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ASYMPTOTICS OF A TAU-FUNCTION AND TOEPLITZ DETERMINANTS WITH SINGULAR GENERATING FUNCTIONS

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ABSTRACT

We compute the short distance asymptotics of a tau-function appearing in the work of Sato, Miwa, and Jimbo on holonomic quantum fields. We show that these asymptotics are determined by the Widom operator. This same operator is fundamental in the asymptotics of Toeplitz determinants with singular generating functions.

1. Introduction and Summary of Results

In their study of holonomic quantum fields, Sato, Miwa, and Jimbo[1] proved the following result about τ-functions:

Theorem 1. Define

$$\tau(t;\theta,\lambda) = \exp\left(-\sum_{k=1}^{\infty}\frac{\lambda^{2k}}{k}e_{2k}(t;\theta)\right) \tag{1.1}$$

where

$$e_{2k}(t;\theta) = \int_0^\infty dx_1\cdots\int_0^\infty dx_{2k}\prod_{j=1}^{2k}\frac{\exp(-\frac{1}{2}t(x_j+x_j^{-1}))}{x_j+x_{j+1}}\left(\frac{x_1x_3\cdots x_{2k-1}}{x_2x_4\cdots x_{2k}}\right)^\theta, \tag{1.2}$$

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$t \in \mathbf{R}^+$, $-1 < \theta < 1$, and $x_{2k+1} = x_1$; then

$$\tau(t;\theta,\lambda) = \exp\left[-\frac{1}{2}\int_t^\infty \left[s\left((\frac{d\psi}{ds})^2 - \sinh^2\psi\right) - \frac{\theta^2}{s}\tanh^2\psi\right] ds\right] \tag{1.3}$$

where $\psi = \psi(t;\theta,\lambda)$ satisfies the differential equation

$$\frac{d^2\psi}{dt^2} + \frac{1}{t}\frac{d\psi}{dt} = \frac{1}{2}\sinh 2\psi + \frac{\theta^2}{t^2}\tanh\psi(1-\tanh^2\psi) \tag{1.4}$$

with boundary condition

$$\psi(t;\theta,\lambda) \sim 2\lambda K_\theta(t) \quad \text{as} \quad t \to \infty \tag{1.5}$$

and $K_\theta(t)$ is the modified Bessel function.

We remark that the notation (1.1) has been changed slightly from Ref. 1, that the right-hand side of (1.1) can be rewritten as $\det(I - \lambda^2 T_\beta)$ (see, e.g., Appendix B), and that (1.4) can be transformed to Painlevé V (PV) by the change of variables $y(x) = \tanh^2\psi(t)$, $x = t^2$;[a] and in fact, this PV is reducible to PIII. We also note that the $\theta = 0$ case is closely related to the 2-point scaling functions of the 2D Ising model[2,3] and the Federbush model 2-point function[4,5] can be expressed in terms of $\tau(t; 2\beta, i\sin\pi\beta/\beta)$ where β is the coupling constant (in Ruijsenaars,[5] his "λ" is our "β").

While the large $t \to \infty$ behavior of τ follows easily from (1.1), the short distance behavior $t \to 0^+$ is more difficult to determine. In physical terms, the leading short distance behavior is the conformal (or massless) limit. In 1982, Jimbo[6] in his analysis of the connection problem for Painlevé functions, proved the following result:

Theorem 2. Let $\tau(t;\theta,\lambda)$ be defined by the series (1.1). Assume that $2\pi^2\lambda^2 - \cos\pi\theta$ is not a real number ≥ 1, and let σ be defined by

$$\pi^2\lambda^2 = \sin\frac{\pi}{2}(\sigma+\theta)\sin\frac{\pi}{2}(\sigma-\theta), \tag{1.6}$$

$0 \leq \Re\sigma < 1$, then the short distance behavior $t \to 0^+$ of $\tau(t;\theta,\lambda)$ is given by

$$\tau(t;\theta,\lambda) \sim \tau_0(\theta,\lambda) t^{\frac{1}{2}(\sigma^2-\theta^2)}. \tag{1.7}$$

(In Ref. 6, $\lambda_J = 2\pi\lambda$, $t_J = t^2/4$).

Jimbo actually proved a stronger result than (1.7) as he provided a method which, in principle, gives any number of correction terms to (1.7). However, he did not determine the constant $\tau_0(\theta,\lambda)$.

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[a]The parameters in PV are $\alpha = \frac{1}{2}\theta^2$, $\beta = 0$, $\gamma = \frac{1}{2}$, and $\delta = 0$.

In the case of the Federbush model, where $\sigma = 0$, Ruijsenaars[5] also proved (1.7), but did not provide the correction terms that are in Jimbo or the constant $\tau_0(\theta, \lambda)$. In unpublished notes dated February 1984, Ruijsenaars derived an infinite series expression (the n^{th} order term being some $(2n-1)$-dimensional integral) for $\tau_0(\theta, \lambda)$ but did not evaluate the series.

More recently, the expression (1.7) was proved in the case $\theta = 0$ and the constant was determined.[7] The purpose of this paper is to generalize the results given in Ref. 7 for $\theta \neq 0$. The next theorem summarizes our results.

Theorem 3. Let $\tau(t; \theta, \lambda)$ be defined by the series (1.1) and define

$$2\alpha = \sigma, \quad 2\beta = \theta. \tag{1.8}$$

Then the constant

$$\tau_0(\theta, \lambda) = 2^{-2(\alpha^2-\beta^2)} \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)G(1-\alpha+\beta)G(1-\alpha-\beta)}{G(1+2\alpha)G(1-2\alpha)} \tag{1.9}$$

where G is the Barnes G-function[8] defined by[b]

$$G(s+1) = (2\pi)^{s/2} e^{-s/2-(1+\gamma)s^2/2} \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right)^k e^{-s+s^2/2k}. \tag{1.10}$$

One of the interesting features of the constant term $\tau_0(\theta, \lambda)$ is that it is closely related to the constant term found in the asymptotic formula for a particular Toeplitz determinant. Consider the function $\varphi(t)$ defined by

$$\varphi(t) = (\cot \frac{t}{2})^{2\alpha} \begin{cases} e^{i\beta\pi}, & \text{if } 0 < t < \pi \\ e^{-i\beta\pi}, & \text{if } \pi < t < 2\pi \end{cases} \tag{1.11}$$

and then define

$$D_N[\varphi] = \det(\varphi_{i-j}), \; 0 \le i, j \le N, \tag{1.12}$$

where φ has Fourier coefficients φ_j. The asymptotic behavior of these determinants with this type of singular generating function was first studied by Fisher and Hartwig[9] in 1968. They conjectured that

$$D_0 = \lim_{N \to \infty} D_N[\varphi] N^{-2(\alpha^2-\beta^2)} \tag{1.13}$$

existed and later it was shown in a series of papers[10–16] that

$$D_0 = 2^{2(\alpha^2-\beta^2)} \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)G(1-\alpha+\beta)G(1-\alpha-\beta)}{G(1+2\alpha)G(1-2\alpha)}. \tag{1.14}$$

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[b] The G-function satisfies the functional equation $G(s+1) = \Gamma(s)G(s)$ and has the special values $G(1) = 1$, $G(\frac{1}{2}) = \exp\left(\frac{3}{2}\zeta'(-1) - \frac{1}{4}\log\pi + \frac{1}{24}\log 2\right)$.

Thus, we have

$$\tau_0(\theta,\lambda) = 2^{-4(\alpha^2-\beta^2)}D_0 = 2^{-4(\alpha^2-\beta^2)} \lim_{N\to\infty} D_N[\varphi]N^{-2(\alpha^2-\beta^2)}. \tag{1.15}$$

Glimpses of the connection between the asymptotics of τ-functions, Toeplitz determinants with singular generating functions, and the Barnes G-function, can also be seen in the work of Lenard[10] and Jimbo, et. al.[17] in their analysis of impenetrable bosons; in Wu[18] and Wu, et. al.[2] on Ising correlations; in Widom,[19] Dyson[20] and Jimbo, et. al.[17] in random matrices (see also Chap. 12 of Ref. 21); and in McCoy, Perk and Shrock[22] in work on a critical field transverse Ising model. For a more detailed account of this and the current status of the Fisher-Hartwig conjecture see Ref. 23. It is also interesting to observe that Sarnak,[24] Voros[25] and Fay[26] have found that the Barnes G-function appears in determinants of Laplacians on compact Riemann surfaces. These determinants are of interest to string theorists (see, e.g., Ref. 27).

One of the earliest papers dealing with the Fisher-Hartwig conjecture is due to Widom.[11] In this paper Widom showed that the inversion of Toeplitz matrices can be reduced to inverting a pair of infinite matrix equations. These infinite matrices were then approximated by certain integral operators $K_{\alpha,\beta}$ defined on $L_{2,\beta} = L_2(0,\infty,(x+1)^{-2\beta}dx)$. The operator $K_{\alpha,\beta}$ is given by the kernel

$$K_{\alpha,\beta}(x,y) = c_{\alpha,\beta}\int_0^\infty \frac{dz}{(z+x+1)^{1-2\beta}(z+y+1)^{1+2\beta}} \tag{1.16}$$

where

$$c_{\alpha,\beta} = \pi^{-2}\Gamma(1+2\beta)\Gamma(1-2\beta)\sin\pi(\alpha+\beta)\sin\pi(\alpha-\beta). \tag{1.17}$$

Perhaps the most significant aspect of the present paper is the fact that the short distance asymptotics (1.7) of the τ-function are also determined by the Widom operator. This indicates that the connections between these various asymptotic formulas take place on a very fundamental level.

