm{E}})^{-1}\bar{A}^{\mathrm{E}}G^{\mathrm{E}}$. Using eqs. (3.10), (4.26) and (4.30) we find that

$$F^+ = \tfrac{1}{2}s\left[A^{\mathrm{E}}, \Lambda_{\mathrm{E}}^+\right] + I - L + \mathrm{O}(R^{-1}) := F_{\mathrm{E}}^+ + \mathrm{O}(R^{-1}), \tag{4.32}$$

where $\Lambda_{\mathrm{E}}^+ = C_+^{\mathrm{E}}(1)$ in the canonical basis. Using eqs. (4.5), (4.6) and (4.30)–(4.32) we get

$$\beta^{(1)} = (\tilde{A}^{\mathrm{E}})^{-1}A^{\mathrm{E}} + \mathrm{O}(R^{-2}), \qquad \beta^{(2)} = R(\tilde{A}^{\mathrm{E}})^{-1} + \mathrm{O}(R^{-1}),$$

$$\beta^{(3)} = R(\tilde{A}^{\mathrm{E}})^{-1}F_{\mathrm{E}}^+ + \mathrm{O}(1). \tag{4.33}$$

Using eq. (4.13) along with (4.33) then gives

$$A_j = E_j\left[\frac{Rs}{2a_j^{\mathrm{E}}}A^{\mathrm{E}} - I - F_{\mathrm{E}}^{+}\right] + \mathrm{O}(R^{-1}),$$

$$A_j' = \tilde{E}_j\left[\frac{Rs}{2\bar{a}_j^{\mathrm{E}}}\tilde{A}^{\mathrm{E}} + I + F_{\mathrm{E}}^{+} - \frac{2}{\bar{a}_j^{\mathrm{E}}}\tilde{A}^{\mathrm{E}}\right] + \mathrm{O}(R^{-1}). \tag{4.34}$$

Using

$$\frac{1}{b-a_j} = \frac{1}{Ru}\left[1 + \frac{a_j^{\mathrm{E}}}{Ru} + \mathrm{O}(R^{-2})\right],$$

$$\frac{1}{b-R^2/\bar{a}_j} = -\frac{\bar{a}_j^{\mathrm{E}}}{R^2}\left[1 + \frac{\bar{a}_j^{\mathrm{E}}}{R} + \mathrm{O}(R^{-2})\right], \tag{4.35}$$

eqs. (4.12) and (4.29) we finally obtain

$$\lim_{R\to\infty}\left[u^{-1}I + RA(Ru)\right] = \tfrac{1}{2}su^{-2}A^{\mathrm{E}} - \tfrac{1}{2}s\tilde{A}^{\mathrm{E}} - u^{-1}F_{\mathrm{E}}^{+}. \tag{4.36}$$

This implies that as $R \to \infty$ the fuchsian system (4.29) approaches

$$\left[u\frac{\mathrm{d}}{\mathrm{d}u} + \tfrac{1}{2}su\tilde{A}^{\mathrm{E}} - \tfrac{1}{2}su^{-1}A^{\mathrm{E}} + F_{\mathrm{E}}^{+}\right]\tilde{f}^{\mathrm{E}}(u) = 0, \tag{4.37}$$

which is the result in SMJ III for the euclidean case (after a change of variable $u \to u^{-1}$).

## 5. τ-functions

In the last section it was shown that the extended system of equations corresponded to a fuchsian system of equations given by theorem 4.1. We now require that the monodromy group remain constant when the $a_j$ (and $\bar{a}_j$) are varied. This requirement is consistent with theorem 4.1 since this theorem implies that the local monodromy matrices are independent of $a_j$ (and $\bar{a}_j$). The necessary and sufficient conditions for isomonodromy are the $A_j$ and $A_j'$ $(j=1,\dots,n)$ satisfy the Schlesinger equations [9, 11, 15, 16]. It is a major result of SMJ II (see also ref. [11]) that there exists a quantum field operator $\Phi_{\mathrm{RH}}(a,l)$ such that the logarithmic derivative of the vacuum expectation value

