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Systems of Partial Differential Equations for a Class of Operator Determinants

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I. Introduction

We consider one-dimensional integral operators with kernel of the form

$$K(x,y) = \frac{1}{x-y} \sum_{\alpha,\beta=1}^{n} c_{\alpha,\beta}\, \varphi_\beta(x)\, \varphi_\alpha(y)$$

acting on functions on the union of intervals

$$J = \bigcup_{k=1}^{m} [a_{2k-1}, a_{2k}].$$

If the φ_α belong to C^1 and the matrix $(c_{\alpha,\beta})$ is antisymmetric then the kernel is symmetric and belongs to C^1, and the corresponding operator K is trace class. Modestly generalizing the principal result of [4], where n was equal to 2, we show here that if the functions φ_α satisfy a differential identity of a certain form then there is a system of partial differential equations, with the a_i as independent variables, associated with the determinant $\det(I-K)$. More exactly, the solution of these equations determine the logarithmic derivatives of the determinant with respect to the a_k.

Write C for the matrix $(c_{\alpha,\beta})_{\alpha,\beta=1}^n$, write $\varphi(x)$ for the vector function $(\varphi_\alpha(x))_{\alpha=1}^n$ and denote the inner product in $\mathbb{R}^n$ by $\langle\cdot,\cdot\rangle$. Then the kernel may be written

$$\frac{\langle C\varphi(x), \varphi(y)\rangle}{x-y}. \tag{1}$$

If $R(x,y)$ is the resolvent kernel for K, the kernel of the operator $K(I-K)^{-1}$, then it is an easy fact (see below) that

$$\frac{\partial}{\partial a_k} \log\det(I-K) = (-1)^{k+1} R(a_k,a_k). \tag{2}$$

We define the function $Q(x) = (Q_\alpha(x))_{\alpha=1}^n$ (which depends also on $a_1,\dots,a_{2m}$) by

$$Q := (I-K)^{-1}\varphi$$

(we restrict ourselves to (a_k) such that $I-K$ is invertible) and set

$$q_k := Q(a_k).$$

These vector functions of $a_1,\dots,a_{2m}$ will be the quantities of principal interest and will be among the dependent variables in our system of equations. We shall see that for any kernel of the form (1) we have the representation

$$R(a_k,a_k) = \langle C\,\frac{\partial q_k}{\partial a_k}, q_k\,\rangle, \tag{3}$$

so the q_k determine $R(a_k,a_k)$, and that

$$\frac{\partial q_j}{\partial a_k} = (-1)^k\,\frac{\langle\,C\,q_j, q_k\,\rangle}{a_j - a_k}\,q_k, \qquad (j \neq k), \tag{4}$$

which is the first part of our system of equations. To complete it we need formulas for the derivatives $\partial q_k/\partial a_k$, and this requires introduction of more dependent variables and the differential identity for $\varphi(x)$ alluded to above.

Given n-vectors $v = (v_\alpha)$ and $w = (w_\alpha)$ we denote by $v\otimes w$ the $n\times n$ matrix with α,β entry $v_\alpha\,w_\beta$. The extra dependent variables are the matrices

$$U_i := \int_J x^i\,Q(x)\otimes\varphi(x)\,dx.$$

We also define $Q_i := (I-K)^{-1}\,M^i\,\varphi$, where M denotes multiplication by x, and

$$q_{i,k} := Q_i(a_k).$$

These last are not new since, as we shall show,

$$q_{i+1,k} = a_k\,q_{i,k} + {U_i}^t\,C\,q_k, \tag{5}$$

(^{t} denotes transpose) and this determines the $q_{i,k}$ in terms of the U_i and $q_{0.k} = q_k$. The differentiation formula

$$\frac{\partial U_i}{\partial a_k} = (-1)^k\,q_k\otimes q_{i,k} \tag{6}$$

will be the second part of our system of equations.