The use of the Widom operator in computing (1.7) differs from the method found in Ref. 7 for $\theta = 0$. Because of the difference in approach, the constant $\tau_0(0,\lambda)$ has a different form in Ref. 7 than given in the present paper. In Appendix C, we give a short proof of the equality of the two expressions using properties of the Barnes G-function. Also in Appendix A, the inversion of $I - K_{\alpha,\beta}$ is described using a factorization technique. Finally, we point out in proving Theorem 3, we give an independent proof of the exponent $\frac{1}{2}(\sigma^2-\theta^2)$ appearing in Theorem 2.

2. Preliminaries and Exponent Evaluation

We define

$$E(t;\theta,\lambda) = \sum_{k=1}^\infty \frac{\lambda^{2k}}{k} e_{2k}(t;\theta) \tag{2.1}$$

where $e_{2k}(t;\theta)$ are given by (1.2). We will prove that

$$e_{2k}(t;\theta) = e_{-1,2k}\log(1/t) + e_{0,2k} + o(1), \quad \text{as} \quad t \to 0^+, \tag{2.2}$$

$$E(t;\theta,\lambda) = E_{-1}\log(1/t) + E_0 + o(1), \quad \text{as} \quad t \to 0^+, \tag{2.3}$$

where

$$E_{-1} = \sum_{k=1}^{\infty} e_{-1,2k}\frac{\lambda^{2k}}{k} \quad \text{and} \quad E_0 = \sum_{k=1}^{\infty} e_{0,2k}\frac{\lambda^{2k}}{k}. \tag{2.4}$$

The expressions (2.4) will follow from (2.2) once we show that the $o(1)$ terms in (2.2) remain $o(1)$ when summed in (2.1). For the moment we take (2.4) as the definitions of E_{-1} and E_0, and later in §7 we will prove that the error terms do indeed remain $o(1)$. A proof that the series (2.1) converges for $|\lambda| < \cos\pi\beta/\pi$ and for all $t > 0$ is given in Appendix B.

To begin, we introduce the Mellin transform of $e_{2k}(t;\theta)$:

$$\begin{aligned}\hat{e}_{2k}(z;\theta) &= \int_0^\infty t^{z-1}e_{2k}(t;\theta)\,dt \\ &= 2^z\Gamma(z)\int_0^\infty dx_1\cdots\int_0^\infty dx_{2k}\prod_{j=1}^{2k}(x_j+x_{j+1})^{-1}\Big(\sum_{j=1}^{2k}(x_j+1/x_j)\Big)^{-z} \\ &\quad\times\left(\frac{x_1x_3\cdots x_{2k-1}}{x_2x_4\cdots x_{2k}}\right)^{\theta}.\end{aligned}$$

We make a change of variables $x_j = \rho\delta_j$, $\delta_{2k} = 1-\delta_1-\cdots-\delta_{2k-1}$, so that (the Jacobian is ρ^{2k-1})

$$\begin{aligned}\hat{e}_{2k}(z;\theta) &= 2^z\Gamma(z)\int_0^1 d\delta_1\int_0^{1-\delta_1}d\delta_2\cdots\int_0^{1-\delta_1-\cdots-\delta_{2k-2}}d\delta_{2k-1}\prod_{j=1}^{2k}(\delta_j+\delta_{j+1})^{-1} \\ &\quad\times\left(\frac{\delta_1\delta_3\cdots\delta_{2k-1}}{\delta_2\delta_4\cdots\delta_{2k}}\right)^{\theta}\int_0^\infty \rho^{z-1}\left[\rho^2+\Delta_{2k}(\delta)\right]^{-z}\end{aligned}$$

where $\Delta_{2k}(\delta) = 1/\delta_1+\cdots+1/\delta_{2k}$. Performing the ρ integration results in

$$\begin{aligned}\hat{e}_{2k}(z;\theta) &= 2^{z-1}\Gamma^2(z/2)\int_0^1 d\delta_1\int_0^{1-\delta_1}d\delta_2\cdots\int_0^{1-\delta_1-\cdots\delta_{2k-2}}d\delta_{2k-1} \\ &\quad\times\prod_{j=1}^{2k}(\delta_j+\delta_{j+1})^{-1}\left(\frac{\delta_1\delta_3\cdots\delta_{2k-1}}{\delta_2\delta_4\cdots\delta_{2k}}\right)^{\theta}\Delta_{2k}(\delta)^{-z/2} \\ &= 2^{z-1}\Gamma^2(z/2)\vartheta_{2k}(z,\theta)\end{aligned} \tag{2.5}$$

where $\delta_{2k} = 1-\delta_1-\cdots-\delta_{2k-1}$, $\delta_{2k+1} = \delta_1$, and $\vartheta_{2k}(z,\theta)$ is defined by the last equality. To get some preliminary feel for these functions, we evaluate $\hat{e}_2(z;\theta)$ explicitly:

$$\hat{e}_2(z;\theta) = 2^{z-1}\Gamma^2(z/2)\Gamma(z/2+\theta+1)\Gamma(z/2-\theta+1)/\Gamma(z+2).$$

In general we will show that

$$\hat{e}_{2k}(z;\theta) = \frac{\hat{e}_{-2,2k}}{z^2} + \frac{\hat{e}_{-1,2k}}{z} + O(1)$$

as $z \to 0$. The Mellin inversion formula implies that $\hat{e}_{-2,2k} = e_{-1,2k}$ and $\hat{e}_{-1,2k} = e_{0,2k}$ (see §7).

Now consider that

$$\begin{aligned}\tilde{J}_{2k}(\theta) &= \int_0^\infty dx_1 \cdots \int_0^\infty dx_{2k} e^{-x_1 - x_{2k}} \prod_{j=1}^{2k-1} (x_j + x_{j+1})^{-1} \left(\frac{x_1 x_3 \cdots x_{2k-1}}{x_2 x_4 \cdots x_{2k}}\right)^\theta \\ &= \vartheta_{2k}(0,\theta)\end{aligned} \tag{2.6}$$

where the second equality results from the change of variables $x_j = \rho\delta_j$ as above followed by an integration over ρ. To evaluate $\tilde{J}_{2k}(\theta)$, first recall the Mellin convolution theorem

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F(\zeta)G(1-\zeta)\,d\zeta$$

where $F(\zeta) = \int_0^\infty x^{-\zeta} f(x)\,dx$ and similarly for $G(\zeta)$. Choosing $f(x) = x^{-\theta}/(x+y)$ and $g(x) = e^{-x}$, it follows from the convolution theorem that

$$\begin{aligned}\tilde{J}_{2k}(\theta) &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \left[\frac{\pi}{\sin\pi(\zeta+\theta)}\right]^k \left[\frac{\pi}{\sin\pi\zeta}\right]^k d\zeta \\ &= \frac{1}{2}\pi^{2k-1}\int_{-\infty}^\infty \frac{1}{(\cosh\pi\xi)^k(\cosh\pi(\xi - i\theta))^k}\,d\xi.\end{aligned} \tag{2.7}$$

Referring back to (2.4) and (2.6) we see that

$$\begin{aligned}E_{-1} &= 2\sum_{k=1}^\infty \frac{\lambda^k}{k}\tilde{J}_{2k}(\theta) \\ &= -\frac{1}{\pi}\int_{-\infty}^\infty \log\left(1 - \frac{\pi^2\lambda^2}{\cosh\pi\xi\cosh\pi(\xi - i\theta)}\right) d\xi\,.\end{aligned} \tag{2.8}$$

To evaluate this integral, first observe that

$$\frac{\partial E_{-1}}{\partial\lambda} = 2\pi\lambda\int_{-\infty}^\infty \frac{1}{\cosh\pi\xi\cosh\pi(\xi - i\theta) - \pi^2\lambda^2}\,d\xi\,.$$

Under the hypothesis that $(\cos\pi\theta - 2\pi^2\lambda^2)^2 < 1$, an elementary calculation gives (and using the parametrization (1.6))

$$\partial E_{-1}/\partial\lambda = 4\pi\lambda\sigma/\sin\pi\sigma \quad \text{and} \quad \partial E_{-1}/\partial\sigma = \sigma$$

since $\partial\lambda/\partial\sigma = \sin\pi\sigma/4\pi\lambda$. Since $\sigma = \theta$ at $\lambda = 0$, an integration gives

$$E_{-1} = \frac{1}{2}(\sigma^2 - \theta^2)\,, \tag{2.9}$$

which yields the exponent given in Theorem 2.