$$\tau(a_1,\dots,a_n) := \langle \Phi_{\mathrm{RH}}(a_1,l_1)\dots\Phi_{\mathrm{RH}}(a_n,l_n)\rangle$$

is a one-form, denoted here by $\omega_{\mathrm{RH}}$, that can be expressed in terms of solutions $A_j(a_1, \ldots, a_n)$ of the Schlesinger equations. The $\omega_{\mathrm{RH}}$ resulting from our fuchsian system is a function of $R$, the radius of the Poincaré disk (or the curvature of the manifold). In this section we will show that $\lim_{R \to \infty} \omega_{\mathrm{RH}}$ is equal to $\omega_{\mathrm{B}}$, the closed one-form defined by SMJ III that is associated to the monodromy preserving deformation of the (elliptic) Klein–Gordon equation in $\mathbf{R}^2$. Since the $R \to \infty$ limit of the fuchsian system in theorem 4.1 is given by eq. (4.47), the limiting $\tau$-function also corresponds to the $\tau$-function arising from the deformation equations of (4.47). That the $\tau$-function corresponding to (4.47) is the same as the $\tau$-function for the Klein–Gordon equation in SMJ III is shown in ref. [6] (see their example 5.3). Therefore, we conclude (though this argument is not a proof) that the $\tau$-function obtained from the fuchsian system corresponds to the vacuum expectation value of $n$ interacting bosonic fields existing in the Poincaré disk. Stated somewhat differently, we are claiming that the $\tau$-function associated with the Klein–Gordon equation in the hyperbolic plane is the same as the "Riemann-Hilbert/ Schlesinger" $\tau$-function associated with the transformed Klein–Gordon equation. We now present the details of the calculation.

As a first step, it is useful to transform the variable $b$ (of theorem 4.1) defined on a circle of radius $R$ to a variable $\xi$ on the real line by

$$\xi = -i \frac{b + iR}{b - iR} . \tag{5.1}$$

Under this transformation, the fuchsian system in theorem 4.1 becomes

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi} = \left[ \frac{A_0}{\xi - i} + \frac{A_0'}{\xi + i} + \sum_{j=1}^{n} \frac{A_j}{\xi - \xi_j} + \frac{A_j'}{\xi - \bar{\xi}_j} \right] \tilde{f} \tag{5.2}$$

where

$$\xi_j = -i \frac{a_j + iR}{a_j - iR}, \qquad A_0' = \left( 2 - \frac{Rs}{2} \right) I, \tag{5.3}$$

the matrices $A_0$, $A_j$ and $A_j'$ are still given by (4.13), (4.14) and (5.3) imply that

$$A_0 + A_0' + \sum_{j=1}^{n} (A_j + A_j') = 0, \tag{5.4}$$

which means that there is no singularity at infinity. We now require that the fuchsian system (5.2) satisfy the condition of isomonodromy. This then results in the Schlesinger equations [9, 11, 15, 16]:

$$\mathrm{d}A_i = -\sum_{j=1}^{n} [A_i, A_j] \,\mathrm{d}\log\left(\frac{\xi_i - \xi_j}{\xi_0 - \xi_j}\right) - \sum_{j=1}^{n} [A_i, A'_j] \,\mathrm{d}\log\left(\frac{\xi_i - \bar{\xi}_j}{\xi_0 - \bar{\xi}_j}\right),$$

$$\mathrm{d}A'_i = -\sum_{j=1}^{n} [A'_i, A_j] \,\mathrm{d}\log\left(\frac{\bar{\xi}_i - \xi_j}{\xi_0 - \xi_j}\right) - \sum_{j=1}^{n} [A'_i, A'_j] \,\mathrm{d}\log\left(\frac{\bar{\xi}_i - \bar{\xi}_j}{\xi_0 - \bar{\xi}_j}\right), \quad (5.5)$$

where $\xi_0$ is the point where the initial conditions are specified and d denotes exterior differentiation with respect to $\xi_j$ and $\bar{\xi}_j$ ($j = 1, \dots, n$). Associated with the above deformation equations is a "Riemann–Hilbert/Schlesinger" closed one-form $\tilde{\omega}$ given by (SMJ II, see also [6, 7, 9, 11, 13])*

