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Proofs of Two Conjectures Related to the Thermodynamic Bethe Ansatz

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Abstract: We prove that the solution to a pair of nonlinear integral equations arising in the thermodynamic Bethe Ansatz can be expressed in terms of the resolvent kernel of the linear integral operator with kernel

$$\frac{e^{-(u(\theta)+u(\theta'))}}{\cosh\frac{\theta-\theta'}{2}}\,.$$

I. Introduction

Thermodynamic Bethe Ansatz techniques were introduced in the pioneering analysis of Yang and Yang [11] of the thermodyamics of a nonrelativistic, one-dimensional Bose gas with delta function interaction. Later this method was extended to a relativistic system with a factorizable S-matrix to give an exact expression for the ground state energy of this system on a cylindrical space of circumference R [5, 12]. This was done by relating the ground state energy to the free energy of the same system on an infinite line at temperature $T = 1/R$. In all cases one expresses the various quantities of interest in terms of "excitation energies" $\varepsilon_a(\theta)$ which are solutions of nonlinear integral equations of the form

$$\varepsilon_a(\theta) = u_a(\theta) - \sum_b \int \phi_{ab}(\theta-\theta')\log(1+z_b\,e^{-\varepsilon_b(\theta')})\,\frac{d\theta'}{2\pi} \quad (a=1,2,\ldots)\,,$$

where $\phi_{ab}(\theta)$ are expressible in terms of the 2-body S-matrix, z_a are activities, and for relativistic systems $u_a(\theta) = m_a R\cosh\theta$. These nonlinear integral equations are the so-called *thermodynamic Bethe Ansatz* (*TBA*) *equations*. Solving the TBA equations is another matter. The methods used are either numerical or perturbative and there are, as far as the authors are aware, no known explicit solutions to the TBA equations.

It thus came as a surprise when Cecotti et al. [2] (see also [3]), in their analysis of certain $N = 2$ supersymmetric theories [1], discovered that a certain quantity (the "supersymmetric index"), expressible in terms of the solution of the pair of "TBA-like" integral equations

$$\varepsilon(\theta) = t \cosh\theta - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1+\eta^2(\theta'))}{\cosh(\theta-\theta')} d\theta' ,$$

$$\eta(\theta) = 2\lambda \int_{-\infty}^{\infty} \frac{e^{-\varepsilon(\theta')}}{\cosh(\theta-\theta')} d\theta' ,$$

is also expressible in terms of a Painlevé III function with independent variable t. (These TBA-like equations are a "small mass" perturbation of a system of TBA equations; see (5.7)–(6.8) in [2].) Using results of McCoy et al. [6] on Painlevé III, they expressed this supersymmetric index as an infinite series related to the resolvent of the integral operator with kernel

$$\frac{e^{-\frac{t}{2}(\cosh\theta+\cosh\theta')}}{\cosh\frac{\theta-\theta'}{2}} .$$

Zamolodchikov [13] then conjectured that the system of nonlinear equations could actually be solved in terms of this resolvent kernel. More precisely, if we denote the operator by K, the kernel of the operator $K(I-\lambda^2K^2)^{-1}$ by $R_+(\theta,\theta')$, and set $R_+(\theta) := R_+(\theta,\theta)$ then the system should be satisfied if

$$e^{-\varepsilon(\theta)} = R_+(\theta)$$

and η is defined by the second equation. In fact he conjectured that this should hold for operators with kernels of the more general form

$$\frac{e^{-(u(\theta)+u(\theta'))}}{\cosh\frac{\theta-\theta'}{2}} \tag{1.1}$$

if the first equation is replaced by

$$\varepsilon(\theta) = 2\,u(\theta) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1+\eta^2(\theta'))}{\cosh(\theta-\theta')} d\theta' .$$

In addition he conjectured that, with the same function η,

$$R_-(\theta) = \frac{1}{\pi} R_+(\theta) \int_{-\infty}^{\infty} \frac{\arctan\eta(\theta')}{\cosh^2(\theta-\theta')} d\theta' ,$$

where $R_-(\theta,\theta')$ is the kernel of $K^2(I-\lambda^2K^2)^{-1}$ and $R_-(\theta) := R_-(\theta,\theta)$. (We state everything in terms of kernels here; in the cited work the functions $R_\pm$ were given by infinite series. That they are the same follows from the Neumann series representation for the kernel of $(I-\lambda^2K^2)^{-1}$.)