3. E_0 and the Widom operator $K_{\alpha,\beta}$

To determine the $e_{0,2k}$ of (2.4) we must compute the residue of $\hat{e}_{2k}(z;\theta)$. This will follow if we can compute

$$\begin{aligned}\vartheta'_{2k}(0,\theta) = -\frac{1}{2}\int_0^1 d\delta_1 \int_0^{1-\delta_1} d\delta_2 \cdots \int_0^{1-\delta_1-\cdots-\delta_{2k-2}} d\delta_{2k-1} \prod_{j=1}^{2k}(\delta_j+\delta_{j+1})^{-1} \\ \times \log\Delta_{2k}(\delta)\left(\frac{\delta_1\delta_3\cdots\delta_{2k-1}}{\delta_2\delta_4\cdots\delta_{2k}}\right)^{\theta}.\end{aligned} \tag{3.1}$$

To begin, consider

$$\begin{aligned}J_{2k}(\theta) = \int_0^\infty dx_1 \cdots \int_0^\infty dx_{2k} e^{-x_1-x_{2k}} \prod_{j=1}^{2k-1}(x_j+x_{j+1})^{-1}\left(\frac{x_1x_3\cdots x_{2k-1}}{x_2x_4\cdots x_{2k}}\right)^{\theta} \\ \times \log\left(\frac{(x_1+\cdots+x_{2k})\Delta_{2k}(x)}{x_1+x_{2k}}\right),\end{aligned} \tag{3.2}$$

make the by now standard change of variable $x_j = \rho\delta_j$ in $J_{2k}(\theta)$, perform the ρ integration, to find

$$J_{2k}(\theta) = \gamma\vartheta_{2k}(0,\theta) - 2\vartheta'_{2k}(0,\theta) \tag{3.3}$$

where γ is Euler's constant. Write

$$J_{2k}(\theta) = J_{2k}^{(1)}(\theta) + J_{2k}^{(2)}(\theta) \tag{3.4}$$

where

$$\begin{aligned}J_{2k}^{(1)}(\theta) = \int_0^\infty dx_1 \cdots \int_0^\infty dx_{2k} e^{-x_1-x_{2k}} \prod_{j=1}^{2k-1}(x_j+x_{j+1})^{-1}\left(\frac{x_1x_3\cdots x_{2k-1}}{x_2x_4\cdots x_{2k}}\right)^{\theta} \\ \times \log\Delta_{2k}(x)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}J_{2k}^{(2)}(\theta) = \int_0^\infty dx_1 \cdots \int_0^\infty dx_{2k} e^{-x_1-x_{2k}} \prod_{j=1}^{2k-1}(x_j+x_{j+1})^{-1}\left(\frac{x_1x_3\cdots x_{2k-1}}{x_2x_4\cdots x_{2k}}\right)^{\theta} \\ \times \log\left(\frac{x_1+\cdots+x_{2k}}{x_1+x_{2k}}\right).\end{aligned} \tag{3.6}$$

We now apply the identity

$$\log(\frac{x}{y}) = \int_0^\infty \frac{d\xi}{\xi} \left(e^{-\xi y} - e^{-\xi x}\right) \tag{3.7}$$

to the logarithms appearing in the integrands of $J_{2k}^{(i)}(\theta)$ to obtain

$$J_{2k}^{(i)}(\theta) = \lim_{\varepsilon \to 0^+} \left(J_{2k}^{\prime(i)}(\varepsilon) - J_{2k}^{\prime\prime(i)}(\varepsilon)\right), \; i = 1, 2, \tag{3.8}$$

where

$$\begin{aligned} J_{2k}^{\prime(1)}(\varepsilon) = &\int_\varepsilon^\infty \frac{d\xi}{\xi} e^{-\xi} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{2k} e^{-x_1} \prod_{j=1}^{2k-1} (x_j + x_{j+1})^{-1} \\ &\times \left(\frac{x_1 x_3 \cdots x_{2k-1}}{x_2 x_4 \cdots x_{2k}}\right)^\theta e^{-x_{2k}}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} J_{2k}^{\prime\prime(1)}(\varepsilon) = &\int_\varepsilon^\infty \frac{d\xi}{\xi} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{2k} (e^{-\xi/x_1}/x_1) e^{-x_1 - \cdots - x_{2k}} \prod_{j=1}^{2k-1} (x_j + x_{j+1})^{-1} \\ &\times \left(\frac{x_2 x_4 \cdots x_{2k}}{x_1 x_3 \cdots x_{2k-1}}\right)^\theta (e^{-\xi/x_{2k}}/x_{2k}), \end{aligned} \tag{3.10}$$

$$\begin{aligned} J_{2k}^{\prime(2)}(\varepsilon) = &\int_\varepsilon^\infty \frac{d\xi}{\xi} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{2k} e^{-(1+\xi)x_1} \prod_{j=1}^{2k-1} (x_j + x_{j+1})^{-1} \\ &\times \left(\frac{x_1 x_3 \cdots x_{2k-1}}{x_2 x_4 \cdots x_{2k}}\right)^\theta e^{-(1+\xi)x_{2k}}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} J_{2k}^{\prime\prime(2)}(\varepsilon) = &\int_0^{1/\varepsilon} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{2k} e^{-\xi x_1} e^{-x_1 - \cdots - x_{2k}} \prod_{j=1}^{2k-1} (x_j + x_{j+1})^{-1} \\ &\times \left(\frac{x_1 x_3 \cdots x_{2k-1}}{x_2 x_4 \cdots x_{2k}}\right)^\theta e^{-\xi x_{2k}}. \end{aligned} \tag{3.12}$$

In obtaining the expression $J_{2k}^{\prime\prime(1)}(\varepsilon)$ we made the change of variables $x_j \to \xi/x_j$ after applying the $\log(x/y)$ identity; and similarly $x_j \to \xi x_j$ followed by $\xi \to 1/\xi$ for $J_{2k}^{\prime\prime(2)}(\varepsilon)$.

Comparing (3.9) with (2.6) we see immediately that

$$\begin{aligned} J_{2k}^{\prime(1)}(\varepsilon) &= \left(\int_\varepsilon^\infty \frac{d\xi}{\xi} e^{-\xi}\right) \tilde{J}_{2k}(\theta) \\ &= \left(\log\frac{1}{\varepsilon} - \gamma + o(1)\right) \tilde{J}_{2k}(\theta). \end{aligned} \tag{3.13}$$

A similar Mellin convolution calculation as in the exponent evaluation shows that

$$\begin{aligned} J_{2k}^{\prime(2)}(\varepsilon) &= \left(\int_{\varepsilon}^{\infty} \frac{d\xi}{\xi}\frac{1}{1+\xi}\right)\tilde{J}_{2k}(\theta) \\ &= \log\frac{1}{\varepsilon}\,\tilde{J}_{2k}(\theta) + o(1) \quad \text{as} \quad \varepsilon \to 0^{+}. \end{aligned} \tag{3.14}$$

Referring back to (2.5), we see that

$$\hat{e}_{-1,2k} = 2\left(\vartheta_{2k}'(0,\theta) + \vartheta_{2k}(0,\theta)(\log 2 - \gamma)\right).$$

By (2.6) and (3.3) we can rewrite $\hat{e}_{-1,2k}$ as

$$\hat{e}_{-1,2k} = -J_{2k}(\theta) + (2\log 2 - \gamma)\tilde{J}_{2k}(\theta),$$

and then using (3.8), (3.13), and (3.14) this becomes

$$\hat{e}_{-1,2k} = \lim_{\varepsilon\to 0+}\left[-2\log\frac{1}{\varepsilon}\,\tilde{J}_{2k}(\theta) + 2\log 2\,\tilde{J}_{2k}(\theta) + J_{2k}^{\prime\prime(1)}(\varepsilon) + J_{2k}^{\prime\prime(2)}(\varepsilon)\right].$$

Thus

$$\begin{aligned} E_0 &= \sum_{k=1}^{\infty}\frac{\lambda^{2k}}{k}e_{0,2k} \\ &= \lim_{\varepsilon\to 0+}\left[\frac{1}{2}(\log 2 + \log\varepsilon)(\sigma^2 - \theta^2) + \sum_{k=1}^{\infty}\frac{\lambda^{2k}}{k}\left(J_{2k}^{\prime\prime(1)}(\varepsilon) + J_{2k}^{\prime\prime(2)}(\varepsilon)\right)\right] \end{aligned} \tag{3.15}$$

where we used (2.8) and (2.9).

We now relate (3.10) and (3.12) to the Widom operator $K_{\alpha,\beta}$ (we henceforth use the α and β notation (1.8) to conform with Widom's notation). We begin by first recording the (trivial) identities:

$$\frac{1}{x+y} = \int_0^{\infty} e^{-\tau(x+y)}\,d\tau \tag{3.16}$$

and

$$\frac{e^{-(x+y)}}{x+y} = \int_1^{\infty} e^{-\tau(x+y)}\,d\tau. \tag{3.17}$$

In (3.12) we use identity (3.16) applied to the factors

$$(x_{2j-1} + x_{2j})^{-1},\ j = 1, 2, \ldots, k,$$

and identity (3.17) applied to the factors

$$e^{-(x_{2j}+x_{2j+1})}(x_{2j} + x_{2j+1})^{-1},\ j = 1, 2, \ldots, k-1.$$

Interchanging the order of the x_j integrations with the τ_j integrations (those integrations coming from applying the identities), one can explicitly carry out the x_j integrations. Defining

$$\psi_2(\tau;\beta,\xi) = \Gamma(1+2\beta)(\tau+\xi+1)^{-1-2\beta}, \tag{3.18}$$

recalling the definition of the Widom kernel $K_{\alpha,\beta}(x,y)$, and the parametrization (1.6), we find that

$$\lambda^{2k-2} J_{2k}^{\prime\prime(2)}(\varepsilon) = \int_0^{1/\varepsilon} \left(\psi_2(\cdot,\beta,\zeta), K_{\alpha,\beta}^{k-1}\psi_2(\cdot,-\beta,\zeta)\right) d\zeta \tag{3.19}$$

where $(f,g) = \int_0^\infty f(x)\bar{g}(x)\,dx$.

