From Newton to Einstein

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1. INTRODUCTION. In 1687 Sir Issac Newton (1642–1727) published *Philosophae Naturalis Principia Mathematica* (known as the *Principia* by those who do not speak Latin), in which he explained the observed motion of the planets in the sky. In particular, he derived Kepler's laws of motion from the assumption that the sun *pulls* on a planet with a *force* that varies inversely with the square of the distance from the sun to the planet. The brilliance of this work lies in the fact that Newton had to invent the meaning of the word *force*, and in so doing he related the change of motion to the force applied through what we now refer to as *Newton's Second Law*:

$$\text{Force} = \text{Mass} \times \text{Acceleration}. \tag{1.1}$$

Newton then postulated that every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two, and whose magnitude varies directly as the product of their masses, and varies inversely as the square of the distance between them. Thus a planet of mass M_p and the sun of mass M_s separated by a distance r each experience an attractive force of magnitude F given by the formula

$$F = \frac{G_0 M_p M_s}{r^2}, \tag{1.2}$$

where G_0 is the universal gravitational constant. From these strikingly simple assumptions, Newton was able to prove mathematically that the planets must obey the celebrated laws of Johannes Kepler (1571–1630); laws that Kepler had earlier formulated on the basis of detailed observational studies of the motions of the heavenly bodies, namely:

(1) *The planets move in elliptical orbits about the sun with the sun fixed at one focus of the ellipse.*
(2) *The velocity of a planet varies in such a way that the line joining the planet to the sun sweeps out equal areas in equal times.*
(3) *The square of the time required by a planet for one revolution around the sun is proportional to the cube of its mean distance from the sun.*

Newton unified all of the planetary laws of motion which were known in his lifetime: laws that were written down by Kepler in the first decade of the seventeenth century and until Newton were understood only as empirical observations. Thus, planetary motion was explained by the assumption that celestial bodies *pull* on each other (across millions of miles of empty space) with a *force* proportional to one over the separation distance squared. This point of view stood as the ultimate explanation of why the stars and planets in the sky move the way they do, and the fundamental starting points, (1.1) and (1.2), were elevated to the status of *Laws of Nature*. That is, until Albert Einstein (1879–1955) entered the scene in 1916 with his paper *Die Grundlagen der allgemeinen Relativitätstheorie* (*The Foundation of the General Theory of Relativity*). Einstein took the point of view that heavenly bodies don't pull on each other across empty space, but rather the massive objects in the universe cause space itself to be *curved*, and the motions of the planets are explained as bodies moving along *straight lines* in a *curved space*. In fact, it is actually *spacetime* that is curved, and in Einstein's theory the curvature of spacetime evolves dynamically in an elaborate manner determined by the stars and planets in the universe. Einstein made mathematically precise sense of this, and used his constructions to show rigorously that with his assumptions, the planets would almost move in ellipses around the sun, but that there would be a small correction. In 1916 this correction to Newtonian theory was too small to observe in all the planets except Mercury (today this effect has been observed in other planetary orbits, but it is most pronounced in the case of Mercury (see [**7**, **8**])). Einstein showed that if his theory were correct, then the *perihelion* of the orbit of the planet Mercury, the point at which the orbit was closest to the sun, would not be the same in every orbit as Newton's theory predicted, but would precess an angular distance of 43 seconds of an arc per century. This had been observed exactly to be the case in 1859 by Joseph Le Verrier (1811–1877)*, and this gave the first experimental evidence that Newton's theory was only an approximation to Einstein's more general theory. In fact, beyond our solar system the predictions of Einstein's theory diverge dramatically from Newton's predictions. Indeed, Einstein's theory implies the formation of *black holes* in extremely massive stars. These are stars in which everything sufficiently close, including *light*, is sucked into the center of the star. It is no wonder that at the moment of his derivation of the perihelion shift predicted by his theory, Einstein is quoted as saying that his excitement was so great as to give him "palpitations of the heart"! ([**6**], pg. 253).

Both Newton's and Einstein's predictions involve the study of ordinary differential equations. The fundamental ODE is the equation that describes how the radius of the orbit, i.e. the distance from the sun, varies as a function of time along a planet's orbit. In fact, it will be simpler to study the ODE that describes how $1/r$ varies as a function of angle θ (Astronomically, it is angular changes that can be measured most accurately with a telescope). In this paper we will derive this ODE in the case of Newton's assumptions (1.1) and (1.2). We will then write down the corresponding ODE which Einstein gets from his theory. We will observe that this ODE approximates the one Newton gets, but with a small perturbation. We will then use the principle of conservation of energy to determine the qualitative structure of the orbits predicted by these ODE's. The analysis of Einstein's ODE gives an elementary qualitative picture of what happens in a black hole and how black holes arise in the theory of gravitation. Finally, an asymptotic expansion of

*Actually, the observed perihelion advance is 574 arcseconds/century of which 531 arcseconds/century are accounted for due to the perturbing effect of the other planets on the Mercury-Sun system. Le Verrier found that the largest contribution comes from Venus, 278 arcseconds, and next Jupiter at 153 arcseconds. The Earth's effect is third with 90 arcseconds and the remaining planets contribute about 10 arcseconds. Thus the total contribution coming from Newtonian celestial mechanics calculations is about 531 arcseconds per century. The remaining 43 arcseconds/century is called the *anomalous perihelion shift* and it is this that is unaccounted for by Newtonian theory. A compilation of a decade's worth of data (1966–1976) by a group at MIT gave the anomalous part of Mercury's perihelion precession to be 43.11 $\pm$ 0.21 arcseconds per century (see [**8**]).

Einstein's ODE will enable us to estimate the difference between the predicted orbits, and we will obtain Einstein's famous result that in the case of Mercury, a precession in the amount of 42.98 arcseconds/century is predicted to occur in the perihelion of the orbit of planet Mercury when Einstein's equation is taken in place of Newton's. To within experimental error this is equal to 43.11 ± 0.21 arcseconds/century which is the observed anomalous precession in Mercury's orbit [**8**].

