Asymptotics of Level-Spacing Distributions for Random Matrices

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Asymptotic formulas for the probability of finding exactly n eigenvalues in an interval of length s, for large s and fixed n, are given for random matrices taken from the Gaussian ensembles ($\beta = 1, 2, 4$). These exact results are compared with the predictions of a continuum Coulomb gas model due to Dyson.

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The theory of random matrices as developed by Hsu, Wigner, Dyson, Mehta, and others found its first application to physics in the level-spacing fluctuations of the excitation spectra of various nuclei [1]. More recently, the various level-spacing distribution functions for these random matrix ensembles have been shown to be good models for the level-spacing fluctuations of the spectra for quantum chaotic systems [2]. Thus it is important to know precisely the theoretical predictions of the various random matrix ensembles. Of course, there is a great deal known concerning the theory of random matrices, see, e.g., [3–5], but even so the asymptotics of various level-spacing distribution functions have not been computed. It is the purpose of this Letter to give new asymptotics formulas for the level-spacing distributions $E_\beta(n;s)$ for fixed n and large s for the three ensembles β=1, 2, and 4.

Recall that $E_\beta(n;s)$ is the probability that an interval of length s contains exactly n eigenvalues where the matrix is chosen from the Gaussian ensembles $\beta = 1$ (Gaussian orthogonal ensemble), $\beta = 2$ (Gaussian unitary ensemble), $\beta = 4$ (Gaussian sympletic ensemble) or from the corresponding circular ensembles. These probabilities are all expressible in terms of either Fredholm determinants or derivatives of these determinants [3, 5]: Let K be the integral operator on $L^2([0,\pi s])$ with kernel $K(x,y) = \frac{1}{\pi}\sin(x-y)/(x-y)$, $K_\pm$ the integral operators with kernels $K(x,y) \pm K(x,-y)$ and $D(s;\lambda) = \det(1-\lambda K)$, $D_\pm(s;\lambda) = \det(1-\lambda K_\pm)$ the respective Fredholm determinants, $0 < \lambda \le 1$, then

$$E_2(n;s) = \frac{(-1)^n}{n!}\frac{\partial^n D(s;\lambda)}{\partial\lambda^n}\Big|_{\lambda=1}. \quad (1)$$

If we define the analogous quantities $E_\pm(n;s)$, then [3, 5] $E_1(0;s) = E_+(0;s)$,

$$E_+(n;s) = E_1(2n;s) + E_1(2n-1;s), \quad n > 0, \quad (2)$$

$$E_-(n;s) = E_1(2n;s) + E_1(2n+1;s), \quad n \ge 0, \quad (3)$$

and

$$E_4(n;s) = \tfrac{1}{2}[E_+(n;2s) + E_-(n;2s)], \quad n \ge 0. \quad (4)$$

Jimbo *et al.* [6] have shown that

$$D(s;\lambda) = \exp\left(\int_0^{\pi s} \frac{\sigma(x;\lambda)}{x}\,dx\right), \quad (5)$$

where $\sigma(x;\lambda)$ is the solution, holomorphic at the origin, to

$$(x\sigma'')^2 + 4(x\sigma' - \sigma)[x\sigma' - \sigma + (\sigma')^2] = 0, \quad (6)$$

satisfying the boundary condition $\sigma(x;\lambda) \sim -\frac{\lambda}{\pi}x$ as $x \to 0$ and $' = \frac{d}{dx}$. The $D_\pm(s;\lambda)$ are given by

$$\ln D_\pm(s;\lambda) = \frac{1}{2}\ln D(s;\lambda) \pm\frac{1}{2}\int_0^s\left(-\frac{d^2}{dx^2}\ln D(x;\lambda)\,dx\right)^{1/2}. \quad (7)$$

Introducing

$$\sigma_\pm(x;\lambda) = x\frac{d}{dx}\ln D_\pm(x/\pi;\lambda), \quad (8)$$

the differentiated version of (7) is

$$\left(\frac{\sigma_-(x;\lambda) - \sigma_+(x;\lambda)}{x}\right)^2 = -\frac{d}{dx}\frac{\sigma(x;\lambda)}{x}. \quad (9)$$

Since we want derivatives of the above Fredholm determinants, we introduce

$$\sigma_n(x) = \frac{\partial^n\sigma(x;\lambda)}{\partial\lambda^n}\Big|_{\lambda=1} \quad (10)$$

and the analogous quantities $\sigma_{\pm,n}(x)$.