Recalling that

$$\int_0^\infty x^{\nu-1}e^{-\beta/x-\gamma x}\,dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma})$$

($\Re\beta > 0, \Re\gamma > 0$) where $K_\nu(z)$ is the modified Bessel function, we similarly find that

$$\lambda^{2k-2} J_{2k}^{\prime\prime(1)}(\varepsilon) = \int_\varepsilon^\infty \left(\psi_1(\cdot;-\beta,\xi), K_{\alpha,\beta}^{k-1}\psi_1(\cdot;\beta,\xi)\right) d\xi \tag{3.20}$$

where

$$\psi_1(\tau;\beta,\xi) = 2(1+\tau)^\beta K_{2\beta}(2\sqrt{\xi(1+\tau)}). \tag{3.21}$$

If we define

$$I_i(\varepsilon) = \sum_{k=1}^{\infty} \lambda^{2k-2} J_{2k}^{\prime\prime(i)}(\varepsilon),\ i = 1,2, \tag{3.22}$$

then

$$I_1(\varepsilon) = \int_\varepsilon^\infty \left(\psi_1(\cdot;-\beta,\xi), (I-K_{\alpha,\beta})^{-1}\psi_1(\cdot;\beta,\xi)\right) d\xi \tag{3.23}$$

and

$$I_2(\varepsilon) = \int_\varepsilon^\infty \frac{1}{\xi}\left(\psi_2(\cdot;\beta,\frac{1}{\xi}), (I-K_{\alpha,\beta})^{-1}\psi_2(\cdot;-\beta,\frac{1}{\xi})\right) d\xi. \tag{3.24}$$

Equations (3.23) and (3.24) express (except for the elementary terms appearing in (3.15)) the derivative of the constant E_0 with respect to λ (equivalently α) in terms of integrals of inner products involving the Widom resolvent.

4. Evaluation of $I_1(\varepsilon)$

We begin by introducing the Mellin transform of $I_1(\varepsilon)$:

$$
\begin{aligned}
\hat{I}_1(z) &= \int_0^\infty \varepsilon^{z-1} I_1(\varepsilon)\, d\varepsilon \\
&= \frac{1}{z}\int_0^\infty \xi^z \left(\psi_1(\cdot;-\beta,\xi),(I-K_{\alpha,\beta})^{-1}\psi_1(\cdot;\beta,\xi)\right)\, d\xi \\
&= \frac{4}{z}\int_0^\infty d\xi\, \xi^z \int_1^\infty dx\, x^{-\beta} K_{2\beta}(2\sqrt{\xi x}) \\
&\quad\times \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} ds\, x^{-s} \frac{\Gamma(1-s-\alpha-\beta)\Gamma(1-s+\alpha-\beta)}{\Gamma(1-s)\Gamma(1-2\beta-s)} \\
&\quad\times \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} dt\, \frac{\Gamma(t-\alpha+\beta)\Gamma(t+\alpha+\beta)}{\Gamma(t)\Gamma(t+2\beta)(t-s)} \int_1^\infty dy\, y^{t+\beta-1} K_{2\beta}(2\sqrt{\xi y})
\end{aligned}
$$

where we used (3.21), results from Appendix A (see (A.12)), and interchanged the order of integration. Furthermore, the s-contour is indented at $t=s$ so that $\Re s < \Re t$. Interchanging the order of integration and performing the ξ integration by using eqn (36) in §7.14.2 of Ref. 28, we get

$$
\begin{aligned}
\hat{I}_1(z) &= \frac{\Gamma^2(1+z)\Gamma(1+2\beta+z)\Gamma(1-2\beta+z)}{z\Gamma(2+2z)} \\
&\quad\times \int_1^\infty dx\, \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} ds\, x^{-2\beta-s-z-1} \frac{\Gamma(1-s-\alpha-\beta)\Gamma(1-s+\alpha-\beta)}{\Gamma(1-s)\Gamma(1-2\beta-s)} \\
&\quad\times \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} dt\, \frac{\Gamma(t-\alpha+\beta)\Gamma(t+\alpha+\beta)}{\Gamma(t)\Gamma(t+2\beta)(t-s)} \\
&\quad\times \int_1^\infty dy\, y^{t+2\beta-1}\, {}_2F_1(1+2\beta+z, 1+z; 2+2z; 1-\frac{y}{x}).
\end{aligned}
$$

Interchanging orders of integration and making the change of variables $v=x$, and $u=-1+y/x$, we obtain

$$
\begin{aligned}
\hat{I}_1(z) &= \frac{\Gamma^2(1+z)\Gamma(1+2\beta+z)\Gamma(1-2\beta+z)}{z\Gamma(2+2z)} \\
&\quad\times \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} ds\, \frac{\Gamma(1-s-\alpha-\beta)\Gamma(1-s+\alpha-\beta)}{\Gamma(1-s)\Gamma(1-2\beta-s)} \\
&\quad\times \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} dt\, \frac{\Gamma(t-\alpha+\beta)\Gamma(t+\alpha+\beta)}{\Gamma(t)\Gamma(t+2\beta)(t-s)} \\
&\quad\times \int_{-1}^\infty du\, (1+u)^{t+2\beta-1}\, {}_2F_1(1+2\beta+z, 1+z; 2+2z; -u) \\
&\quad\times \int_{1/(u+1)}^\infty dv\, v^{-s-z+t-1}.
\end{aligned}
$$

The v integration can now be done leaving

$$\hat{I}_1(z) = \frac{\Gamma^2(1+z)\Gamma(1+2\beta+z)\Gamma(1-2\beta+z)}{z\Gamma(2+2z)}$$
$$\times \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} ds\, \frac{\Gamma(1-s-\alpha-\beta)\Gamma(1-s+\alpha-\beta)}{\Gamma(1-s)\Gamma(1-2\beta-s)}$$
$$\times \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} dt\, \frac{\Gamma(t-\alpha+\beta)\Gamma(t+\alpha+\beta)}{\Gamma(t)\Gamma(t+2\beta)(t-s)(s+z-t)}$$
$$\times \int_{-1}^{\infty} du\,(u+1)^{s+z+2\beta-1}\, {}_2F_1(1+2\beta+z, 1+z; 2+2z; -u).$$

We break the u-integration into intervals $(-1,0)$ and $(0,\infty)$. For the interval $(-1,0)$ we use the (generalized) Euler transform (after letting $u \to -u$)

$$ {}_3F_2(c, a_1, a_2; d, b_1; z) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_2F_1(a_1, a_2; b_1; tz)\,dt$$

and for the interval $(0,\infty)$ we make use of the result (see, e.g., Ref. 29, pg. 315)

$$\int_0^{\infty} \frac{1}{(1+x)^{\rho}}\,{}_2F_1(a,b;c;-x)\,dx =$$
$$\frac{1}{\rho-1}\,{}_3F_2(a,b,1;c,2-\rho;1) + \frac{\Gamma(c)\Gamma(a-1+\rho)\Gamma(b-1+\rho)\Gamma(1-\rho)}{\Gamma(a)\Gamma(b)\Gamma(c-1+\rho)}$$
$$\times\, {}_3F_2(a-1+\rho, b-1+\rho, \rho; c-1+\rho, \rho; 1).$$

Combining these terms gives (note the cancellation and the fact that the second ${}_3F_2$ function reduces to ${}_2F_1$ evaluated at 1 which in turn is expressible in terms of gamma functions)

$$\hat{I}_1(z) = \frac{1}{z}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} ds\,\Gamma(1-s-\alpha-\beta)\Gamma(1-s+\alpha-\beta)\Gamma(z+2\beta+s)\Gamma(s+z)$$
$$\times \frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} dt\,\frac{\Gamma(t-\alpha+\beta)\Gamma(t+\alpha+\beta)}{\Gamma(t)\Gamma(t+2\beta)(t-s)(s+z-t)}.$$

We can now evaluate the t-integral by closing the contour in the right half plane $\Re t \geq 1/2$ where the integrand is holomorphic except for a simple pole at $t = s+z$. The result is

$$\hat{I}_1(z) = \frac{z^{-2}}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} \Gamma(1-s-\alpha-\beta)\Gamma(1-s+\alpha-\beta)\Gamma(s+z-\alpha+\beta)\Gamma(s+z+\alpha+\beta)\,ds.$$

This integral can be evaluated by appealing to Barnes' First Lemma:

$$\frac{1}{2\pi i}\int_C \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)\,ds = \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} \tag{4.1}$$

where the contour C separates the poles of $\Gamma(\alpha+s)\Gamma(\beta+s)$ from the poles of $\Gamma(\gamma-s)\Gamma(\delta-s)$. Doing this the final result is

$$\hat{I}_1(z) = \frac{\Gamma(1-2\alpha+z)\Gamma(1+2\alpha+z)\Gamma^2(1+z)}{z^2\Gamma(2+2z)}$$

$$= \frac{\Gamma(1-2\alpha)\Gamma(1+2\alpha)}{z^2} + \frac{\Gamma(1-2\alpha)\Gamma(1+2\alpha)}{z}\left[\psi(1-2\alpha)+\psi(1+2\alpha)-2\right]$$

$$+O(1) \quad \text{as} \quad z \to 0$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$. This implies that

$$I_1(\varepsilon) = \frac{2\pi\alpha}{\sin 2\pi\alpha}\log\frac{1}{\varepsilon} + \frac{2\pi\alpha}{\sin 2\pi\alpha}\left(\psi(1-2\alpha)+\psi(1+2\alpha)-2\right) + o(1) \tag{4.2}$$

as $\varepsilon \to 0^+$ (recall that $\Gamma(1+z)\Gamma(1-z) = \pi z/\sin\pi z$).