$$\tilde{\omega} = \tfrac{1}{2} \sum_{i,j=1, i \neq j}^{n} \left[\mathrm{Tr}(A_i A_j) \,\mathrm{d}\log(\xi_i - \xi_j) + \mathrm{Tr}(A'_i A'_j) \,\mathrm{d}\log(\bar{\xi}_i - \bar{\xi}_j)\right]$$
$$+ \sum_{i,j=1}^{n} \mathrm{Tr}(A_i A'_j) \,\mathrm{d}\log(\xi_i - \bar{\xi}_j). \quad (5.6)$$

This differs from the one in SMJ II by an exact one-form. Now we want to make the connection between this one-form and the $\tau$-function. For the $\tau$-function to describe expectation values of operators in a quantum field theory, we would want the one-point functions to be a constant, independent of the position of the field operator. This then implies that the one-form corresponding to the $n = 1$ case should be normalized to zero. For the case $n = 1$, $\tilde{\omega}$ is given by

$$\tilde{\omega} = \left(\frac{Rs}{2} + l_1 - 2\right)\left(\frac{Rs}{2} - l_1\right) \mathrm{d}\log(\xi_1 - \bar{\xi}_1) \qquad (n = 1), \quad (5.7)$$

where we have used theorem 4.1. We note that eq. (5.7) is an exact one-form. In order to define a normalized one-form $\omega$, we have to subtract from $\tilde{\omega}$ in eq. (5.6) an exact form at each of the singular points $\xi_i$ given by eq. (5.7). This then results in the following closed one-form:

$$\omega = \tfrac{1}{2} \sum_{i,j=1, i \neq j}^{n} \left[\mathrm{Tr}(A_i A_j) \,\mathrm{d}\log(\xi_i - \xi_j) + \mathrm{Tr}(A'_i A'_j) \,\mathrm{d}\log(\bar{\xi}_i - \bar{\xi}_j)\right]$$
$$+ \sum_{i,j=1}^{n} \mathrm{Tr}(A_i A'_j) \,\mathrm{d}\log(\xi_i - \bar{\xi}_j) - \sum_{j=1}^{n} \left(\frac{Rs}{2} + l_j - 2\right)\left(\frac{Rs}{2} - l_j\right) \mathrm{d}\log(\xi_j - \bar{\xi}_j). \quad (5.8)$$

* We drop the subscript "RH" from now on.

This $\omega$ is connected to the $\tau$-function locally by

$$\omega = \mathrm{d}\log\tau\,, \tag{5.9}$$

where the $\tau$-function describes the vacuum expectation value of a product of quantum field operators properly normalized.

Now we will investigate the one-form $\omega$ as $R\to\infty$. We first note that $A_i$ is of the form $E_iM_i$ and that $A_i'$ is of the form $\tilde{E}_iM_i'$. Therefore

$$\begin{aligned}
\mathrm{Tr}(A_iA_j) &= \mathrm{Tr}(E_iM_iE_jM_j) = (M_i)_{ij}(M_j)_{ji}\,,\\
\mathrm{Tr}(A_i'A_j') &= \mathrm{Tr}\big(\tilde{E}_iM_i'\tilde{E}_jM_j'\big) = \big(GM_i'G^{-1}\big)_{ij}\big(GM_j'G^{-1}\big)_{ji}\,,\\
\mathrm{Tr}(A_iA_j') &= \mathrm{Tr}\big(E_iA_i\tilde{E}_jM_j'\big) = \big(M_iG^{-1}\big)_{ij}\big(GM_j'\big)_{ji}\,.
\end{aligned} \tag{5.10}$$

Using eqs. (4.24) and (5.3) we find that

$$\xi_j = i + \frac{2a_j^{\mathrm{E}}}{R} - \frac{2i\big(a_j^{\mathrm{E}}\big)^2}{R^2} + \mathrm{O}(R^{-3})\,. \tag{5.11}$$