We prove these conjectures here. One of the main ingredients is the fact that the equations are in a sense equivalent to relations among the analytic continuations of the functions $R_\pm(\theta)$ into a strip. (See formula (6.8) of [13].) Another is a particularly convenient representation for these functions in terms of other functions, which we call $Q(\theta)$ and $P(\theta)$. (That these latter functions are fundamental is known from earlier work [4, 8–10].) These representations are stated in Lemma 1, and in

Lemmas 2 and 3 we state precise versions of the equivalence alluded to before. In Lemmas 4 and 5 we derive general properties of functions in the range of the operator K in order to derive, as Lemma 6, some basic properties of the functions $Q(\theta)$ and $P(\theta)$.

If we try to prove the desired relation among the analytic continuations of $R_{\pm}(\theta)$ we find that we have to prove a certain crucial identity involving $Q(\theta)$ and $P(\theta)$ which is by no means obvious. But once conjectured it is not hard to prove, given the previous preparatory work, and that is stated as the proposition which follows the lemmas. We show that a certain combination of these functions, which is clearly analytic in the strip, extends by periodicity to an entire function. Combining this fact with the use of Liouville's theorem, we deduce the identity.

It should be mentioned that the main part of the argument may only be used if u belongs to a restricted class, but the result for general u follows by an approximation argument. This will be presented in the Appendix, as will proofs of some of the lemmas and some facts about Fourier transforms we shall use.

II. Preliminaries

We shall assume throughout that u is continuous and bounded from below and that

$$0 < \lambda < e^{2 \min u}/2\pi \, . \tag{2.1}$$

This assures that the series defining $R_{\pm}(\theta)$ converge uniformly and that the operator λK, acting on any of the usual function spaces, has norm less than 1, so that $I - \lambda^2 K^2$ is invertible. (This follows from (4.5) below.) Since the parameter λ may be incorporated into u we may assume that in fact $\lambda = 1$. If we set

$$E(\theta) := \sqrt{2}\, e^{-u(\theta)}\, e^{\theta/2} \, , \tag{2.2}$$

then the kernel of K is given by

$$\frac{E(\theta)E(\theta')}{e^{\theta} + e^{\theta'}} \, . \tag{2.3}$$

Our functions Q and P are defined by

$$Q := (I - K^2)^{-1} E, \qquad P := (I - K^2)^{-1} K E \, .$$

Lemma 1. *We have the representations*

$$R_{+}(\theta) = \frac{Q(\theta)^2 - P(\theta)^2}{2\, e^{\theta}}, \qquad R_{-}(\theta) = \frac{Q'(\theta) P(\theta) - P'(\theta) Q(\theta)}{e^{\theta}} \, . \tag{2.4}$$

Proof. We use the notations $[A,B] := AB - BA$, $\{A,B\} := AB + BA$ and write $X \otimes Y$ for the operator with kernel $X(\theta)\, Y(\theta')$ and M for multiplication by e^{θ}. Then we have immediately

$$\{M,K\} = E \otimes E \, ,$$

from which it follows also that

$$[M,K^2] = E \otimes KE - KE \otimes E \, ,$$

and then that

$$[M,(I-K^2)^{-1}] = Q\otimes P - P\otimes Q\,.$$

Since

$$K^2(I-K^2)^{-1} = (I-K^2)^{-1} - I\,, \tag{2.5}$$

we deduce that the kernel of this operator is given by the formula

$$R_-(\theta,\theta') = \frac{Q(\theta)P(\theta') - P(\theta)Q(\theta')}{e^\theta - e^{\theta'}}\,.$$

The second part of (2.4) follows.

To derive the first part we use the general identity

$$\{A,BC\} = \{A,B\}\,C - B\,[A,C]$$

with $A = M$, $B = K$ and $C = (I-K^2)^{-1}$ together with the formulas above to deduce

$$\{M,K(I-K^2)^{-1}\} = E\otimes Q - K(Q\otimes P - P\otimes Q)\,.$$

Of course $KQ = P$, and applying (2.5) to E gives

$$KP = Q - E\,,$$

and so the right side of the previous identity simplifies to $Q\otimes Q - P\otimes P$. This gives the representation for the kernel of $K(I-K^2)^{-1}$,

$$R_+(\theta,\theta') = \frac{Q(\theta)Q(\theta') - P(\theta)P(\theta')}{e^\theta + e^{\theta'}}\,,$$

and the first part of (2.4) follows.

Recall that a function f defined on $\mathbf{R}$ is said to belong to the Wiener space W if its Fourier transform $\hat{f}$ belongs to L_1. Such a function is necessarily continuous and vanishes at $\pm\infty$. A sufficient condition that $f\in W$ is that f and f' belong to L_2. (See the Appendix.)