5. Evaluation of $I_2(\varepsilon)$

Using (A.5), (3.18), and standard integrals we have

$$G(s) = \frac{\Gamma(1-2\beta)}{1-2\beta-s}\, {}_2F_1(1-2\beta, 1-2\beta-s; 2-2\beta-s, -\xi)$$

$$= \sum_{k\geq 0} \frac{\Gamma(1-2\beta+k)}{1-2\beta+k-s}\frac{1}{k!}(-\xi)^k, \qquad |\xi| < 1.$$

This $G(s)$ should not be confused with the Barnes G-function. $G(s)$ has poles at $s = 1-2\beta+k$, $k = 0, 1, 2, \ldots$ so that

$$G_-(s) = \frac{\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)\Gamma(1-2\beta-s)}{\Gamma(2-2\beta-s)}$$

$$\times\, {}_3F_2(1-\alpha-\beta, 1+\alpha-\beta, 1-2\beta-s; 1, 2-2\beta-s, -\xi)$$

as follows from (A.9). Hence (A.11) in this case becomes

$$F_-(s) = \frac{\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)\Gamma(1-s-\alpha-\beta)\Gamma(1+\alpha-\beta-s)}{\Gamma(1-s)\Gamma(2-2\beta-s)}$$

$$\times\, {}_3F_2(1-\alpha-\beta, 1+\alpha-\beta, 1-2\beta-s; 1, 2-2\beta-s; -\xi).$$

Thus the inner product becomes

$$\Gamma(1+2\beta)\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)$$

$$\times \int_0^\infty dx\,(x+\xi+1)^{-1-2\beta}\frac{1}{2\pi i}\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} ds\,\frac{\Gamma(1-s-\alpha-\beta)\Gamma(1-s+\alpha-\beta)}{\Gamma(1-s)\Gamma(1-2\beta-s)}$$

$$\times (x+1)^{-s}\frac{1}{1-2\beta-s}\, {}_3F_2(1-\alpha-\beta, 1+\alpha-\beta, 1-2\beta-s; 1, 2-2\beta-s; -\xi).$$

Interchanging the order of the x and s integrations, integrating over x results in the inner product being equal to

$$\Gamma(1+2\beta)\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)\int_{\frac{1}{2}-\beta-i\infty}^{\frac{1}{2}-\beta+i\infty} ds\,\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)$$
$$\times\frac{1}{1-2\beta-s}\,{}_3F_2(1-\alpha-\beta,1+\alpha-\beta,1-2\beta-s;1,2-2\beta-s;-\xi)$$
$$\times\frac{\Gamma(1-s-\alpha-\beta)\Gamma(1-s+\alpha-\beta)}{\Gamma(1-s)\Gamma(1-2\beta-s)}\frac{1}{2\beta+s}\,{}_2F_1(1+2\beta,2\beta+s;1+2\beta+s;-\xi).$$

Now

$$\frac{1}{1-2\beta-s}\,{}_3F_2(1-\alpha-\beta,1+\alpha-\beta,1-2\beta-s;1,2-2\beta-s;-\xi)$$

has simple poles at $s=1-2\beta+n$, $n=0,1,\ldots$ which are removed in the integrand by the factor $1/\Gamma(1-2\beta-s)$. Similarly,

$$\frac{1}{2\beta+s}\,{}_2F_1(1+2\beta,2\beta+2;1+2\beta+s;-\xi)$$

has simple poles at $s=-2\beta-n$, $n=0,1,\ldots$. The integration over s can be done by closing the contour in the left half plane $\Re s\le 1/2$ with the result that the inner product is

$$\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)\sum_{k=0}^{\infty}\frac{\Gamma(1-\alpha+\beta+k)\Gamma(1+\alpha+\beta+k)}{\Gamma(2+k)\Gamma(1+k)}$$
$$\times(-\xi)^k\,{}_3F_2(1-\alpha-\beta,1+\alpha-\beta,1+k;1,2+k;-\xi).$$

Now

$${}_3F_2(1-\alpha-\beta,1+\alpha-\beta,1+k;1,2+k;-\xi)=$$
$$\frac{\Gamma(2+k)}{\Gamma(1+k)\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)}\sum_{j=0}^{\infty}\frac{\Gamma(1-\alpha-\beta+j)\Gamma(1+\alpha-\beta+j)}{\Gamma^2(j+1)(1+k+j)}(-\xi)^j=$$
$$\int_0^1 dx\,\frac{\Gamma(2+k)}{\Gamma(1+k)\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)}$$
$$\times\sum_{j=0}^{\infty}\frac{\Gamma(1-\alpha-\beta+j)\Gamma(1+\alpha-\beta+j)}{\Gamma^2(j+1)}(-\xi)^j x^{k+j}.$$

The inner product thus reduces to

$$\int_0^1\sum_{k=0}^{\infty}\frac{\Gamma(1-\alpha+\beta+k)\Gamma(1+\alpha+\beta+k)}{\Gamma^2(1+k)}(-\xi x)^k$$
$$\times\sum_{j=0}^{\infty}\frac{\Gamma(1-\alpha-\beta+j)\Gamma(1+\alpha-\beta+j)}{\Gamma^2(1+j)}(-\xi x)^j$$
$$=c_1\int_0^1 {}_2F_1(1-\alpha+\beta,1+\alpha+\beta;1,-\xi x)\,{}_2F_1(1-\alpha-\beta,1+\alpha-\beta;1;-\xi x)\,dx,$$

where $c_1 = \Gamma(1+\alpha+\beta)\Gamma(1-\alpha-\beta)\Gamma(1+\alpha-\beta)\Gamma(1-\alpha+\beta)$, and hence

$$\begin{aligned} I_2(\varepsilon) = c_1 \int_0^{1/\varepsilon} d\xi \int_0^1 dx\ {}_2F_1(1-\alpha+\beta, 1+\alpha+\beta; 1; -\xi x) \\ \times {}_2F_1(1+\alpha-\beta, 1-\alpha-\beta; 1; -\xi x). \end{aligned}$$

Now let $\xi x \to x$ so that

$$I_2(\varepsilon) = c_1 \int_0^{1/\varepsilon} \frac{d\xi}{\xi} \int_0^{\xi} dx\ {}_2F_1(1-\alpha+\beta, 1+\alpha+\beta; 1; -x)\,{}_2F_1(1+\alpha-\beta, 1-\alpha-\beta; 1; -x).$$

Integrating by parts gives

$$\begin{aligned} I_2(\varepsilon) = {} & c_1 \log\frac{1}{\varepsilon} \int_0^{1/\varepsilon} {}_2F_1(1-\alpha+\beta, 1+\alpha+\beta; 1; -x) \\ & \times {}_2F_1(1+\alpha-\beta, 1-\alpha-\beta; 1; -x)\, dx \\ & - c_1 \int_0^{1/\varepsilon} \log x\ {}_2F_1(1-\alpha+\beta, 1+\alpha+\beta; 1; -x) \\ & \times {}_2F_1(1+\alpha-\beta, 1-\alpha-\beta; 1; -x)\, dx. \end{aligned}$$

For x large the hypergeometric functions are at most $|x|^{-1\pm\alpha-\beta}$ and $|x|^{-1\pm\alpha+\beta}$ so the product is at most $|x|^{-2\pm\alpha}$. If $|\alpha| < 1/2$, this is $x^{-(1+\delta)}$, $0 < \delta < 1$ so we can replace the integral from 0 to $1/\varepsilon$ by the integral from 0 to ∞ and the error is $o(1)$. Thus

$$\begin{aligned} I_2(\varepsilon) = {} & \log\frac{1}{\varepsilon}\, c_1 \int_0^{\infty} {}_2F_1(1-\alpha+\beta, 1+\alpha+\beta; 1; -x) \\ & \times {}_2F_1(1+\alpha-\beta, 1-\alpha-\beta; 1; -x)\, dx \\ & - c_1 \int_0^{\infty} \log x\ {}_2F_1(1-\alpha+\beta, 1+\alpha+\beta; 1, -x) \\ & \times {}_2F_1(1+\alpha-\beta, 1-\alpha-\beta; 1, -x)\, dx \\ & + o(1) \end{aligned} \tag{5.1}$$

as $\varepsilon \to 0^+$.

We now evaluate the integrals involving the hypergeometric functions. To this end we define

$$I(s) = \int_0^{\infty} x^{s-1}\,{}_2F_1(1-\alpha+\beta, 1+\alpha+\beta; 1; -x)\,{}_2F_1(1+\alpha-\beta, 1-\alpha-\beta; 1, -x)\, dx \tag{5.2}$$

(we want $I(1)$ and $I'(1)$). We begin by first recalling

$$\int_0^{\infty} x^{s-1}\ {}_2F_1(a, b; c; -x)\, dx = \frac{\Gamma(c)\Gamma(a-s)\Gamma(b-s)\Gamma(s)}{\Gamma(a)\Gamma(b)\Gamma(c-s)}, \tag{5.3}$$

and the inverse transform

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\ {}_2F_1(a, b; c; -x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} \Gamma(-s) x^s\, ds, \tag{5.4}$$

where the contour separates the poles of $\Gamma(a+s)\Gamma(b+s)$ from the poles of $\Gamma(-s)$. We now substitute (5.4) for the second hypergeometric function appearing in (5.2), interchange orders of integration, and use (5.3) to obtain

$$I(s) = \frac{1}{c_1}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(1-\alpha-\beta+t)\Gamma(1+\alpha-\beta+t)\Gamma(s+t)}{\Gamma(1+t)\Gamma(1-s-t)} \times \Gamma(1-\alpha+\beta-s-t)\Gamma(1+\alpha+\beta-s-t)\Gamma(-t)\,dt. \tag{5.5}$$

At $s=1$ this reduces to

$$I(1) = \frac{1}{c_1}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(1-\alpha-\beta+t)\Gamma(1+\alpha-\beta+t)\Gamma(-\alpha+\beta-t)\Gamma(\alpha+\beta-t)\,dt = \frac{\Gamma(1-2\alpha)\Gamma(1+2\alpha)}{c_1} \tag{5.6}$$

by Barnes' First Lemma.