Once we assume the ODE that comes from Einstein's theory, our treatment is entirely self-contained. The actual derivation of the ODE in Einstein's theory involves an in-depth study of differential geometry and physics which is beyond the scope of this paper. It is remarkable, though, that once the fundamental ODE's are established, both Newton's and Einstein's predictions can be derived by methods taught in an undergraduate course in differential equations.

For an in-depth discussion of the history of this subject, the reader is referred to the book *Subtle is the Lord* by Pais [**6**]. A brief but informative discussion can also be found in the first chapter of *Gravitation and Cosmology* by Weinberg [**7**] (see also [**4**]). An introductory account of the experimental tests of general relativity can be found in *Was Einstein Right*? by Will [**8**]. A comprehensive study of black holes can be found in *The Mathematical Theory of Black Holes* by Chandrasekhar [**5**].

2. THE FUNDAMENTAL ODE'S. We will first derive the fundamental ODE predicted by Newton's theory. So assume that Newton's Laws (1.1) and (1.2) hold. We derive an ODE for the distance r as a function of the angle θ, and the final form of the ODE will be obtained by making the substitution $u = 1/r$. This will give us an ODE that must be satisfied along every trajectory that corresponds to a solution of (1.1) and (1.2). To start, let $\mathbf{r}_p$ and $\mathbf{r}_s$ denote the positions of the planet and sun, respectively, with respect to some (inertial) coordinate system. Then combining (1.1) and (1.2) (and accounting for the direction of the force) we have

$$M_p \ddot{\mathbf{r}}_p = -\frac{G_0 M_p M_s}{|\mathbf{r}_p - \mathbf{r}_s|^3}(\mathbf{r}_p - \mathbf{r}_s), \tag{2.1}$$

$$M_s \ddot{\mathbf{r}}_s = -\frac{G_0 M_p M_s}{|\mathbf{r}_p - \mathbf{r}_s|^3}(\mathbf{r}_s - \mathbf{r}_p). \tag{2.2}$$

The dot here and throughout denotes differentiation with respect to the time t. We introduce $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_s$, the vector that points from the sun to the planet, and the center of mass $\mathbf{r}_0 = (M_p \mathbf{r}_p + M_s \mathbf{r}_s)/(M_p + M_s)$. Adding (2.1) and (2.2) shows that the center of mass $\mathbf{r}_0$ moves freely (that is, its time dependence is $\mathbf{c}_1 t + \mathbf{c}_2$, $\mathbf{c}_1$ and $\mathbf{c}_2$ are vector constants). Subtracting (2.2) from (2.1), expressing $\mathbf{r}_p$ and $\mathbf{r}_s$ in terms of $\mathbf{r}$ and $\mathbf{r}_0$, and using $\ddot{\mathbf{r}}_0 = 0$, we obtain

$$\ddot{\mathbf{r}} = -\frac{G}{|\mathbf{r}|^3}\mathbf{r}, \tag{2.3}$$

where we set

$$G = G_0(M_s + M_p) \approx G_0 M_s.$$

Note that the constant G is essentially independent of the planet considered because for all planets $M_p/M_s \ll 1$. Thus (2.3) is an equation that holds for every planet. Since $M_p/M_s \ll 1$, the center of mass is essentially at the sun; and so, we may think of the sun at the origin and (2.3) describes the motion of the planet about the fixed sun. Since (2.3) is a second order (nonlinear) ODE, the vector valued function $\mathbf{r}(t)$ that satisfies (2.3) is determined by the initial conditions

$$\mathbf{r}(0) = \mathbf{r}_0 \quad \text{and} \quad \dot{\mathbf{r}}(0) = \dot{\mathbf{r}}_0.$$

As a consequence of (2.3), the orbit $\mathbf{r}(t)$ must lie in a fixed plane containing the sun. To see this, let $\mathbf{r}$ and $\dot{\mathbf{r}}$ be given at time t, and $\mathbf{M} = \mathbf{r} \times \dot{\mathbf{r}}$ denote the cross product of $\mathbf{r}$ and $\dot{\mathbf{r}}$. Since $\mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to both $\mathbf{r}$ and $\dot{\mathbf{r}}$, it suffices to show that $\mathbf{M}$ is constant in t. Using the Leibniz rule for the cross product, we obtain

$$\dot{\mathbf{M}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0,$$

because $\ddot{\mathbf{r}}$ is parallel to $\mathbf{r}$ by (2.3), and the cross product of parallel vectors vanishes. Thus the entire trajectory lies in the plane perpendicular to $\mathbf{M}$. Let $\mathbf{r} = (x, y)$ denote Cartesian coordinates in this plane with the sun at the origin, and let r and θ denote the corresponding polar coordinates. Now a given trajectory $\mathbf{r}(t) = (x(t), y(t))$ that satisfies (2.3) determines the functions $r(t)$ and $\theta(t)$ through the relations $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. We now find the ODE that this trajectory in polar coordinates satisfies. To this end, note that (2.3) reads

$$\ddot{x} = -\frac{G}{r^3}x, \tag{2.4}$$

$$\ddot{y} = -\frac{G}{r^3}y, \tag{2.5}$$

and using the substitution $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,$$
$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta.$$

Differentiating again and using (2.4) and (2.5) we have

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta = -\frac{G}{r^2}\cos\theta, \tag{2.6}$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta = -\frac{G}{r^2}\sin\theta. \tag{2.7}$$

Now multiplying (2.6) by $\cos\theta$, (2.7) by $\sin\theta$, and adding the result we obtain

$$\ddot{r} - r\dot{\theta}^2 = -\frac{G}{r^2}; \tag{2.8}$$

and multiplying (2.6) by $\sin\theta$, (2.7) by $\cos\theta$, and subtracting we obtain