Using eqs. (4.44), (5.10) and (5.11) we get

$$\mathrm{Tr}(A_iA_j)\,\mathrm{d}\log(\xi_i-\xi_j) = (F_{\mathrm{E}}^{+})_{ij}(F_{\mathrm{E}}^{+})_{ji}\,\mathrm{d}\log\big(a_i^{\mathrm{E}}-a_j^{\mathrm{E}}\big) + \mathrm{O}(R^{-1}),\qquad i\neq j, \tag{5.12a}$$

$$\mathrm{Tr}(A_i'A_j')\,\mathrm{d}\log\big(\bar{\xi}_i-\bar{\xi}_j\big) = \big(GF_{\mathrm{E}}^{+}G^{-1}\big)_{ij}\big(GF_{\mathrm{E}}^{+}G^{-1}\big)_{ji}\,\mathrm{d}\log\big(\bar{a}_i^{\mathrm{E}}-\bar{a}_j^{\mathrm{E}}\big) + \mathrm{O}(R^{-1}),\qquad i\neq j, \tag{5.12b}$$

$$\begin{aligned}
\sum_{i,j=1}^{n}\mathrm{Tr}(A_iA_j')\,\mathrm{d}\log\big(\xi_i-\bar{\xi}_j\big) = &-\left(\frac{iRs^2}{4}-is\right)\mathrm{Tr}(\mathrm{d}A^{\mathrm{E}}-\mathrm{d}\bar{A}^{\mathrm{E}})\\
&-\tfrac{1}{4}s^2\,\mathrm{Tr}[A^{\mathrm{E}}\,\mathrm{d}A^{\mathrm{E}}+\bar{A}^{\mathrm{E}}\,\mathrm{d}\bar{A}^{\mathrm{E}}+G^{-1}\bar{A}^{\mathrm{E}}G\,\mathrm{d}A^{\mathrm{E}}\\
&+A^{\mathrm{E}}G^{-1}\,\mathrm{d}\bar{A}^{\mathrm{E}}G] + \mathrm{O}(R^{-1}),
\end{aligned} \tag{5.12c}$$

$$\begin{aligned}
&\sum_{j=1}^{n}\left(\frac{Rs}{2}+l_j-2\right)\left(\frac{Rs}{2}-l_j\right)\mathrm{d}\log\big(\xi_j-\bar{\xi}_j\big)\\
&\quad= -\left(\frac{iRs^2}{4}-is\right)\mathrm{Tr}(\mathrm{d}A^{\mathrm{E}}-\mathrm{d}\bar{A}^{\mathrm{E}}) - \frac{s^2}{4}\mathrm{Tr}(A^{\mathrm{E}}+\bar{A}^{\mathrm{E}})(\mathrm{d}A^{\mathrm{E}}+\mathrm{d}\bar{A}^{\mathrm{E}}) + \mathrm{O}(R^{-1}).
\end{aligned} \tag{5.12d}$$

The above results when substituted into eq. (5.8) give the following:

$$\lim_{R\to\infty} \omega = \tfrac{1}{2}\mathrm{Tr}\left(F_{\mathrm{E}}^{+}\Theta + \Theta^{*}GF_{\mathrm{E}}^{+}G^{-1}\right)$$
$$+\tfrac{1}{4}s^{2}\,\mathrm{Tr}\left(\mathrm{d}(A^{\mathrm{E}}\bar{A}^{\mathrm{E}}) - G^{-1}\bar{A}^{\mathrm{E}}G\,\mathrm{d}A^{\mathrm{E}} - A^{\mathrm{E}}G^{-1}\,\mathrm{d}\bar{A}^{\mathrm{E}}G\right)$$
$$:= \omega_{\mathrm{B}}\,, \tag{5.13}$$

where

$$\begin{aligned} \Theta_{ij} &= (F_{\mathrm{E}}^{+})_{ij}\,\mathrm{d}\log\left(a_i^{\mathrm{E}} - a_j^{\mathrm{E}}\right); && i\neq j \\ &= 0, && i=j; \\ \Theta_{ij}^{*} &= \left(GF_{\mathrm{E}}^{+}G^{-1}\right)_{ij}\mathrm{d}\log\left(\bar{a}_i^{\mathrm{E}} - \bar{a}_j^{\mathrm{E}}\right), && i\neq j; \\ &= 0, && i=j. \end{aligned} \tag{5.14}$$

This limiting one-form $\omega_{\mathrm{B}}$ is the bosonic one-form in SMJ III (see also refs. [6, 7]).