Before presenting our results on the asymptotics of $E_\beta(n;s)$ for fixed $n > 0$ and large s, we briefly recall the situation for $n = 0$:

$$\ln D_{\pm}(s;1) = -\frac{1}{16}\pi^2 s^2 \mp \frac{1}{4}\pi s - \frac{1}{8}\ln \pi s \pm \frac{1}{4}\ln 2 + \frac{1}{12}\ln 2 + \frac{3}{2}\zeta'(-1) + o(1) \qquad (11)$$

as $s \to \infty$ where ζ is the Riemann zeta function. This result, along with a systematic procedure to calculate the correction terms, was first derived by Dyson [7]. Dyson's method is not entirely self-contained in that he must use a result from [8] to fix the constants $\pm\frac{1}{4}\ln 2$, and he uses a scaling argument (of the type discussed below) on a result from [9] to fix the constant $\frac{1}{12}\ln 2 + \frac{3}{2}\zeta'(-1)$. We note that even though the result [9] is rigorous, the scaling argument must be considered heuristic since it uses the statement of the theorem in a region where it has yet to be proved.

Although known to the experts, we would like to point out the sensitivity of these results to the parameter λ being set to 1. This dependence is best discussed in terms of the differential equation (6) where it is an instance of the general problem of *connection formulas*. In this context the problem is, given the small x boundary condition, to find asymptotic formulas as $x \to \infty$ where all constants not determined by a local analysis at ∞ are given as functions of the parameter λ. If we assume an asymptotic solution for large x of the form $\sigma(x) \sim ax^p$, then (6) implies either $p = 1$ or $p = 2$ and if $p = 2$ then necessarily $a = -\frac{1}{4}$. The connection problem for (6) has been studied by McCoy and Tang [10] who show that for $0 < \lambda < 1$ one has $\sigma(x,\lambda) = a(\lambda)x + b(\lambda) + o(1)$ as $x \to \infty$ with $a(\lambda) = \frac{1}{\pi}\ln(1-\lambda)$, $b(\lambda) = \frac{1}{2}a^2(\lambda)$. Since these formulas make no sense at $\lambda = 1$, it is reasonable to guess that $\sigma(x;1) \sim -\frac{1}{4}x^2$. This corresponds to just the first-order asymptotics in (11). Given this, and only this, it is a simple matter using (6) to compute recursively the correction terms to this leading asymptotic behavior:

$$\sigma(x;1) = -\frac{1}{4}x^2 - \frac{1}{4} + \sum_{n=1}^{\infty}\frac{c_{2n}}{x^{2n}} \qquad (12)$$

($c_2 = -\frac{1}{4}$, $c_4 = -\frac{5}{2}$, etc.). Using (12) in (5) and (7) one can efficiently generate the large s expansions for $D(s;1)$ and $D_{\pm}(s;1)$ except that neither the constant involving the zeta function nor the constants $\pm\frac{1}{4}\ln 2$ are determined. We mention that for $0 < \lambda < 1$ the asymptotics of $D(s;\lambda)$ are also completely known [10, 11].

We now state our results: First, for $n > 2$

$$\sigma_n(x) = -\frac{n!}{(2^3\pi)^{n/2}}\frac{\exp(nx)}{x^{n/2-1}}\left[1 + \frac{1}{8}(7n-4)\frac{1}{x} + \frac{7}{128}(7n^2+12n-16)\frac{1}{x^2} + O\left(\frac{1}{x^3}\right)\right] \qquad (13)$$

as $x \to \infty$. For $n = 1, 2$ the above is correct for the leading behavior but for $n = 1$ the correction terms have coefficients $\frac{5}{8}$ and $\frac{65}{128}$, respectively, and for $n = 2$ the above formula gives the coefficient for $1/x$ but the coefficient for $1/x^2$ is $\frac{65}{32}$. Similarly, for $n > 1$ $\sigma_{+,n}(x)$ has the expansion beginning exactly as does the right-hand side of (13) and

$$\sigma_{-,n}(x) = -\frac{n!}{(2^7\pi)^{n/2}}\frac{\exp(nx)}{x^{3n/2-1}}\left[1 + \frac{1}{8}(19n-12)\frac{1}{x} + \frac{1}{128}(361n^2+204n-304)\frac{1}{x^2} + O\left(\frac{1}{x^3}\right)\right]. \qquad (14)$$