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0. \tag{2.9}$$

Statements (2.8) and (2.9) hold so long as $r \neq 0$. Assuming this, (2.9) tells us that

$$r^2\dot{\theta} = H, \tag{2.10}$$

where H is a constant determined by the initial conditions. Without loss of generality we assume that H is positive. We can use (2.10) to solve for $\dot{\theta}$ in terms of r, substitute into (2.8), and obtain the following ODE that relates r and t:

$$\ddot{r} = \frac{H^2}{r^3} - \frac{G}{r^2}, \qquad r \neq 0. \tag{2.11}$$

We now use (2.11) to obtain an ODE that is satisfied by r as a function θ. Indeed, (2.10) shows that $\dot{\theta} = H/r^2 \neq 0$ when $H \neq 0$ and $r \neq 0$, so in this case θ is a monotone function of time along trajectories. Let us now assume that $H \neq 0$, $r \neq 0$, let $r = r(\theta)$ give r as a function of θ along a trajectory, and let prime denote differentiation with respect to θ. In this case the chain rule gives

$$\dot{r} = r'\dot{\theta} \quad \text{and} \quad \ddot{r} = r''\dot{\theta}^2 + r'\ddot{\theta}.$$

But $\dot{\theta} = H/r^2$ implies $\ddot{\theta} = -(2H/r^3)r'\dot{\theta} = -(2H^2/r^5)r'$, so we can obtain $\ddot{r} = r''(H^2/r^4) - r'^2(2H^2/r^5)$. Substituting this into (2.11) gives us an ODE for r as a function of θ:

$$\frac{H^2}{r^2}r'' - 2\frac{H^2}{r^3}r'^2 = \frac{H^2}{r} - G. \tag{2.12}$$

We now use one final clever trick to simplify this ODE. We make the definition $u = 1/r$, and substitute u in favor of r in (2.12) using the identities

$$r' = -u'/u^2 \quad \text{and} \quad r'' = -u''/u^2 + 2u'^2/u^3. \tag{2.13}$$

This gives the final remarkably simple linear constant coefficient ODE

$$u'' + u = \frac{G}{H^2}. \tag{2.14}$$

Equation (2.14) is known as *Binet's equation*, and it tells how $u = 1/r$ varies as a function of θ along the trajectory of a planet (assuming Newton's laws are correct). In (2.14) we have transformed a nonlinear equation into a linear one which we can solve explicitly. To summarize, (2.14) is the fundamental ODE predicted by Newton's theory for the orbit of the sun-planet system.

The predictions of Einstein's theory for a sun-planet system are similar. In Einstein's theory a derivation analogous to the derivation above leads to the conclusion that trajectories also lie in a fixed plane containing the sun and equation (2.10) is still satisfied, but the equation that $u = u(\theta)$ satisfies is no longer (2.14), but is instead the following nonlinear ODE which is a perturbation of (2.14) (cf. [**1**], pg. 207):

$$u'' + u = \frac{G}{H^2} + \frac{3G}{c^2}u^2. \tag{2.15}$$

Here c is the speed of light expressed in the units of time and length that G is expressed in. This equation is the same as the equation (2.14) except for the term $(3G/c^2)u^2$, which we might expect is small because the constant c^2 is in the denominator, and the speed of light is very large. In the case of a star in which M_s is large enough so that $G = G_0 M_s$ is on the order of c^2, this term will not be small; and consequently, we expect the orbits of planets to be significantly different from those predicted from Newton's theory. Indeed, Einstein's theory predicts the existence of *black holes* when the density of the star is sufficiently large.

Our analysis of Einstein's ODE in the next section will show that all planets near enough to the star (with low enough energy) will ultimately be sucked into the center of the star as they follow trajectories of (2.15). This contrasts strikingly with the conclusions of Newton's ODE, which predicts that the corresponding planets would enter stable elliptical orbits which would rotate around the star forever. In Einstein's full theory, one can show that when the density of a star is sufficiently large there is a distance, called the *Schwarzschild radius*; and that objects of *all* energies, including *light*, will be drawn into the star when the distance to the star falls within this radius (the Schwarzchild radius for the sun lies well inside the surface of the sun). Thus radiation emitted from such a star cannot be seen, and hence the name *black hole*. This general result cannot be obtained from the ODE (2.15) alone. In fact the xt-coordinates in terms of which (2.15) is expressed do not separate space and time uniformly, curvature effects become dominant, and (2.15) is not a good approximation to Einstein's theory for distances near the Schwarzschild radius. In fact, the fundamental ODE (2.15) was obtained as an approximation to the *Schwarzschild solution*, an exact solution to the Einstein field equations, under the condition that G/Hc is small*. Even though our analysis of (2.15) is not strictly valid close to the center of very massive stars, the next section gives a nice qualitative indication of how black holes arise in the theory of gravitation.

In Section 3 we determine the qualitative properties of solutions of (2.14) and (2.15) using the principle of conservation of energy, and in the final section we will show that the extra term $(3G/c^2)u^2$ in Einstein's equation (2.15) gives rise to the observed anomalous precession in the perihelion of the orbit of the planet Mercury.

3. STRUCTURE OF SOLUTIONS. First we discuss the solutions of the ODE (2.14). We rewrite (2.14) as

$$u'' = -u + \frac{G}{H^2}. \tag{3.1}$$

This ODE is linear and has the general solution

$$u = \frac{G}{H^2} + D\cos(\theta + K), \tag{3.2}$$

where D and K are arbitrary real constants. It is easily verified that (3.2) defines an ellipse, a hyperbola, or a parabola depending upon whether $|D| < G/H^2$, $|D| > G/H^2$, or $|D| = G/H^2$, respectively. We now verify the qualitative properties of the solutions of (3.1) using the principle of conservation of energy. We could get this information directly from (3.2), but we wish to use a method which applies also to the study of Einstein's ODE which is nonlinear.