## 6. The $n = 2$ case

In this section we will discuss the $n=2$ case in some detail in order to understand the two-point function of the underlying field theory. Our starting point will be the fuchsian system (5.2). We will focus our interest on the one-form associated with the $n=2$ case. Referring to eq. (5.8) we note that $\omega$ is independent of the singularities at $\xi = \pm i$ and those singularities can be placed anywhere in the $\xi$-plane. We will make a particular choice and place both of them at $\bar{\xi}_1$. Then (5.2) for the case of $n=2$ reads

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}\xi} = \left[\frac{A_1}{\xi-\xi_1} + \frac{A_2}{\xi-\xi_2} + \frac{A_1 + A_0 + A_0'}{\xi-\bar{\xi}_1} + \frac{A_2'}{\xi-\bar{\xi}_2}\right]\tilde{f}. \tag{6.1}$$

The next step is to perform an $\mathrm{SL}_2(\mathbf{C})$ transformation which takes $\xi_1 \to 0$, $\bar{\xi}_1 \to \infty$ and $\bar{\xi}_2 \to 1$ by

$$x := \frac{\bar{\xi}_2 - \bar{\xi}_1}{\bar{\xi}_2 - \xi_1}\,\frac{\xi - \xi_1}{\xi - \bar{\xi}_1}\,. \tag{6.2}$$

This takes (6.1) into

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}x} = \left[\frac{A_1}{x} + \frac{A_2}{x-t} + \frac{A_2'}{x-1}\right]\tilde{f} \tag{6.3}$$

where

$$t := \left|\frac{\xi_2 - \xi_1}{\xi_2 - \bar{\xi}_1}\right|^2 = \tanh^2 \tfrac{1}{2}r(\xi_2, \xi_1)\,, \tag{6.4}$$

and $r(\xi_2, \xi_1)$ is the radial part of the geodesic polar coordinate of $\xi_2$ centered around $\xi_1$; i.e., it is the hyperbolic distance between $\xi_1$ and $\xi_2$.

At this point we note that (6.3) is exactly the starting point of the analysis of the Painlevé equation of the sixth kind ($P_{VI}$) in appendix C of ref. [7]. We will use the results of the analysis of $P_{VI}$ in ref. [7] and in so doing will adopt their notation and their equation numbers. Comparing eq. (6.3) with (C.46) in ref. [7]* we make the following identification:

$$A_1 \leftrightarrow (A_0)_{MJ}, \qquad A_2 \leftrightarrow (A_t)_{MJ}, \qquad A_2' \leftrightarrow (A_1)_{MJ}. \tag{6.5}$$

On comparing the traces of $A_1$, $A_2$ and $A_2'$ in theorem 4.1 with eqs. (C.47) and (C.48) we find

$$\theta_0 = \tfrac{1}{2}Rs + l_1 - 2, \qquad \theta_1 = \tfrac{1}{2}Rs - l_2,$$
$$\theta_t = \tfrac{1}{2}Rs + l_2 - 2, \qquad \theta_\infty = \tfrac{1}{2}Rs - l_1. \tag{6.6}$$

Referring to eq. (5.8) we find that the one-form $\omega$ is given by

$$\omega = \mathrm{Tr}\left[\frac{A_0}{t} + \frac{A_1}{t-1}\right]A_t\,\mathrm{d}t - \frac{\theta_1\theta_t}{t-1}\,\mathrm{d}t \tag{6.7}$$

where we have used the notation of ref. [7] for the matrices (see eq. (6.5)). The $\omega$ in eq. (6.7) differs by an exact one-form from the one in eq. (C.57) because of the normalization condition discussed in sect. 5. If we identify $\omega$ with $\mathrm{d}\log\tau$ where $\tau$ is the normalized two-point function, then we get the following connection between $\sigma$ in eq. (C.61) and $\tau$:

$$\sigma = t(t-1)\frac{\mathrm{d}\log\tau}{\mathrm{d}t} + at + b, \tag{6.8}$$

where

$$a = -\tfrac{1}{4}(l_1 + l_2 - 2)^2, \qquad b = \frac{1}{2}\left[\left(\frac{Rs}{2} - 1\right)^2 + (l_1 - 1)(l_2 - 1)\right]. \tag{6.9}$$