To obtain formulas for $\partial q_k/\partial a_k$ in terms of the q_k and U_i we use the basic assumption on $\varphi(x)$. This assumption is that there exist a scalar polynomial $m(x)$ and a matrix $A(x)$ with polynomial entries connected with the matrix C by

$$A(x)^t\,C + C\,A(x) = 0 \tag{7}$$

such that

$$m(x)\,\varphi'(x) = A(x)\,\varphi(x). \tag{8}$$

Of course this is the same as taking $m(x)=1$ and $A(x)$ a rational matrix function, but our final equations are expressed most simply in terms of the coefficients of the polynomials $m(x)$ and $A(x)$. If

$$m(x) = \sum_{l\geq 0} m_l\,x^l, \qquad A(x) = \sum_{l\geq 0} A_l\,x^l, \tag{9}$$

then these equations are

$$m(a_k)\,\frac{\partial q_k}{\partial a_k} = A(a_k)\,q_k + \sum_{\substack{i+j=l-1\\ l\geq 1}} [U_j\,C\,(A_l-(j+1)\,m_{l+1}) - A_l\,U_j\,C]\,q_{i,k}$$

$$-\sum_{j\neq k}(-1)^j\,m(a_j)\frac{\langle\, C\,q_j, q_k\,\rangle}{a_j-a_k}\,q_j. \tag{10}$$

Thus, our unknown functions are the q_k and the U_i, and the complete system of equations consists of (4), (6) and (10). We see from the last that we need include as unknowns only the U_i with $i\leq \max(\deg A-1, \deg m-2)$. In particular, if A is constant and m is linear there need be no U_i at all.

II. Derivation of the System of Equations

We may assume until the derivation of (10) that the φ_α are C^1 functions defined on all of $\mathbb{R}$ and have compact support. We denote by χ the characteristic function of J and denote now by K the operator on $L_2(\mathbb{R})$ having kernel $K(x,y)\,\chi(y)$. The determinant $\det(I-K)$ is the same with this definition of K. Moreover since $K(x,y)$ is a C^1 function with compact support K is even a trace class operator on the Sobolev space $H_1(\mathbb{R})$. The mapping $(a_1,\ldots,a_{2m})\mapsto K$ from $\mathbb{R}^{2m}$ to the space of trace class operators on $H_1(\mathbb{R})$ is strongly differentiable and it is immediate that

$$\frac{\partial K}{\partial a_k} \doteq (-1)^k\,K(x,a_k)\,\delta(y-a_k), \tag{11}$$

where "$\doteq$" means "has kernel" and δ is the Dirac distribution. It follows from this that

$$\frac{\partial}{\partial a_k}\log\det(I-K) = -\mathrm{tr}\,(I-K)^{-1}\,\frac{\partial K}{\partial a_k} = (-1)^{k+1}\,R(a_k,a_k),$$

which is (2). (Note that now $R(x,y)$ is defined for all $x,y\in\mathbb{R}$ and is discontinuous in y. The quantity $R(a_k,a_k)$ denotes the limit of $R(a_k,y)$ as $y\to a_k$ from the interior of J. Similarly for other like quantities.)

We have

$$[M,K]\doteq \langle\, C\,\varphi(x), \varphi(y)\,\rangle\,\chi(y),$$

where the brackets denote commutator, from which it follows upon left and right multiplication by $(I-K)^{-1}$, and using the symmetry of $K(x,y)$, that

$$[M,(I-K)^{-1}] \doteq \langle\, C\,Q(x), Q(y)\,\rangle\,\chi(y). \tag{12}$$

($Q(x)$ is defined precisely as before, but now has domain $\mathbb{R}$.) Hence

$$R(x,y) = \frac{\langle\, C\,Q(x), Q(y)\,\rangle}{x-y}\,\chi(y), \qquad (x\neq y).$$

In particular we have

$$R(a_j,a_k) = \frac{\langle\, C\, q_j, q_k\,\rangle}{a_j - a_k}, \qquad (j \neq k) \tag{13}$$

and

$$R(a_k,a_k) = \langle\, C\, Q'(a_k), q_k\,\rangle. \tag{14}$$

Here the prime denotes differentiation with respect to x. Of course Q is a function of $a_1,\ldots,a_{2m}$ as well as x, and $\langle\, C\, Q(x), q_k\,\rangle$ vanishes identically when $x = a_k$ since C is antisymmetric. It follows that the last identity is equivalent to

$$R(a_k,a_k) = -\frac{\partial}{\partial a_k}\langle\, C\, Q(x), q_k\,\rangle\,|_{x=a_k}\,. \tag{15}$$

We define $\rho(x,y) := R(x,y) + \delta(x-y)$. This is the kernel of $(I-K)^{-1}$. It follows from (11) that

$$\frac{\partial}{\partial a_k}(I-K)^{-1} \doteq (-1)^k\, R(x,a_k)\,\rho(a_k,y),$$

and applying this to φ gives

$$\frac{\partial Q}{\partial a_k} = (-1)^k\, R(x,a_k)\, q_k. \tag{16}$$

Using this to evaluate the right side of (15) gives (3).