For $\sigma_{\pm,1}(x)$ the above formulas are correct except that the coefficients of the $1/x^2$ terms are $\frac{37}{128}$ and $\frac{309}{128}$, respectively. Introducing

$$r_\beta(n;s) = \frac{E_\beta(n;s)}{E_\beta(0;s)}, \quad \beta = 1, 2, 4, +, -,$$

we have

$$r_2(n;s) = B_{2,n}\frac{\exp(n\pi s)}{s^{n^2/2}}\left[1 + \frac{n}{8}(2n^2+7)\frac{1}{\pi s} + \frac{n^2}{128}(4n^4+48n^2+229)\frac{1}{(\pi s)^2} + O\left(\frac{1}{s^3}\right)\right], \qquad (15)$$

where

$$B_{2,n} = 2^{-n^2-n/2}\pi^{-(n^2+n)/2}(n-1)!\,(n-2)!\cdots 2!\,1!\,,$$

$$r_\pm(n;s) = B_{\pm,n}\frac{\exp(n\pi s)}{s^{n^2\mp n/2}}\left[1 + n\left(n^2 \mp \frac{3}{4}n + \frac{5}{8}\right)\frac{1}{\pi s} + \frac{n}{128}(64n^5 \mp 96n^4 + 276n^3 \mp 220n^2 + 297n \mp 116)\frac{1}{(\pi s)^2} + O\left(\frac{1}{s^3}\right)\right], \qquad (16)$$

with

$$B_{+,n} = \pi^{-n^2}2^{-2n^2+n/2}(2n-2)!\,(2n-4)!\cdots 2!\,0!\,, \quad B_{-,n} = \pi^{-n^2-n}2^{-2n^2-3n/2}(2n-1)!\,(2n-3)!\cdots 3!\,1!\,.$$

From (2)–(4) and (16) it follows that as $s \to \infty$

$$r_1(2n;s) \sim r_+(n;s), \quad r_1(2n+1;s) \sim r_-(n;s)\frac{E_-(0;s)}{E_+(0;s)}, \qquad (17)$$

and

$$r_4(n;s) \sim r_-(n;2s) \; . \tag{18}$$

Though (13) is proved for $n \in \mathbf{N}$, (14) and (15) have been proved only for $1 \le n \le 10$ and (16) for $1 \le n \le 9$. The difficulty in proving (14)–(16) for general n is due to the large amount of cancellation.

There is further support for the validity of these results for all n. Formula (5.4.30) of [3] expresses $r_2(n;s)$ as a sum of products involving the eigenvalues of K; there is a single dominant term in this sum, which corresponds to the n largest eigenvalues, and substituting Slepian's asymptotic formula for the eigenvalues [(12.1.15) in [3]] into this term gives the first-order asymptotics in (15). Although this is quick it would be difficult to make a proof out of it.

We now sketch our methods. Differentiating (6) with respect to λ and setting $\lambda = 1$, we obtain the first variational equation for $\sigma_1(x)$:

$$L_{\sigma_0}\sigma_1 = 0 \; , \tag{19}$$

where

$$L_\sigma = x^2\sigma'' \, \mathrm{D}^2 + [4x(x\sigma' - \sigma) + 6x(\sigma')^2 - 4\sigma\sigma'] \, \mathrm{D}$$
$$-2[(\sigma')^2 + 2(x\sigma' - \sigma)]$$

and $\mathrm{D} = \frac{d}{dx}$. Using (12) we find two linearly independent solutions to (19), $\varepsilon_\pm(x)$, which for large x behave as $\varepsilon_\pm(x) \sim \sqrt{x}e^{\pm x}$. Hence $\sigma_1(x) = c_1\varepsilon_+(x) + c_2\varepsilon_-(x)$, but the constants c_1 and c_2 *are not determined* from this analysis. Note this also shows, at least heuristically, that in "going beyond all orders" in (12) there will be terms proportional to $\varepsilon_-(x)$.