To obtain $I'(1)$ we expand the integrand of (5.5) in s about the point $s=1$. The result is that

$$I'(1) = \frac{1}{c_1}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(1-\alpha-\beta+t)\Gamma(1+\alpha-\beta+t)\Gamma(-\alpha+\beta-t)\Gamma(\alpha+\beta-t) \times\left[-\psi(-\alpha+\beta-t)-\psi(\alpha+\beta-t)+\psi(1+t)+\psi(-t)\right]dt. \tag{5.7}$$

Consider

$$\begin{aligned} A &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(1-\alpha-\beta+t)\Gamma(1+\alpha-\beta+t)\Gamma(-\alpha+\beta-t)\Gamma(\alpha+\beta-t) \times\psi(-\alpha+\beta-t)\,dt \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(1-\alpha-\beta+t)\Gamma(1+\alpha-\beta+t)\Gamma(\alpha+\beta-t)\Gamma'(-\alpha+\beta-t)\,dt. \end{aligned}$$

This last integral can be evaluated by differentiating Barnes' First Lemma with respect to one of the parameters, say δ appearing in (4.1). Doing this and simplifying the result gives

$$A = \frac{\pi}{\sin 2\pi\alpha}\left(2\alpha\psi(1-2\alpha)-2\alpha\right). \tag{5.8}$$

This same formula for the differentiated Barnes' lemma gives

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(1-\alpha-\beta+t)\Gamma(1+\alpha-\beta+t)\Gamma(-\alpha+\beta-t)\Gamma'(\alpha+\beta-t)\,dt = \frac{\pi}{\sin 2\pi\alpha}\left(2\alpha\psi(1+2\alpha)-2\alpha\right). \tag{5.9}$$

The remaining integral is (use $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$)

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\pi^2}{\sin\pi(\alpha+\beta-t)\sin\pi((-\alpha+\beta-t)}\left(\psi(1+t)+\psi(-t)\right).$$

If one integrates

$$\frac{\pi^2 z}{\sin\pi(\alpha+\beta-z)\sin\pi(-\alpha+\beta-z)}\left(\psi(1+z)+\psi(-z)\right)$$

on the rectangular contour with vertices iN, $iN+1$, $-iN+1$, and $-iN$, then in the limit $N\to\infty$ one finds that the remaining integral is

$$\begin{aligned}&\frac{\pi}{\sin 2\pi\alpha}\left[(\alpha+\beta)\left(\psi(1+\alpha+\beta)+\psi(1-\alpha-\beta)\right)\right.\\&\quad\left.+(\alpha-\beta)\left(\psi(1-\alpha+\beta)+\psi(1+\alpha-\beta)\right)-4\alpha\right].\end{aligned}\tag{5.11}$$

Combining (5.8), (5.9), and (5.11) gives $I'(1)$ which along with the expression for $I(1)$ gives our final expression for $I_2(\varepsilon)$:

$$\begin{aligned}I_2(\varepsilon)&=\frac{2\pi\alpha}{\sin 2\pi\alpha}\log\frac{1}{\varepsilon}+\frac{\pi}{\sin 2\pi\alpha}\left[2\alpha\psi(1-2\alpha)+2\alpha\psi(1+2\alpha)-(\alpha+\beta)\right.\\&\quad\times\left(\psi(1+\alpha+\beta)+\psi(1-\alpha-\beta)\right)-(\alpha-\beta)\left(\psi(1-\alpha+\beta)+\psi(1+\alpha-\beta)\right)\Big]\\&\quad+o(1)\end{aligned}\tag{5.12}$$

as $\varepsilon\to 0^+$.

6. Evaluation of E_0 in terms of Barnes G-function

Referring to (3.15) and (3.23) we see that we must integrate the sum $I_1(\varepsilon)+I_2(\varepsilon)$ with respect to the parameter λ. The coefficient of the $\log\varepsilon$ term of this sum is $-4\pi\alpha/\sin 2\pi\alpha$. If we call A the coefficient of the $\log\varepsilon$ term of the infinite series appearing in (3.15), then an elementary calculation (very similar to the one occurring before (2.9)) shows that $A=-2\alpha^2+2\beta^2$. Thus the $\log\varepsilon$ terms in (3.15) cancel, as they must. From (3.15) we therefore have

$$E_0=2(\alpha^2-\beta^2)\log 2+E_0'$$

where E_0' is the contribution from the infinite series. From (4.2) and (5.12) we know $\partial E_0'/\partial\lambda$, and thus by a change of variables we know $\partial E_0'/\partial\alpha$. Integrating with respect to the parameter α we get (when $\lambda=0$, $E_0'=0$ and $\alpha=\beta$)

$$\begin{aligned}E_0'=\int_\beta^\alpha&\Big[-4x+4x\psi(1-2x)+4x\psi(1+2x)\\&-(x+\beta)\left(\psi(1+x+\beta)+\psi(1-x-\beta)\right)\\&-(x-\beta)\left(\psi(1-x+\beta)+\psi(1+x-\beta)\right)\Big]\,dx\end{aligned}\tag{6.1}$$

The apparently difficult integrals are those involving a ψ-function times the variable x. It is here that the Barnes G-function enters the problem. Alexeiewsky's integral[8] is

$$\int_0^z \log\Gamma(x+a)\,dx = \frac{z}{2}\log 2\pi - \frac{z}{2}(z+2a-1) + (z+a-1)\log\Gamma(z+a) - \log\frac{G(z+a)}{G(a)} - (a-1)\log\Gamma(a),$$

from which one can easily derive

$$\int_\beta^\alpha x\psi(x+a)\,dx = -\frac{\alpha-\beta}{2}\log 2\pi + \frac{1}{2}(\alpha^2-\beta^2) + \frac{1}{2}(\alpha-\beta)(2a-1) - (a-1)\log\frac{\Gamma(\alpha+a)}{\Gamma(\beta+a)} + \log\frac{G(\alpha+a)}{G(\beta+a)}. \tag{6.2}$$

Applying (6.2) and the integral obtained from (6.2) by letting $x \to -x$ and then $\alpha \to -\alpha$, $\beta \to -\beta$ to E_0', we find

$$E_0 = 2(\alpha^2-\beta^2)\log 2 - \log\frac{G(1+\alpha+\beta)G(1-\alpha-\beta)G(1-\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)G(1-2\alpha)},$$

from which Theorem 3 follows immediately.

7. Error Estimates

In this section we derive an integral representation for $e_{2k}(t;\theta)$ from which the assertions after eqn. (2.4) will follow. In the inverse Mellin transform of $e_{2k}(t;\theta)$, we deform the contour to the imaginary axis except for a small semicircle of radius ε centered at the origin and lying in the right half plane. From this we arrive at

$$e_{2k}(t;\theta) = \frac{\hat{e}_{-2,2k}}{\pi\varepsilon} + \frac{1}{2}(\hat{e}_{-2,2k}\log(1/t) + \hat{e}_{-1,2k}) + \frac{1}{\pi}\int_\varepsilon^\infty \cos(y\log t)\Re(\hat{e}_{2k}(iy;\theta))\,dy + \frac{1}{\pi}\int_\varepsilon^\infty \sin(y\log t)\Im(\hat{e}_{2k}(iy;\theta))\,dy. \tag{7.1}$$

Now

$$\frac{1}{\pi}\int_\varepsilon^\infty \frac{\hat{e}_{-2,2k}}{y^2}\cos(y\log t)\,dy + \frac{1}{\pi}\int_\varepsilon^\infty \frac{\hat{e}_{-1,2k}}{y}\sin(y\log t)\,dy = \frac{\hat{e}_{-2,2k}}{\pi\varepsilon} - \frac{1}{2}(\hat{e}_{-2,2k}\log(1/t) + \hat{e}_{-1,2k}) + o(1) \tag{7.2}$$

as $\varepsilon \to 0^+$ (since we take $\log t < 0$). Subtracting (7.2) from (7.1) and combining the integrals, we can then let $\varepsilon \to 0^+$ to obtain

$$e_{2k}(t;\theta) = \hat{e}_{-2,2k}\log(1/t) + \hat{e}_{-1,2k} + \frac{1}{\pi}\int_0^\infty \cos(y\log t)\left(\Re(\hat{e}_{2k}(iy;\theta)) + \frac{\hat{e}_{-2,2k}}{y^2}\right)dy + \frac{1}{\pi}\int_0^\infty \sin(y\log t)\left(\Im(\hat{e}_{2k}(iy;\theta)) + \frac{\hat{e}_{-1,2k}}{y}\right)dy \tag{7.3}$$

The Riemann-Lebesgue lemma shows that the integrals tend to zero as $t \to 0^+$ and hence $\hat{e}_{-2,2k} = e_{-1,2k}$ and $\hat{e}_{-1,2k} = e_{0,2k}$ as asserted earlier.