Writing $F(u) = -u + G/H^2$, equation (3.1) becomes

$$u'' = F(u). \tag{3.3}$$

For equations of this type, the *energy* $E(u, u') = u'^2/2 + P(u)$ is constant along solutions $u = u(\theta)$. Here $u'^2/2$ is called the *kinetic energy* associated with (3.3); and $P(u)$, the *potential energy*, satisfies $P'(u) = -F(u)$. To check that $E = E(u(\theta), u'(\theta))$ is constant along solutions, we simply differentiate with respect to θ:

$$E'(\theta) = u'u'' + P'(u)u' = u'u'' - F(u)u' = 0.$$

Thus, if our initial conditions for (3.1) are

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u'_0$$

for some constants u_0 and u'_0, then $E(u(\theta), u'(\theta)) = E(u_0, u'_0) = E$ for all θ.

*As a historical note, this exact solution was derived by Karl Schwarzschild (1873–1916) in December 1915 while serving in the German army on the eastern front. This work was communicated to the Berlin Academy by Einstein on January 13, 1916, shortly before Schwarzschild's untimely death [5, p. 136].

The positivity of the kinetic energy implies,

$$E \geq P(u(\theta)) \quad \text{for all } \theta, \tag{3.4}$$

and so the solution cannot take on values of u where $P(u) > E$. Thus the energy controls "ahead of time" the possible values of u that a solution $u(\theta)$ of (3.1) can assume. In technical terms, we say that (3.4) is an *apriori estimate* for (3.1). A graph of P will thus indicate to us the types of solutions that are possible for a given initial value of the energy E. Since P is any antiderivative of F with respect to u, we can take P to be

$$P(u) = \frac{1}{2}u^2 - \frac{G}{H^2}u. \tag{3.5}$$

P, a quadratic function, is sketched in Figure 1.

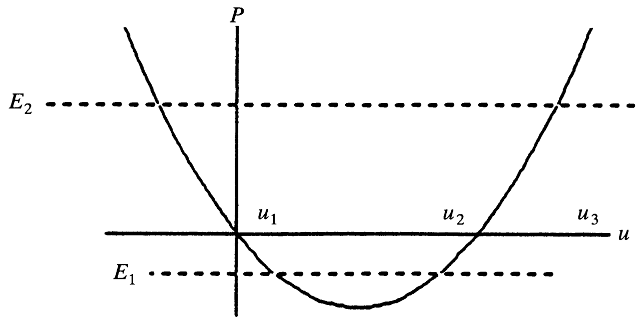

Figure 1

We see that P takes a minimum value of $-G^2/2H^4$ at $u = G/H^2$, so $E_0 = -G^2/2H^4$ is the smallest possible value that the energy E of an orbit can have because $E \geq P$ all along the orbit. For trajectories having $E = E_0$, $u = 1/r = G/H^2$ is constant, so the orbit must be a circle of radius $r = H^2/G$. For trajectories that have energy $E = E_1$, where $-G^2/2H^4 < E_1 < 0$ (see Figure 1), $E \geq P$ implies that the possible values of u taken on in the trajectory lie between the two values u_1 and u_2 which satisfy $P(u_1) = P(u_2) = E_1$. These trajectories correspond to the elliptical orbits in the plane that move between $r = 1/u_1$ and $r = 1/u_2$, the major axis of this ellipse occurring at the value $\theta = \theta_1$ which satisfies $u(\theta_1) = u_1$, and the minor axis occurring at $\theta = \theta_2$ satisfying $u(\theta_2) = u_2$.

The trajectories with energies $E = E_2$, $0 < E_2 < \infty$, are restricted to taking on values of u between 0 and u_3 in Figure 1 (recall that $u = 1/r$ and hence must be nonnegative) with $P(u_3) = E_2$. Such trajectories correspond to hyperbolic orbits that come closest to the sun at $r = 1/u_3$, and then go off to infinity as u tends to zero and $r = 1/u$ tends to infinity. Similarly, the $E = 0$ orbit is the lowest energy orbit for which r tends to infinity, and the nearest this trajectory comes to the sun is $r = H^2/G$. This trajectory corresponds to a parabolic orbit. Note that in the arguments given above for obtaining qualitative structure of orbits at various energies, we used the important observation that the angular velocity u' can be zero only at values of u where $P(u) = E$. This means that u, and hence r, is a strictly increasing or decreasing function of θ when u is in one of the intervals determined by the values of u where $P(u) = E$; and hence solutions can "turn around" only at these special values.

Note that none of the solutions ever crashes into the sun. Thus there is one solution missing from the above analysis; namely, the trajectory corresponding to an object falling straight into the sun. For such a solution, $\theta = constant$, and thus we lost this one solution when we made the assumption $H = r^2\dot{\theta} \neq 0$. Note also that the above energy analysis told us that a trajectory in Newton's theory behaves like an ellipse, hyperbola, or parabola, but it did not tell us the exact shape of an orbit. For Newton's equation we can find a simple formula (3.2) for the trajectories; and we can verify directly from the formula that the orbits truly describe conic sections in the xy-plane. In the following analysis of Einstein's equation, we do not have the luxury of an elementary formula for the solutions, and we will use the energy method to understand the behavior of the orbits.

We now discuss Einstein's ODE (2.15) which we write as

$$u'' = -u + \frac{G}{H^2} + \frac{3G}{c^2}u^2 = F(u). \tag{3.6}$$

This is a nonlinear equation, and the energy E associated with (3.6) is given by

$$E(u,u') = \tfrac{1}{2}u'^2 + P(u),$$

where P is a cubic function of u given by

$$P(u) = \frac{1}{2}u^2 - \frac{G}{H^2}u - \frac{G}{c^2}u^3.$$

One can verify that the critical points in the graph of P are u_+ and u_- given by

$$u_{\pm} = \frac{1}{2B}\left(1 + \sqrt{1 \pm 4\varepsilon}\right),$$

where $A = G/H^2$, $B = 3G/c^2$, and $\varepsilon = AB$ are positive constants. Note that for $\varepsilon \ll 1$, the case for the sun, a Taylor expansion of $\sqrt{1-4\varepsilon}$ shows that it is approximately $1 - 2\varepsilon = 1 - 2AB$, and substituting this into the formula for u_- gives the value $u_- = A$, the critical point in Newton's potential. However, u_+ does not correspond to a critical point in Newton's theory. A graph of P is sketched in Figure 2.