We write Q more explicitly as $Q(x,a_1,\ldots,a_{2m})$, displaying its dependence on the a_k. Thus $q_j = Q(a_j,a_1,\ldots,a_{2m})$. We see immediately that if $j \neq k$ then

$$\frac{\partial q_j}{\partial a_k} = \frac{\partial Q}{\partial a_k}\,|_{x=a_j}\,.$$

Hence (4) follows from (13) and (16). But

$$\frac{\partial q_k}{\partial a_k} = \frac{\partial Q}{\partial a_k}\,|_{x=a_k} + Q'(a_k) = (-1)^k\, R(a_k,a_k)\, q_k + Q'(a_k), \tag{17}$$

where the prime denotes differentiation with respect to x, and so we will have to find a formula for $Q'(x)$.

But first we derive identities (5) and (6). If we use the fact

$$\langle\, u,v\,\rangle\, w = (w \otimes v)\, u, \tag{18}$$

then we see that the former follows immediately upon applying both sides of the operator identity (12) to $M^i\varphi$ and setting $x = a_k$. For the latter, define

$T := \chi(I-K)^{-1}$, where χ denotes multiplication by $\chi(x)$. Then the product formula and (16) give

$$\begin{aligned}\frac{\partial T}{\partial a_k} &= (-1)^k\,(\delta(x-a_k)\,\rho(x-y) + \chi(x)\,R(x,a_k)\,\rho(a_k,y))\\ &= (-1)^k\,(\delta(x-a_k)\,\rho(a_k-y) + \chi(x)\,R(a_k,x)\,\rho(a_k,y))\\ &= (-1)^k\,\rho(a_k,x)\,\rho(a_k,y),\end{aligned}$$

where we have used the facts that $R(x,y)=R(y,x)$ for $x,y\in J$ and that $R(a_k,x)$ vanishes outside J. Applying φ to both sides, tensoring on the right with $M^i\varphi$ and integrating over $\mathbb{R}$ give (6).

Everything up to now applied to an arbitrary kernel of the form (1). Now we are going to use (8) and think of our operator K once again as acting on J, more precisely on $H_1(J)$. It is an easy fact that for an operator L with kernel $L(x,y)$ one has

$$[m\,D, L] \doteq \left(m(x)\frac{\partial}{\partial x} + m(y)\frac{\partial}{\partial y} + m'(y)\right)\,L(x,y).$$

(In the above, m denotes multiplication by $m(x)$ and $D=d/dx$.) We apply this to our operator K with kernel $K(x,y)\,\chi(y)$. From (8) we find that

$$\left(m(x)\frac{\partial}{\partial x} + m(y)\frac{\partial}{\partial y}\right)\,\langle\, C\,\varphi(x),\varphi(y)\,\rangle$$

$$= \langle\, C\,A(x),\varphi(x),\varphi(y)\,\rangle + \langle\, C\,\varphi(x),A(y)\,\varphi(y)\,\rangle = \langle\, C\,(A(x)-A(y))\,\varphi(x),\varphi(y)\,\rangle,$$

where we have used (7). Hence

$$\frac{1}{x-y}\,\left(m(x)\frac{\partial}{\partial x} + m(y)\frac{\partial}{\partial y}\right)\,\langle\, C\,\varphi(x),\varphi(y)\,\rangle$$

$$= \langle\, C\,\frac{A(x)-A(y)}{x-y}\,\varphi(x),\varphi(y)\,\rangle = \sum_{\substack{i+j=l-1\\ l\geq 1}} \langle\, C\,A_l\,x^i\varphi(x),y^j\,\varphi(y)\,\rangle.$$

(Recall the notation (9)). We also have

$$\left(m(x)\frac{\partial}{\partial x} + m(y)\frac{\partial}{\partial y} + m'(y)\right)\,\frac{1}{x-y}$$

$$= \frac{m(y)-m(x)}{(x-y)^2} + \frac{m'(y)}{x-y} = -\sum_{\substack{i+j=l-1\\ l\geq 1}} (j+1)\,m_{l+1}\,x^i\,y^j$$

and

$$\left(m(x)\frac{\partial}{\partial x} + m(y)\frac{\partial}{\partial y}\right)\,\chi(y) = -\sum_{j=1}^{2m}(-1)^j\,m(a_j)\,\delta(a_j-y).$$