The argument that this $o(1)$ error term remains $o(1)$ when summed in (2.1) is essentially identical to the proof given in Ref. 7 for the $\theta = 0$ case. Thus we omit the details.

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Appendix A

The integral operator $K_{\alpha,\beta}$ was first introduced by Widom[11] where a Wiener-Hopf type factorization procedure for inverting $I - K_{\alpha,\beta}$ was also described. The evaluation of the integrals $I_1(\varepsilon)$ and $I_2(\varepsilon)$ was accomplished by use of this inversion formula. We include an account of the procedure for completeness sake. Recall that the Widom operator $K_{\alpha,\beta}$ is defined by (1.16) and (1.17).

Suppose that

$$(I - K_{\alpha,\beta})f = g, \quad f, g \in L_{2,\beta}. \tag{A.1}$$

Then (A.1) can be written as

$$f(x-1)-c_{\alpha,\beta}\int_1^\infty f(y-1)\,dy\int_0^\infty \frac{dz}{(z+x)^{1-2\beta}(z+y)^{1+2\beta}} = g(x-1), \quad x > 1. \tag{A.2}$$

Define

$$F_-(s) = \int_1^\infty x^{s-1}f(x-1)\,dx, \qquad F_+(s) = \int_0^1 x^{s-1}f(x-1)\,dx. \tag{A.3}$$

Notice that $F_-(s)$ $(F_+(s))$ belongs to H_2 of the half-plane $\Re s < 1/2-\beta$ $(\Re s > 1/2-\beta)$ and that $f(x)$ can be recovered from $F_-(s)$ using the inverse Mellin transform. We extend $g(x)$ to zero in the interval $(-1,0)$ and define $f(x-1)$ by (A.2) for $x \in (0,1)$. Thus (A.2) is valid now for $x > 0$.

Next, multiply both sides of (A.2) by x^{s-1}, $-\beta < \Re s < 1/2-\beta$, integrate over x between 0 and ∞, and interchange the x and y integrations. Using the definitions given in (A.3) we obtain

$$\left(1 - \frac{\sin\pi(\alpha+\beta)\sin\pi(\alpha-\beta)}{\sin\pi s\sin\pi(s+2\beta)}\right)F_-(s) + F_+(s) = G(s) \tag{A.4}$$

where

$$G(s) = \int_0^\infty (x+1)^{s-1}g(x)\,dx. \tag{A.5}$$

Since

$$1-\frac{\sin\pi(\alpha+\beta)\sin\pi(\alpha-\beta)}{\sin\pi s\sin\pi(s+2\beta)}=\frac{\sin\pi(s-\alpha+\beta)\sin\pi(s+\alpha+\beta)}{\sin\pi s\sin\pi(s+2\beta)}$$

$$=\frac{\Gamma(s)\Gamma(1-s)\Gamma(s+2\beta)\Gamma(1-s-2\beta)}{\Gamma(s-\alpha+\beta)\Gamma(s+\alpha+\beta)\Gamma(1-s+\alpha-\beta)\Gamma(1-s-\alpha-\beta)} \tag{A.6}$$

(A.4) can be rewritten as

$$\frac{\Gamma(1-s)\Gamma(1-s-2\beta)}{\Gamma(1-s+\alpha-\beta)\Gamma(1-s-\alpha-\beta)}F_-(s)+\frac{\Gamma(s+\alpha+\beta)\Gamma(s-\alpha+\beta)}{\Gamma(s)\Gamma(s+2\beta)}F_+(s)$$
$$=\frac{\Gamma(s+\alpha+\beta)\Gamma(s-\alpha+\beta)}{\Gamma(s)\Gamma(s+2\beta)}G(s). \tag{A.7}$$

Now set

$$\frac{\Gamma(s+\alpha+\beta)\Gamma(s-\alpha+\beta)}{\Gamma(s)\Gamma(s+2\beta)}G(s)=G_-(s)-G_+(s) \tag{A.8}$$

where $G_-(s)$ ($G_+(s)$) belongs to H_2 of the half-plane $\Re s<1/2-\beta$ ($\Re s>1/2-\beta$). The functions $G_\pm(s)$ can be described in integral form by

$$G_\pm(s)=\frac{1}{2\pi i}\int_{\frac12-\beta-i\infty}^{\frac12-\beta+i\infty}\frac{\Gamma(t-\alpha+\beta)\Gamma(t+\alpha+\beta)}{\Gamma(t)\Gamma(t+2\beta)}\frac{G(t)}{t-s}\,dt,\quad \Re s\gtrless\frac12-\beta. \tag{A.9}$$

Substituting (A.8) into (A.7), we have for $\Re s=1/2-\beta$

$$\frac{\Gamma(1-s)\Gamma(1-s-2\beta)}{\Gamma(1-s+\alpha-\beta)\Gamma(1-s-\alpha-\beta)}F_-(s)-G_-(s)=$$
$$-\frac{\Gamma(s+\alpha+\beta)\Gamma(s-\alpha+\beta)}{\Gamma(s)\Gamma(s+2\beta)}F_+(s)-G_+(s) \tag{A.10}$$

and since this implies both sides of (A.10) vanish,

$$F_-(s)=\frac{\Gamma(1-s+\alpha-\beta)\Gamma(1-s-\alpha-\beta)}{\Gamma(1-s)\Gamma(1-s-2\beta)}G_-(s) \tag{A.11}$$

and thus (using Mellin inversion);

$$f(x)=\frac{1}{2\pi i}\int_{\frac12-\beta-i\infty}^{\frac12-\beta+i\infty}\frac{\Gamma(1-s+\alpha-\beta)\Gamma(1-s-\alpha-\beta)}{\Gamma(1-s)\Gamma(1-s-2\beta)}G_-(s)(x+1)^{-s}\,ds. \tag{A.12}$$

Putting all of this together, we see that for finding $f=(I-K_{\alpha,\beta})^{-1}g$ we begin with $g(x)$, find $G(s)$, $G_-(s)$, $F_-(s)$ and then finally $f(x)$.

Appendix B

In this appendix we prove that the series (1.1) converges for $|\lambda| < \cos\pi\beta/\pi$ and for all $t > 0$. To begin we define the integral operator $T_\beta : L^2(0,\infty) \to L^2(0,\infty)$ by

$$T_\beta(x,y) = x^\beta e^{-t(x+1/x)/4} \left(\int_0^\infty \frac{e^{-t(z+1/z)/2}}{(x+z)(z+y)} z^{-2\beta}\, dz \right) y^\beta e^{-t(y+1/y)/4} \tag{B.1}$$

so that

$$e_{2k}(t;2\beta) = \operatorname{Tr}(T_\beta^k) \tag{B.2}$$

and

$$\tau(t;2\beta,\lambda) = \det(I - \lambda^2 T_\beta). \tag{B.3}$$

The operator T_β is Hilbert-Schmidt for all $t > 0$. Let $\lambda_0 \geq \lambda_1 \geq \dots$ denote its eigenvalues and ϕ_0, $\phi_1, \dots$ its respective normalized eigenfunctions. Let $f \in L^2(0,\infty)$, $f(x) \geq 0$, then

$$\begin{aligned}(f, T_\beta f) &= \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3\, f(x_1) x_1^\beta \frac{e^{-\frac{t}{4}(x_1+1/x_1)} e^{-\frac{t}{2}(x_2+1/x_2)}}{(x_1+x_2)(x_2+x_3)} \\ &\quad \times x_2^{-2\beta} e^{-\frac{t}{4}(x_3+1/x_3)} x_3^\beta f(x_3) \\ &\leq \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3\, f(x_1) x_1^\beta \frac{x_2^{-2\beta}}{(x_1+x_2)(x_2+x_3)} x_3^\beta f(x_3).\end{aligned} \tag{B.4}$$

Define

$$M(x_1,x_3) = x_1^\beta \left(\int_0^\infty \frac{x_2^{-2\beta}}{(x_1+x_2)(x_2+x_3)}\, dx_2 \right) x_3^\beta \tag{B.5}$$

and observe that $M(x_1,x_3) \geq 0$, $M(x_3,x_1) = M(x_1,x_3)$, and $M(\lambda x_1, \lambda x_3) = \lambda^{-1} M(x_1,x_3)$. We recall the generalized Hilbert inequality:

Theorem. Let $M(x,y)$ be nonnegative, symmetric, homogeneous of degree -1, and let

$$M_0 = \int_0^\infty M(x,1) x^{-\frac{1}{2}}\, dx.$$

Suppose that $M(x,1)x^{-\frac{1}{2}}$ decreases strictly with x and $M(1,y)y^{-\frac{1}{2}}$ decreases with y. Then

$$\int_0^\infty \int_0^\infty M(x,y) f(x) f(y)\, dx dy < M_0 \int_0^\infty f^2(x)\, dx.$$

The constant M_0 is best possible.