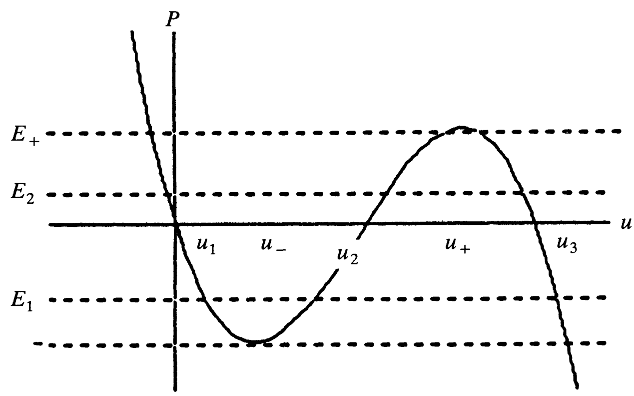

Figure 2

In this figure, $E_- = P(u_-)$, $E_+ = P(u_+)$, and E_i $(i = 1,2)$ are sample values of the energy lying in the intervals determined by E_- and E_+. For fixed E, the states u_i are the values of u where $P(u) = E$. The states u_1, u_2 and u_3 are graphed for energy level $E = E_1$ in Figure 2. The qualitative structure of an orbit depends on which of these intervals the energy of the orbit lies. If $E > E_+$, then $E \geq P$ implies that u' is never zero. Thus, if $u' < 0$, then u tends to zero, r tends to infinity as θ increases, and this corresponds to a trajectory that escapes the sun's gravitational pull in Newtonian theory. Similarly, if $u' > 0$, then $u(\theta)$ will continue out to infinity as θ increases; and hence, r tends to zero and the planet crashes into the sun. Thus, unlike Newton's equation, Einstein's equation predicts that if an object moves toward the sun with enough energy, it will necessarily crash into the sun.

Consider now the orbits in the case with energy E satisfying $E_- < E < E_+$ say, $E = E_1$. Then $E \geq P$ implies that the values of u taken on by the trajectory must either lie within the interval $[u_1, u_2]$, or else within the interval $[u_3, \infty]$, as the dotted lines at energy level E_1 indicates in Figure 2. The case $u(\theta)$ in $[u_1, u_2]$ corresponds to the orbits of Newton's theory. When $u_1 > 0$ and $u(\theta)$ ranges between u_1 and u_2 (exemplified by $E = E_1$ in Figure 2), we obtain a cyclic trajectory that rotates between $r_1 = 1/u_1$ and $r_2 = 1/u_2$, and these correspond to the elliptical orbits of Newton's theory. When $u_1 < 0$ (exemplified by $E = E_2$ in Figure 2), then $u(\theta)$ in $[u_1, u_2]$ implies that $u(\theta)$ actually ranges between 0 and u_2 because $u(\theta) > 0$. Such solutions can move out to the maximum value u where $P(u) = E_2$ (a distance of closest approach), and then they turn around and move monotonically to $u = 0$ (equivalently $r = \infty$). These solutions correspond to the hyperbolic orbits of Newton's theory. Note that nothing we have said implies that the minimum value r_1 for one of the cyclic orbits will be taken on at the same value of θ in every cycle. Indeed, it is the precession of this angle that we will calculate in the next section for the orbit of Mercury.

For the case $E_- < E < E_+$ (again, say, $E = E_1$), $u(\theta) > u_3$, $u' < 0, u(\theta)$ decreases to $u = u_3$ where $u' = 0$, and then "turns around" and $u(\theta)$ increases to infinity. If $u' > 0$ then $u(\theta)$ increases monotonically to infinity. In either case this corresponds to an orbit crashing into the sun. The same is true for trajectories for which $E < E_-$ (see Figure 2). We can conclude that objects close enough to the sun or with low enough energy will necessarily crash into the origin. There are no corresponding trajectories in Newton's theory. At this point it is important to note, however, that our analysis above assumes throughout that the sun is a point mass located at the origin. In fact, the radius of the sun actually occurs at $u < u_+$, so the solutions with $u > u_+$ that crash into the origin are not really observed in our solar system because $u > u_+$ lies inside the surface of the sun*. In contrast, for very massive stars the radius of the star can lie at a value of u well outside of u_+, and one can show that there is in fact a critical value of r, the Schwarzschild radius, inside of which everything, including light, falls into the star, in analogy with the orbits in the last case above. Although the above analysis is a nice indication of the behavior of orbits near a black hole, a complete analysis requires a deeper understanding of general relativity and cannot be obtained from the ODE (3.6) alone (see [5]). As a final comment, note that the solution corresponding to $E = E_-$ is a stable circular orbit at radius $r = 1/u_-$; and solutions sitting at u_+ with energy E_+ are unstable circular orbits, and can just as well fall into the sun as drift away to infinity. Also there is an omitted solution corresponding to an object falling straight into the sun with $\dot{\theta} = 0$, and as in Newton's theory, this solution is not accounted for in (3.6).

*More precisely,

$$r_+ = \frac{1}{u_+} = \frac{2B}{1 + \sqrt{1 + 4\varepsilon}} \leq 2B = \frac{6G}{c^2}$$

has the dimension of length (cf. §4) and corresponds to a radius much smaller than the radius of the sun.