Putting these things together gives

$$[m\,D, K] \doteq \sum_{\substack{i+j=l-1\\ l\geq 1}} \langle\, C\,(A_l - (j+1)m_{l+1})\, x^i\, \varphi(x), y^j\, \varphi(y)\,\rangle\, \chi(y)$$

$$-\sum_{j=1}^{2m} (-1)^j\, m(a_j)\, K(x,a_j)\, \delta(a_j - y),$$

and it follows from this in the usual way that

$$[m\,D, (I-K)^{-1}] \doteq \sum_{\substack{i+j=l-1\\ l\geq 1}} \langle\, C\,(A_l - (j+1)m_{l+1})\, Q_i(x), Q_j(y)\,\rangle\, \chi(y)$$

$$-\sum_{j=1}^{2m} (-1)^j\, m(a_j)\, R(x,a_j)\, \rho(a_j - y).$$

Applying this to φ, using (18) and the symmetry of $K(x,y)$, gives the identity

$$m(x)\,Q'(x) - (I-K)^{-1}\, m\,\varphi' = \sum_{\substack{i+j=l-1\\ l\geq 1}} U_j\, C(A_l - (j+1)m_{l+1}))\, Q_i(x)$$

$$-\sum_{j=1}^{2m} (-1)^j\, m(a_j)\, R(x,a_j)\, q_j. \tag{19}$$

Now by (8), $(I-K)^{-1}\, m\,\varphi' = (I-K)^{-1}\, A\,\varphi$, and so we compute commutators with (multiplication by) A:

$$[A,K] \doteq \frac{A(x) - A(y)}{x-y} \langle\, C\,\varphi(x), \varphi(y)\,\rangle\, \chi(y) = \sum_{\substack{i+j=l-1\\ l\geq 1}} A_l\, \langle\, C\,x^i\, \varphi(x), y^j\, \varphi(y)\,\rangle\, \chi(y),$$

whence

$$[A,(I-K)^{-1}] = \sum_{\substack{i+j=l-1\\ l\geq 1}} A_l\, \langle\, C\,Q_i(x), Q_j(y)\,\rangle\, \chi(y),$$

and this gives

$$(I-K)^{-1} m\,\varphi' = A(x)\, Q(x) - \sum_{\substack{i+j=l-1\\ l\geq 1}} A_l\, U_j\, C\, Q_i(x). \tag{20}$$

Combining (19) and (20) we obtain

$$m(x)\,Q'(x) \;=\; A(x)\,Q(x) + \sum_{\substack{i+j=l-1\\ l\geq 1}} [U_j\,C(A_l - (j+1)m_{l+1}) - A_l\,U_j\,C]\,Q_i(x)$$
$$- \sum_{j=1}^{2m} (-1)^j\, m(a_j)\, R(x,a_j)\, q_j .$$

Equation (10) follows from this, (17) and (13).

III. Remarks

Formula (14) for general kernel of the form (1) is by no means new with us. See, for example, Sec. II of [1]. In this reference, which treats certain kernels which have some features in common with ours, the matrix we would call U_0 also plays an important role.

In [4], where $n = 2$, our matrix C was $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, our vector $\varphi(x)$ was denoted by $(\varphi(x), \psi(x))$, our matrix A by $\begin{pmatrix} A & B \\ -C & -A \end{pmatrix}$ and our matrix U_i by $\begin{pmatrix} u_i & \tilde{v}_i \\ v_i & w_i \end{pmatrix}$. The derivation of the equations given here may seem at first sight simpler than the derivation in [4]. In fact the derivations are essentially the same, but because of its generality we were naturally led to use matrix notation here, and this led to an apparent simplification (and a real compactness of notation).

The simplest case is that of the "sine kernel"

$$\lambda \frac{\sin{(x-y)}}{x-y},$$

when $\varphi(x) = \sqrt{\lambda}\, e^{ix}$ and $\psi(x) = \sqrt{\lambda}\, e^{-ix}$. The equations in this case were first derived in [2], where they were obtained as the deformation equations for an isomonodromy problem. It was also shown there that if J is a single interval the determinant can be expressed in terms of a Painlevé transcendent. In [4] several other kernels were studied in detail, some of which also led to Painlevé functions. Recently Palmer [3] showed that all the equations considered in [4] were deformation equations for isomonodromy problems. It may very well be that the same is true of the more general class of equations considered here.

References

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