With this identification, we have the result that (C.61) is the differential equation for $\sigma(t)$ where $\tau$ is the two-point function of a massive bosonic field theory on $\mathbf{D}_R$.

We now investigate the limit $R \to \infty$ of eq. (C.61) in order to obtain the differential equation for the two-point function on $\mathbf{R}^2$. From the result in sect. 5 we know that as $R \to \infty$, $\omega \to \omega^{\mathrm{E}}$ and therefore $\tau \to \tau^{\mathrm{E}}$ where $\tau^{\mathrm{E}}$ is the euclidean

* From here onwards in this section, we will just refer to the equation numbers in ref. [7] without making an explicit reference to ref. [7] – such equations always begin with the letter C.

two-point function. Next, if in eq. (6.4) we write

$$r = 2\tilde{t}/Rs, \tag{6.10}$$

then as $R \to \infty$, $\tilde{t} \to t^{\mathrm{E}}$, where $t^{\mathrm{E}}$ is the dimensionless distance in the euclidean plane. If we now define

$$\sigma^{\mathrm{E}} = t^{\mathrm{E}} \frac{\mathrm{d} \log \tau^{\mathrm{E}}}{\mathrm{d} t^{\mathrm{E}}} - (t^{\mathrm{E}})^2, \tag{6.11}$$

we find, using eqs. (6.4) and (6.8)–(6.10), the following connection between $\sigma$ and $\sigma^{\mathrm{E}}$:

$$\sigma = -\frac{1}{2}\left(1 - \frac{4}{3}\left(\frac{t^{\mathrm{E}}}{Rs}\right)^2\right)\left(\sigma^{\mathrm{E}} + (t^{\mathrm{E}})^2\right) + \frac{1}{2}\left[\left(\frac{Rs}{2} - 1\right)^2 + (l_1 - 1)(l_2 - 1)\right]$$
$$- \frac{1}{R^2 s^2}(l_1 + l_2 - 2)^2 (t^{\mathrm{E}})^2 + \mathrm{O}(R^{-4}). \tag{6.12}$$

Using eqs. (6.4), (6.10) and (6.12) in (C.61) and taking the $R \to \infty$ limit we find that

$$\left(t^{\mathrm{E}}(\sigma^{\mathrm{E}})'' - (\sigma^{\mathrm{E}})'\right)^2 = 4\left(2\sigma^{\mathrm{E}} - t^{\mathrm{E}}(\sigma^{\mathrm{E}})'\right)\left((\sigma^{\mathrm{E}})'^2 - 4(t^{\mathrm{E}})^2\right)$$
$$+ 4(l_1 - l_2)^2\left((\sigma^{\mathrm{E}})'^2 + 4(t^{\mathrm{E}})^2\right) + 16(l_1 - l_2)^2 t^{\mathrm{E}}(\sigma^{\mathrm{E}})'. \tag{6.13}$$

This is (essentially) the differential equation satisfied by the two-point function $\tau^{\mathrm{E}}$ in $\mathbf{R}^2$. Now we note that eq. (6.13) is exactly the same differential equation as the one in (C.29) which is the equation for the $\tau$-function; or more precisely, the $\sigma$-function related to the $\tau$-function by $\sigma(t) = t\,\mathrm{d}\log\tau(t)/\mathrm{d}t$, associated with $\mathrm{P_{III}}$. But $\mathrm{P_{III}}$ is precisely the deformation equation associated with (4.47) in the $2 \times 2$ case. Therefore we have succeeded in showing that the differential equation for the two-point function on $\mathbf{D}_R$, viz., eq. (C.61) goes into the differential equation for the two-point function on $\mathbf{R}^2$, viz., (6.13) or (C.29) as $R \to \infty$.