4. THE PRECESSION IN THE PERIHELION OF THE ORBIT OF MERCURY. In this section we study the precession that occurs in the cyclical trajectories of Einstein's equation (3.6). Now the solutions of (3.6) should approximate the solutions of Newton's equation (3.1) when the term $(3G/c^2)u^2$ is "small". We then need a way to measure how small this term really is. It is tempting to take G/c^2 as a measure of how small the term is, but a closer look shows that this makes no sense. Indeed, the absolute magnitude of G/c^2 depends on the choice of units in terms of which we decide to measure mass, length and time. To make sense of the size of term $(3G/c^2)u^2$, we must construct a constant which has a value independent of units we choose. Then we can write $(3G/c^2)u^2$ in terms of this constant. Such a constant is called a *dimensionless parameter*. To obtain our dimensionless parameter, we must first determine the dimensions of the constants G, H, and c which appear in our equation. To this end, let L denote the dimension of length, T the dimension of time, and M the dimension of mass. Now let square brackets around a quantity denote the dimensions of that quantity. For example,

$$[c] = L/T$$

since c is a velocity. Letting X and Y denote two quantities, $[\cdot]$ has the property that

$$[X^n Y^m] = [X]^n [Y]^m$$

for any two integers m and n. Thus, for example,

$$[c^2] = L^2/T^2.$$

We now use the following principle to obtain the dimensions of the quantities G and H: *Every term in the same physical equation must have the same dimensions.* We call such an equation dimensionally correct. Indeed, this principle is really expressing the fact that if we have a function which satisfies a given physical equation expressed in one set of units, then the equation expressed in a new set of units should have as its solution the function obtained from the original one by rescaling it according to the dimensions of the solution variable. We now obtain the dimensions of G and H.

Using that an acceleration has units L/T^2, from (2.3) we obtain

$$L/T^2 = [\ddot{\mathbf{r}}] = \left([G]/[|\mathbf{r}|^3]\right)[\mathbf{r}] = [G]/L^2,$$

so solving for $[G]$ yields

$$[G] = L^3/T^2. \tag{4.1}$$

Equation (2.10) implies

$$[H] = L^2/T \tag{4.2}$$

since the unit of $\dot{\theta}$, a frequency, is $1/T$. Using (4.1) and (4.2) we can verify that G/Hc is the simplest dimensionless parameter constructible from G, H, and c.
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As an aside, statement (4.1) asserts that within Newtonian theory there is a universal constant G, independent of the planet considered, which has the dimension L^3/T^2. This might well lead you to guess there is a quantity of dimension L^3/T^2 associated with each planetary orbit that is independent of the planet chosen. Kepler's third law verifies that this intuition is correct, and that the simplest guess for such a quantity (mean distance to the sun cubed divided by the period of the orbit squared) is correct! In short, by dimensional analysis, one could guess Kepler's third law without making any headway whatsoever in rigorously solving Newton's ODE. When one is presented with a complicated equation, this type of intuition can be crucial. It can also be incorrect!

We are now ready to study the perihelic motion which occurs in the cyclical trajectories in Einstein's theory when $G/Hc \ll 1$. But there is a problem. Since G/Hc is dimensionless, it will be the same when evaluated under any choice of units, and thus it is tempting to say that G/Hc is a true measure of how small this last term is. However, the rate at which a solution of Einstein's ODE (2.14) diverges from a solution of Newton's equation (2.15) also depends on the size of the initial conditions, and G/Hc is not a measure of the perturbation which is independent of the starting conditions. To obtain such a dimensionless parameter that accounts for the initial conditions as well, we "nondimensionalize" the ODE's (2.14) and (2.15). To begin, let us fix on the underlying elliptical solution u_0 of (2.14) which corresponds to the orbit of Mercury in the Newtonean theory. The solution of (2.14) is given in (3.2) as

$$u_0 = A + D\cos(\theta + K), \tag{4.3}$$

where $A = G/H^2$, and we assume $|D| < A$, so that (4.3) describes an ellipse in $r\theta$-coordinates. Since rotating the coordinate axes by an angle $-K$ would eliminate the constant K in this formula, we can assume with no loss of generality that $K = 0$, in which case the initial conditions are

$$u(0) = A + D, \qquad u'(0) = 0. \tag{4.4}$$

(To specify the orbit of Mercury, we must obtain the values for D and H from astronomical tables, but we will see that only the value of H affects the perihelion shift.) Since $|D| < A$, we can take A as a dimensional measure of the size of the initial conditions. Now back to our problem: we wish to find a dimensionless measure of the perturbation of solutions of the Einstein ODE (2.15) from the solution of (2.14) that accounts for the size of the initial condition. The idea is to obtain the equations for the dimensionless variable

$$\bar{u} = u/A. \tag{4.5}$$

First, for the Newton equation, substituting $\bar{u}$ into (2.14) gives

$$\bar{u}_0'' = -\bar{u}_0 + 1, \tag{4.6}$$

with initial conditions

$$\bar{u}_0(0) = \frac{A + D}{A}, \qquad \bar{u}_0'(0) = 0. \tag{4.7}$$

Similarly, for the Einstein ODE, substituting $\bar{u}$ into (2.15) and assuming the same initial conditions, gives

$$\bar{u}'' = -\bar{u} + 1 + \varepsilon\bar{u}^2, \tag{4.8}$$

$$\bar{u}(0) = \frac{A + D}{A}, \qquad \bar{u}'(0) = 0, \tag{4.9}$$

as the Einstein prediction for the same planetary orbit, where $\varepsilon = 3G^2/H^2c^2$. Since $[\bar{u}] = [\varepsilon] = 1$, the parameter ε is a dimensionless parameter that reasonably gives an absolute measure of the perturbation of the Einstein solution $\bar{u}$ from the

Newtonian solution $\bar{u}_0$. Conclude that by writing the non-dimensional equations (4.6) and (4.8) for the dimensionless variables $\bar{u}_0$ and $\bar{u}$, we have located a dimensionless perturbation parameter ε that incorporates the size of the initial conditions. Thus, let

$$u_0(\theta) = 1 + d\cos(\theta), \tag{4.10}$$

denote the fixed solution of Newton's ODE (4.6) corresponding to the Mercury solution (4.3), $d = D/A$. When $\varepsilon \ll 1$, the solution to Einstein's ODE (4.8) with the same initial data will remain close to this trajectory at least over changes of angle that are not too great. Thus we write the corresponding solution $\bar{u}$ to Einstein's ODE as $\bar{u} = \bar{u}_0 + \varepsilon v$ so that εv is the perturbation from the Newtonian trajectory. We wish to estimate this perturbation. Thus we plug $\bar{u}_0 + \varepsilon v$ into Einstein's ODE (4.8) and collect like powers of ε. If ε is small, and the trajectory ranges over angles that are not too great, we can ignore all terms with powers of ε smaller than or equal to ε^2. The term corresponding to the first power of ε will provide an equation whose solution is a good estimate for the perturbation of the solution u from the underlying orbit u_0 of Newton's theory. Plugging in we obtain:

$$\bar{u}_0'' + \bar{u}_0 - 1 + \varepsilon\left[v'' + v - u_0^2\right] + (\text{higher order terms in } \varepsilon) = 0. \tag{4.11}$$