If we differentiate (6) n times with respect to λ and then set $\lambda = 1$, we obtain the nth variational equation for $\sigma_n(x)$:

$$L_{\sigma_0}\sigma_n(x) = f_n(x, \sigma_0, \ldots, \sigma_{n-1}) \; , \tag{20}$$

where one can easily write out a formula for f_n. By examining the small x expansions, one can show that the solution to (20) contains no homogeneous part. Thus we have a recursive set of equations for the $\sigma_n(x)$ with only two undetermined constants.

We now describe one method to determine the constant c_1. First observe that $(\partial/\partial\lambda) \ln D(s;\lambda) = -\operatorname{tr}(R_\lambda)$, where R_λ is the resolvent operator. We first introduce the discrete analogs of the operators K and R_1. As usual $T_n[\varphi]$ denotes a Toeplitz matrix of order $n+1$ with generating function φ and χ_A the characteristic function of set A. Think of $\alpha > 0$ as small but, for the moment, fixed; E_α is the major arc of the unit circle joining $e^{i\alpha}$ with $e^{-i\alpha}$ and e_α is the complementary minor arc (e_α contains the point 1). The matrix $T_n[\chi_{e_\alpha}](I - T_n[\chi_{e_\alpha}])^{-1}$ is the discrete analog of the resolvent operator $R_1 = K(I-K)^{-1}$. We claim that for fixed α and $n \to \infty$ we have

$$\operatorname{tr}(T_n[\chi_{e_\alpha}](I - T_n[\chi_{e_\alpha}])^{-1}) \sim \frac{\kappa^{2n}}{4\sqrt{\pi n(\kappa - 1)}}(1 + \epsilon_\alpha) \; , \tag{21}$$

where $\kappa = \tan(\alpha/2) + \sec(\alpha/2)$ and ϵ_α is a constant which approaches 0 as $\alpha \to 0$. Notice that the trace in question equals

$$\operatorname{tr}(T_n[\chi_{E_\alpha}]^{-1} - I) = \operatorname{tr}T_n[\chi_{E_\alpha}]^{-1} - n - 1 \; ,$$

so it suffices to show that $\operatorname{tr} T_n[\chi_{E_\alpha}]^{-1}$ has the claimed asymptotics. We will use the fact that this trace is representable in terms of the orthogonal polynomials on E_α. The reason for this is the following: A theorem of Gohberg and Fel'dman [12] tells us how to find the entries of $T_n[\varphi]^{-1}$ for general φ defined on the unit circle once we know the vectors $T_n[\tilde\varphi]^{-1}v_0$ and $T_n[\varphi]^{-1}v_0$, where v_0 is the first vector of a standard basis and $\tilde\varphi(z) = \varphi(z^{-1})$. If φ is non-negative the components of these vectors are the coefficients (and their complex conjugates, respectively) of the suitably normalized orthogonal polynomial $P_n(z)$ associated with the weight function φ. We deduce that if the highest coefficient of $P_n(z)$ is $(D_{n-1}[\varphi]/D_n[\varphi])^{1/2}$, where $D_n[\varphi] = \det T_n[\varphi]$, then

$$\operatorname{tr} T_n[\varphi]^{-1} = \frac{1}{2\pi}\int_0^{2\pi} \left[2zP_n'(z)\overline{P_n(z)} - (n-1)|P_n(z)|^2\right] d\theta \; . \tag{22}$$

In our case $\varphi = \chi_{E_\alpha}$ and it is well known that polynomials like this have moderate size on the support of the weight function and are much larger (when n is large) on the component of its complement which contains ∞. Thus the major contribution to the integral in (22) will come from the minor arc e_α. As our first leap of faith, we accept that the first-order asymptotics of (22) are obtained by using the first-order asymptotics of P_n valid off E_α and integrating the result over e_α. The asymptotic formula is

$$P_n(z) \sim C_n\Phi(z)^n\sqrt{\Phi'(z)} \; ,$$

where Φ maps the complement of E_α in the complex plane to the exterior of the unit circle, with $\infty \leftrightarrow \infty$, and where the constants C_n are adjusted to conform to the normalization of $P_n(z)$. The function $\Phi(z)$ can be found explicitly, the first-order asymptotics of D_{n-1}/D_n are known, and an asymptotic analysis of the integral (22) then gives (21).