Referring to (B.5) and recalling that $\beta < \frac{1}{2}$, we see that both $M(x,1)x^{-\frac{1}{2}}$ and $M(1,y)y^{-\frac{1}{2}}$ are decreasing. In this case an elementary calculation gives

$$M_0 = \left(\frac{\pi}{\cos\pi\beta} \right)^2,$$

and thus

$$(f, T_\beta f) < \left(\frac{\pi}{\cos \pi\beta}\right)^2 \int_0^\infty f^2(x)\, dx. \tag{B.6}$$

Applying this to $\phi_0(x)$ and recalling that ϕ_0 is normalized we see that

$$\lambda_0 = \lambda_0(t, \beta) < \left(\frac{\pi}{\cos \pi\beta}\right)^2.$$

This holds for all $t > 0$ so

$$\sup_{t>0} \lambda_0(t, \beta) \le \left(\frac{\pi}{\cos \pi\beta}\right)^2. \tag{B.7}$$

The largest eigenvalue λ_0 satisfies for any $f \in L^2(0, \infty)$

$$B_f(t) \equiv \frac{(f, T_\beta f)}{(f, f)} \le \lambda_0(t, \beta). \tag{B.8}$$

We choose

$$f(x) = x^{-\frac{1}{2}} e^{-t(x+1/x)/4}, \tag{B.9}$$

then

$$(f, f) = 2K_0(t) \tag{B.10}$$

and

$$(f, T_\beta f) = \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3\, x_1^{\beta-\frac{1}{2}} e^{-\frac{t}{2}(x_1+1/x_1)} \frac{e^{-\frac{t}{2}(x_2+1/x_2)}}{(x_1+x_2)(x_2+x_3)} \times x_2^{-2\beta} x_3^{\beta-\frac{1}{2}} e^{-\frac{t}{2}(x_3+1/x_3)}. \tag{B.11}$$

We want $\sup_{t>0} B_f(t)$ which is determined by $\lim_{t\to 0^+} B_f(t)$ since $B_f(t)$ is a decreasing function of t.[c] To compute this limit we introduce the Mellin transform of (B.11):

$$2^z \Gamma(z) \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \frac{(x_1 x_3)^{\beta-\frac{1}{2}} x_2^{-2\beta}}{(x_1+x_2)(x_2+x_3)} \left(\sum_{i=1}^3 (x_i + 1/x_i)\right)^{-z}$$

which after the standard change of variables $x_j = \rho\delta_j$, and an integration over the variable ρ becomes

$$2^{z-1}\Gamma^2(z/2) \int_0^1 d\delta_1 \int_0^{1-\delta_1} d\delta_2 \frac{\delta_1^{\beta-\frac{1}{2}} \delta_2^{-2\beta} (1-\delta_1-\delta_2)^{\beta-\frac{1}{2}}}{(\delta_1+\delta_2)(1-\delta_1)} \Delta_3(\delta)^{-z/2}.$$

[c] One way to see the decreasing property is to make the change of variables $x_j = \rho\delta_j$, $j = 1, 2, 3$, $\delta_3 \equiv 1 - \delta_1 - \delta_2$ in (B.11), perform the ρ integration to find that the t-dependent part of of the integrand of $B_f(t)$ is $K_0(\gamma t)/K_0(t)$ where γ, which is some function of δ_1 and δ_2, is always greater than 1. This ratio of Bessel functions for $\gamma > 1$ is a decreasing function of t from which it follows that $B_f(t)$ is a decreasing function of t.

We need only determine the coefficient of the $\log(1/t)$ term of (B.11), and hence only the coefficient of the double pole at $z = 0$ of the above Mellin transform. This coefficient is

$$2\int_0^1 d\delta_1 \frac{\delta_1^{\beta-\frac{1}{2}}}{1-\delta_1}\int_0^{1-\delta_1} d\delta_2\, \frac{\delta_2^{-2\beta}(1-\delta_1-\delta_2)^{\beta-\frac{1}{2}}}{\delta_1+\delta_2}$$

which after the change of variables $\delta_2 = (1-\delta_1)u$ on the inner integral becomes

$$2\int_0^1 d\delta_1\, \delta_1^{\beta-\frac{3}{2}}(1-\delta_1)^{-\beta-\frac{1}{2}}\int_0^1 \frac{u^{-2\beta}(1-u)^{\beta-\frac{1}{2}}}{1-\frac{\delta_1-1}{\delta_1}u}\,du.$$

The inner integral is now Euler's formula for the hypergeometric function. Thus the coefficient is

$$\begin{aligned}
&2\frac{\Gamma(1-2\beta)\Gamma(\beta+\frac{1}{2})}{\Gamma(\frac{3}{2}-\beta)}\int_0^1 \delta_1^{\beta-\frac{3}{2}}(1-\delta_1)^{-\frac{1}{2}-\beta}\,{}_2F_1(1,1-2\beta;\frac{3}{2}-\beta;1-\frac{1}{\delta_1})\,d\delta_1\\
&=2\frac{\Gamma(1-2\beta)\Gamma(\beta+\frac{1}{2})}{\Gamma(\frac{3}{2}-\beta)}\int_0^1 x^{-\beta-\frac{1}{2}}(1-x)^{-\beta-\frac{1}{2}}\,{}_2F_1(\frac{1}{2}-\beta,1-2\beta;\frac{3}{2}-\beta;x)\,dx\\
&=2\frac{\Gamma(\frac{1}{2}+\beta)\Gamma^2(\frac{1}{2}-\beta)}{\Gamma(\frac{3}{2}-\beta)}\,{}_3F_2(\frac{1}{2}-\beta,\frac{1}{2}-\beta,1-2\beta;\frac{3}{2}-\beta,1-2\beta;1)\\
&=2\left(\Gamma(\frac{1}{2}+\beta)\Gamma(\frac{1}{2}-\beta)\right)^2\\
&=2\left(\frac{\pi}{\cos\pi\beta}\right)^2
\end{aligned}$$

Thus we have as $t \to 0^+$

$$(f, T_\beta f) = 2\left(\frac{\pi}{\cos\pi\beta}\right)^2\log(\frac{1}{t}) + O(1),$$

and, of course,

$$(f,f) = 2K_0(t) = 2\log(\frac{1}{t}) + O(1),$$

and hence

$$B_f(t) = \left(\frac{\pi}{\cos\pi\beta}\right)^2 + o(1).$$

Since $\sup_{t>0} B_f(t) = \lim_{t\to 0^+} B_f(t)$, we have

$$\left(\frac{\pi}{\cos\pi\beta}\right)^2 \le \sup_{t>0}\lambda_0(t,\beta),$$

which when combined with (B.7) gives

$$\sup_{t>0}\lambda_0(t,\beta) = \left(\frac{\pi}{\cos\pi\beta}\right)^2. \tag{B.12}$$

Referring to (2.1) and (B.2) we see that if the series appearing in (2.1) converges (which for t large enough and λ small enough it clearly does)

$$E(t;2\beta,\lambda)=\sum_{j\geq 0}\log(1-\lambda^2\lambda_j).$$

The above series can be expanded in a power series in λ provided $|\lambda_0\lambda^2|<1$. For $0\leq\frac{\pi}{\cos\pi\beta}|\lambda|<1$, we have

$$|\lambda_0(t,\beta)\lambda^2|\leq\sup_{t>0}\lambda_0(t,\beta)|\lambda|^2<1$$

by (B.12).

Appendix C

When $\beta=0$, eqn. (1.9) yields the formula

$$\tau_0(0,\lambda)=2^{-2\alpha^2}\frac{G(1+\alpha)^2G(1-\alpha)^2}{G(1+2\alpha)G(1-2\alpha)}. \tag{C.1}$$

However, in Ref. 7, the expression for $\tau_0(0,\lambda)$ is seen to be (use (2.2), (2.5), (3.12), and (4.7) in Ref. 7)

$$\tau_0(0,\lambda)=2^{-6\alpha^2}\frac{\cos\pi\alpha\, G(1/2)^4}{G(\frac{1}{2}+\alpha)^2G(\frac{1}{2}-\alpha)^2}. \tag{C.2}$$

We now show how these are equal. This is done through the use of the duplication formula for the Barnes G-function, a generalization of the duplication formula for the gamma function.[8] This formula is given by

$$G(\frac{1}{2}+x)^2=\frac{e^{3\zeta'(-1)}G(2x)(2\pi)^4}{(2\pi)^{1/2}2^{5/12}\Gamma(x)G(x)^2 2^{2x^2-2x}}. \tag{C.3}$$

Thus

$$\begin{aligned}G(\frac{1}{2}+x)^2G(\frac{1}{2}-x)^2&=\frac{e^{6\zeta'(-1)}G(2x)G(-2x)}{(2\pi)2^{5/6}\Gamma(x)\Gamma(-x)G(x)^2G(-x)^2 2^{4x^2}}\\&=\frac{e^{6\zeta'(-1)}G(1+2x)G(1-2x)\Gamma(x)^2\Gamma(-x)^2}{(2\pi)2^{5/6}\Gamma(x)\Gamma(-x)G(1+x)^2G(1-x)^2 2^{4x^2}\Gamma(2x)\Gamma(-2x)}\\&=\frac{e^{6\zeta'(-1)}G(1+2x)G(1-2x)2\pi x\sin 2\pi x}{(2\pi)2^{5/6}2^{4x^2}G(1-x)^2G(1+x)^2\pi x\sin\pi x}\\&=\frac{e^{6\zeta'(-1)}4G(1+2x)G(1-2x)\cos\pi x}{(2\pi)2^{5/6}2^{4x^2}G(1-x)^2G(1+x)^2}.\end{aligned} \tag{C.4}$$

Recall that $G(1/2)=e^{3\zeta'(-1)/2}\pi^{-1/4}2^{1/24}$, let $x=\alpha$ in (C.4) and substitute (C.4) and the value of $G(1/2)$ in (C.2). An elementary calculation then shows that the RHS of (C.2) is indeed the same as the RHS of (C.1).