We now comment briefly on the long distance behavior of $\tau(r)$ which is equivalently the limit $t \to 1$ (see (6.4)). The analysis follows very closely that of Jimbo [4]. The $\tau$-function has an asymptotic solution of the form:

$$\tau(t) = C(1-t)^{\nu_0}\left[1 + \alpha_1(1-t) + \alpha_2(1-t)^{1-\nu} + \alpha_3(1-t)^{1+\nu}\right.$$
$$\left. + \sum_{j=2}^{\infty} \sum_{k=-j}^{j} \alpha_{jk}(1-t)^{j+k\nu}\right], \tag{6.14}$$

where $C$ is an arbitrary constant and $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_{jk}$, $\nu_0$ and $\nu$ $(0 < \mathrm{Re}\,\nu < 1)$ will be determined in terms of two integration constants since the associated $\sigma$ (see (6.8)) satisfies a second-order differential equation. Upon substitution of eq. (6.14) into (6.18) and the result into (C.61) we get the following results:

$$\nu_0 = \tfrac{1}{4}\nu^2 - \left(1 - \tfrac{1}{2}Rs\right)^2,$$

$$\alpha_1 = \frac{1}{8\nu^2}\left[\nu^2 - 4\left(1 - \tfrac{1}{2}Rs\right)^2\right]\left[\nu^2 - 4(1-l)^2\right],$$

$$\alpha_2\alpha_3 = \frac{1}{256\nu^4(1-\nu^2)^2}\left[\nu^2 - 4\left(1 - \tfrac{1}{2}Rs\right)^2\right]^2\left[\nu^2 - 4(1-l)^2\right]^2. \qquad (6.15)$$

Therefore $\nu$ and, say, $\alpha_2$ are the two free parameters and all others are determined in terms of them. The $\tau$-function as normalized in sect. 5 is the two-point function

$$\langle \Phi(0)\Phi(r)\rangle / \langle \Phi(0)\rangle\langle \Phi(r)\rangle$$

and therefore one expects $\tau \to 1$ as $r \to \infty$. This additional boundary condition can be satisfied only if $\nu_0 = 0$ which then implies $\alpha_1$, $\alpha_2\alpha_3 = 0$. Computation of several higher-order terms in this special case makes it plausible that $\nu_0 = 0$ implies the solution is trivial, i.e., $\tau(r) \equiv 1$. This suggests two possibilities: (a) The form of the $\tau$-function used in eq. (6.14) is not unique and there is another disconnected two-parameter solution which is relevant for a field-theory interpretation. (b) Because we are in a curved manifold the $\tau$-function does not factorize at large distances and therefore it does not have to approach one. There does seem to be a third possibility, viz., adjusting $a$ and $b$ in eq. (6.8) so that $\tau \to 1$ as $r \to \infty$. This is not really a possibility because the $a$ and $b$ in eqs. (6.8) and (6.9) were determined by the normalization condition in sect. 5 which is the one that is consistent with the interpretation of $\tau \to 1$ as $r \to \infty$. We hope to understand this better when we analyze the Dirac operator on $\mathbf{D}_R$ since the theory then has a direct connection to the massive Ising field theory as $R \to \infty$.

## Note added in proof

In the analysis of the large-distance behavior of the two-point function in sect. 6 (see eqs. (6.14) and (6.15) and discussion thereafter) it was assumed that we could have an ordered phase at finite $R$, i.e. $\tau \to 1$ as $r \to \infty$. Since our $\tau$-function agreed exactly with the SMJ $\tau$-function as $R \to \infty$ this seemed reasonable because it could be done in the euclidean plane (SMJ IV). But there is a problem of order of limits here, i.e. $r \to \infty$ and $R \to \infty$. It has been shown by Callan and Wilczek in a recent work [19] that theories in hyperbolic space are always in the disordered phase. Since we are taking the limit $r \to \infty$ at fixed $R$, we will end up in the disordered phase and therefore we cannot impose $\tau \to 1$.

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