The big assumption is, of course, that scaling makes sense, i.e., that if we formally set $\alpha n = \pi s$ in (21) and take the limit as $n \to \infty$, $\alpha \to 0$ with $\alpha n \gg 1$ we get a correct asymptotic formula for $\operatorname{tr} K(I-K)^{-1}$. Doing this gives

$$\operatorname{tr} K(I-K)^{-1} \sim \frac{e^{\pi s}}{2^{3/2}\pi\sqrt{s}} \, , \quad s \to \infty \; . \tag{23}$$

Observe this identifies the constant c_1 as $-1/(2^{3/2}\sqrt{\pi})$. Results for $n = 1$ follow from this and from (9). Statements about general n are proved by an analysis of the recursion relations arising from (20) and from differentiating (9) n times with respect to λ. When we could not prove a statement for general n, the recursion relations were programmed using the software MATHEMATICA.

As we pointed out, two assumptions were made in the above analysis. Rather than attempt to fill these gaps, one of us [13] has given a completely different analysis of the asymptotics of the trace of the resolvent working directly in the continuous setting. This analysis is completely rigorous and gives the result (23). Furthermore, this continuum approach also gives $\frac{d}{ds}\ln D(s;1) \sim -\pi^2 s/4$ which proves $\sigma(x;1) \sim -x^2/4$ as $x \to \infty$ thus providing a self-contained derivation (up to the constant involving the zeta function) of the asymptotics of $D(s;1)$ starting with (5) and (6). In addition, this approach also gives information in the many interval case, where N nonoverlapping intervals of lengths $s_1, \ldots, s_N$ are given and we seek the probability that exactly n_1 eigenvalues are in interval $1, \ldots, n_N$ eigenvalues are in interval N.

Upon communicating some of our results to Dyson, one of us received from Dyson a series of letters [14] in which he constructs a Coulomb gas model [3] for $E_\beta(n;s)$. In this continuum model,

$$E_\beta(n;s) = \exp[-\beta W - (1-\beta/2)S] ,$$

where $W = -\frac{1}{2}\int\int \hat{\rho}(x)\hat{\rho}(y)\,\ln|x-y|\,dx\,dy$ is the total energy, $S = \int \rho\ln\rho\,dx$ is the entropy, $\hat{\rho}(x) = \rho(x) - 1$, and $\rho(x)$ is a continuum charge distribution on the line satisfying $\rho(x) \to 1$ as $x \to \pm\infty$ and $\rho(x) \geq 0$ everywhere. The distribution $\rho(x)$ is chosen to minimize the free energy subject to the condition $\int_{-s/2}^{s/2}\rho(x)\,dx = n$. Analyzing his solution in the limit $1 \ll n \ll s$, Dyson finds $E_\beta(n;s) \sim \exp(-\beta W_c)$, where

$$W_c = \frac{\pi^2 s^2}{16} - \frac{\pi s}{2}(n+\delta) + \frac{1}{4}n(n+\delta) + \frac{1}{4}n(n+2\delta)\left[\ln\left(\frac{4\pi s}{n}\right) + \frac{1}{2}\right] , \tag{24}$$

with $\delta = 1/2 - 1/\beta$.

We now compare these predictions of the continuum model with the exact results. First of all, this continuum prediction does not get the $s^{-1/4}$ (for $\beta = 2$) or the $s^{-1/8}$ (for $\beta = 1,4$) present in all $E_\beta(n;s)$ that come from the $\ln\pi s$ term in (11). Thus it is better to compare with the continuum prediction for $r_\beta(n;s)$. We find that the continuum model gives both the correct exponential behavior *and* the correct power of s for all three ensembles. Thus the power of s involving the n^2 exponent is an *energy effect* and the power of s involving the n exponent is an *entropy effect*. Finally, the continuum model also makes a prediction (for large n) for the $B_{\beta,n}$'s. Here we find that the ratio of the exact result to the continuum model result is approximately $n^{-1/12}$ for $\beta = 2$ and $n^{-1/24}$ for $\beta = 1,4$. This prediction of the continuum model is better than it first appears when one considers that the constants themselves are of order $n^{n^2/2}$ ($\beta = 2$) and n^{n^2} ($\beta = 1,4$).

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