Equation (4.11) is called an *asymptotic expansion* of (4.8). Now $\bar{u}_0'' + \bar{u}_0 - 1 = 0$ because this is Newton's equation, and $\bar{u}_0$ was assumed at the outset to be some given solution of this equation. Neglecting the higher order terms in ε, we obtain an ODE that approximately describes the function v when ε is small:

$$v'' + v - \bar{u}_0^2 = 0. \tag{4.12}$$

Thus for $\bar{u}_0$ known, the ODE (4.12) is a linear, constant coefficient, inhomogeneous equation in v, and we can solve it directly. We conclude that, given a Newtonian trajectory $\bar{u}_0$, we can approximate the Einsteinian perturbation εv from this orbit by solving (4.12). Plugging (4.10) into (4.12) yields the ODE

$$v'' + v = 1 + \frac{d^2}{2} + 2d\cos\theta + \frac{d^2}{2}\cos 2\theta, \tag{4.13}$$

where we have applied the trigonometric identity $\cos^2\theta = (1 + \cos 2\theta)/2$. Now (4.13) is an inhomogeneous linear ODE with constant coefficients, and $v_0 = d'\cos(\theta + K')$ solves the underlying homogeneous equation for arbitrary constants d' and K'. To obtain a particular solution of the inhomogeneous problem, we can apply superposition and write $v = v_1 + v_2 + v_3$ where v_i solve the separate equations:

$$v_1'' + v_1 = 1 + \frac{d^2}{2},$$

$$v_2'' + v_2 = 2d\cos\theta,$$

$$v_3'' + v_3 = \frac{d^2}{2}\cos 2\theta.$$

One can easily verify that the three solutions are

$$v_1 = 1 + \frac{d^2}{2}, \qquad v_2 = d\theta\sin\theta, \qquad v_3 = -\frac{d^2}{6}\cos 2\theta.$$

Thus the general solution of (4.8) is $v = v_0 + v_1 + v_2 + v_3$. Now in order that $\bar{u}_0$

and $\bar{u}$ satisfy the same initial conditions (4.7) and (4.9), $v(\theta)$ must satisfy

$$v(\theta) = 0, \quad v'(\theta) = 0,$$

and thus it is easy to calculate that

$$K' = 0,$$

and

$$d' = -1 - d^2/3.$$

Our approximation for $\bar{u}$ can now be written down:

$$\bar{u} = \bar{u}_0 + \varepsilon v \approx 1 + d\cos\theta + \varepsilon d'\cos(\theta) + \varepsilon + \varepsilon\frac{d^2}{2} + \varepsilon d\theta\sin\theta - \varepsilon\frac{d^2}{6}\cos 2\theta. \tag{4.14}$$

Now (4.14) is a messy formula, but we are only interested in the perihelion shift (the rotation in the angle at which the maximum value of either u or $\bar{u}$ is taken on in successive orbits) for the cyclical trajectory of u. But it is only the nonperiodic terms in (4.14) that can contribute to such a shift, and the only nonperiodic term in (4.14) is the term $\varepsilon d\theta\sin(\theta)$. To see the effect of this term on successive perihelia, rewrite (4.9) as,

$$\bar{u} = 1 + d(\cos\theta + \varepsilon\theta\sin\theta) + \text{periodic terms of order } \varepsilon. \tag{4.15}$$

The periodic terms can change the angle at which the perihelia are taken on, but being periodic, they cannot significantly affect the shift in the position of the perihelia that occur in successive revolutions. To see this more clearly, we can write

$$\cos\theta + \varepsilon\theta\sin\theta \approx \cos\theta\cos(\varepsilon\theta) + \sin\theta\sin(\varepsilon\theta) = \cos(\theta - \varepsilon\theta) \tag{4.16}$$

because, since we are neglecting higher order terms in ε, Taylor's theorem implies that $\cos(\varepsilon\theta) \approx 1$ and $\sin(\varepsilon\theta) \approx \varepsilon\theta$. Using this in (4.15) we obtain

$$\bar{u} = 1 + d\cos(\theta - \varepsilon\theta) + \text{periodic terms of order } \varepsilon. \tag{4.17}$$

We now claim that the shift in the perihelion during one cycle is affected by the periodic terms of order ε only in an amount that is second order in ε; and so neglecting higher order terms, the shift observed in (4.17) after one revolution will be the same as the shift observed in the function $\bar{u} = 1 + d\cos(\theta - \varepsilon\theta)$ after one revolution. We postpone the proof of this claim until the end of this section, where we show that the claim is a special case of a general principle. The function $1 + d\cos(\theta - \varepsilon\theta)$ takes on successive maxima when $\theta = 2n\pi/(1 - \varepsilon) \approx 2n\pi(1 + \varepsilon)$. Therefore the shift in the angle at which the perihelia occur after one revolution (ignoring terms quadratic in ε) is estimated as

$$\Delta\theta = 2\pi\varepsilon = 6\pi\frac{G^2}{H^2c^2}, \tag{4.18}$$

since $\varepsilon = 3G^2/H^2c^2$. To apply this formula to the Mercury-Sun system, we must have numerical values for G, c, and H, the latter applying to Mercury's orbit. From **[3]** we obtain current experimental values for G and c:

$$G = 1.32712497 \times 10^{26}\ \text{cm}^3/\text{sec}^2,$$

$$c = 2.99792458 \times 10^{10}\ \text{cm/sec}.$$

The quantity H is somewhat more difficult to find because it is difficult to measure directly by astronomical observations. In constrast, the lengths of the major and minor axes of the almost elliptical orbit of Mercury are readily observable since these are obtained from measurements of the closest and farthest distances that the planet comes to the sun. We claim that

$$G/H^2 \approx \frac{1}{L}, \tag{4.19}$$

where $L = a(1 - e)$, a is the length of the semi-major axis (the average of the major and minor axes of the ellipse) and e is the eccentricity for the elliptical orbit of Mercury (cf. [7]). We leave the verification of this claim until the end. Assuming (4.19), (4.18) becomes

$$\Delta\theta \approx 6\pi \frac{G}{c^2 L}. \tag{4.20}$$

From [2] we find that $a = 5.7909 \times 10^{12}$ cm and $e = 0.205628$, which implies that $L = 5.5460 \times 10^{12}$ cm. Thus equation (4.20) gives

$$\begin{aligned} \Delta\theta &= 5.0187 \times 10^{-7} \text{ radians per revolution} \\ &= 2.8755 \times 10^{-5} \text{ degrees per revolution} \\ &= 0.103518 \text{ seconds of arc per revolution.} \end{aligned}$$

Since there are 415.2 revolutions in a century, we obtain that the precession in the perihelion of the orbit of the planet Mercury in one century is predicted by Einstein's theory to be approximately $415.2\Delta\theta$, which is approximately 42.98 seconds of an arc per century.

All that remains is to verify (4.19), and to prove our claim that the periodic terms of order ε contribute order ε^2 in the perihelion shift. For (4.19), note that in Newtonian theory the orbit of Mercury is an ellipse with major and minor axes given by

$$r_\pm = (1 \pm e)a.$$

At $u = u_\pm = 1/r_\pm$, $u' = 0$ since $u_\pm$ are critical points of $u = u(\theta)$. Thus evaluating the energy integral at these values gives two equations:

$$E = \frac{1}{2}u_\pm^2 - \frac{G}{H^2}u_\pm - \frac{G}{c^2}u_\pm^3.$$

Subtracting these two equations and canceling a common factor of $(u_+ - u_-)$ we find

$$\frac{G}{H^2} = \frac{1}{2}(u_+ - u_-) - \frac{G}{c^2}\left(u_+^2 + u_+u_- + u_-^2\right).$$

In terms of a and e,

$$u_+ + u_- = \frac{2}{L},$$

$$u_+^2 + u_+u_- + u_+^2 = \frac{3 + e}{L^2},$$

so that

$$\frac{G}{H^2} = \frac{1}{L}\left(1 - \frac{G}{c^2 L}(3 + e^2)\right). \tag{4.21}$$

For Mercury, $G/c^2L = 2.7 \times 10^{-8}$, so we may neglect this term in (4.21) to obtain equation (4.20).

Finally, our claim that the periodic terms of order ε in (4.17) contribute errors in the perihelion shift of order ε^2 follows directly from the following:

Theorem. *Let F and f be smooth, real valued, 2π-periodic functions of θ and set*

$$G(\theta) = F(\theta - \varepsilon\theta) + \varepsilon f(\theta).$$

Assume that $|\varepsilon| \ll 1$, that θ_0 satisfies $G(\theta_0) = 0$ and that $F'(\theta_0 - \varepsilon\theta_0) = a \neq 0$. Then

$$G(\theta_0 + 2\pi + \Delta\theta) = 0,$$

where

$$\Delta\theta = 2\pi\varepsilon + \textit{terms of order } \varepsilon^2.$$

Our claim follows when we let $\theta_0 = 0$, $F(\theta) = \cos\theta$, and $f(\theta) =$ the periodic terms of order ε.

Proof: Let $\theta = \theta_0 + 2\pi + \Delta\theta$ for $|\Delta\theta| \ll 1$, and set $f'(\theta_0) = b$, then

$$F(\theta - \varepsilon\theta) = F(\theta_0 + 2\pi + \Delta\theta - \varepsilon(\theta_0 + 2\pi + \Delta\theta)).$$

Using Taylor's theorem to expand $F(\theta - \varepsilon\theta)$ about the point $\theta_0 - \varepsilon\theta_0 + 2\pi$, we obtain

$$F(\theta - \varepsilon\theta) = F(\theta_0 - \varepsilon\theta_0 + 2\pi) + a(\Delta\theta - 2\pi\varepsilon) + \text{Error}_1$$

where $|\text{Error}_1| \leq \text{const}(|\Delta\theta| + |\varepsilon|)^2$. Similarly,

$$\begin{aligned} \varepsilon f(\theta) &= \varepsilon f(\theta_0 + 2\pi) + \varepsilon b\Delta\theta + \cdots \\ &= \varepsilon f(\theta_0 + 2\pi) + \text{Error}_2 \end{aligned}$$

where $|\text{Error}_2| \leq \text{const}(|\Delta\theta| + |\varepsilon|)^2$. Thus,

$$\begin{aligned} G(\theta) &= F(\theta - \varepsilon\theta) + \varepsilon f(\theta) \\ &= F(\theta_0 - \varepsilon\theta_0 + 2\pi) + \varepsilon f(\theta_0 + 2\pi) + a(\Delta\theta - 2\pi\varepsilon) + \text{Error}_3 \\ &= a(\Delta\theta - 2\pi\varepsilon) + \text{Error}_3 \end{aligned}$$

where $|\text{Error}_3| \leq \text{const}(|\Delta\theta| + |\varepsilon|)^2$ and we have used the fact that G is 2π-periodic and $G(\theta_0) = 0$. Therefore, $G(\theta)$ will be zero when

$$\Delta\theta = 2\pi\varepsilon + \text{Error}_3.$$

But this implies $|\Delta\theta| \leq \text{const}\ \varepsilon$, so $|\text{Error}_3| \leq \text{const}\ \varepsilon^2$, and so we conclude that

$$\Delta\theta = 2\pi\varepsilon + \text{terms of order } \varepsilon^2,$$

which verifies the claim.

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