We use the notation $\mathcal{S}_a$ to denote the strip $|\Im\,\theta| < a$ in the complex θ-plane, and $A(\mathcal{S}_a)$ to denote those functions g which are bounded and analytic in the strip, continuous on its closure, and for which $g(\theta+iy)\to 0$ as $\theta\to+\infty$ through real values when $y\in\mathbf{R}$ is fixed and satisfies $|y|<a$.

The proofs of the next three lemmas will be found in the Appendix.

Lemma 2. *Assume $f\in W$. If*

$$g(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\theta')}{\cosh(\theta-\theta')}\,d\theta'\,, \tag{2.6}$$

then $g\in A(\mathcal{S}_{\pi/2})$ and its boundary functions satisfy

$$g(\theta+i\pi/2) + g(\theta-i\pi/2) = f(\theta) \tag{2.7}$$

for real θ. Conversely, if $g\in A(\mathcal{S}_{\pi/2})$ and if (2.7) *holds then so does* (2.6).

Lemma 3. *Assume $f, f' \in W$. If*

$$g(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(\theta')}{\cosh^2(\theta - \theta')}\, d\theta' ,$$

then $g \in A(\mathcal{S}_{\pi/2})$ and its boundary functions satisfy

$$g(\theta + i\pi/2) - g(\theta - i\pi/2) = i f'(\theta)$$

for real θ. Conversely, if $g \in A(\mathcal{S}_{\pi/2})$ and the second relation holds then so does the first.

Lemma 4. *Assume $f \in L_1$ and set $h(x)$ equal to $f(\log x)/x$ for $x > 0$ and equal to 0 for $x \leqq 0$. If $h \in W$ and*

$$g(\theta) = \int_{-\infty}^{\infty} \frac{f(\theta')}{e^{\theta} + e^{\theta'}}\, d\theta' ,$$

then $g \in A(\mathcal{S}_{\pi})$.

These are the basic ingredients we shall use. We specialize at first to the case where u is a Laurent polynomial in e^{θ},

$$u(\theta) = \sum_{k=-m}^{n} a_k\, e^{k\theta} \tag{2.8}$$

for some $m, n > 0$ with $a_{-m}, a_n > 0$. Thus $u(\theta)$ is an entire function of period $2\pi i$, and $E(\theta)$ and its derivative tend exponentially to 0 as $\theta \to \pm\infty$ through real values. It follows from this that if $f \in L_2$ then Kf and its derivative are exponentially small at $\pm\infty$. In particular, any function in the range of K satisfies the hypothesis of the next lemma.

Lemma 5. *If f is a bounded function on $\mathbf{R}$ with bounded derivative then $Kf(\theta)/E(\theta)$ extends to a function in $A(\mathcal{S}_{\pi})$. The boundary functions of Kf satisfy*

$$Kf\,(\theta + i\pi) + Kf\,(\theta - i\pi) = 4\pi v(\theta) f(\theta)$$

for real θ, where

$$v(\theta) := e^{-(u(\theta) + u(\theta + i\pi))} .$$

Proof. If we look at the expression (2.3) for the kernel of K we see that $Kf(\theta)/E(\theta)$ is of the form of the function g of Lemma 4 if $f(\theta)$ there is replaced by our $E(\theta)f(\theta)$. It is easy to see that if our f satisfies the stated conditions then the function h in the statement of Lemma 4 belongs to L_2 and has an L_2 derivative, so $h \in W$ and the conclusion of the lemma holds. Thus $Kf/E \in A(\mathcal{S}_{\pi})$. For the boundary function identity we use the expression (1.1) for the kernel of K. If we make the substitutions $\theta \to 2\theta$, $\theta' \to 2\theta'$, we see that $e^{u(2\theta)}(Kf)(2\theta) = 2\,e^{\theta}\,(Kf)(2\theta)/E(2\theta)$ is exactly of the form of the function g in the statement of Lemma 2 if the function $f(\theta)$ there is replaced by our present $4\pi e^{-u(2\theta)} f(2\theta)$. Applying the identity stated there and using the periodicity of u give the identity stated here.

We apply this to the functions $Q(\theta)$ and $P(\theta)$.

Lemma 6. *The functions*

$$\frac{Q(\theta)}{E(\theta)} - 1 \quad \textit{and} \quad \frac{P(\theta)}{E(\theta)} \tag{2.9}$$

belong to $A(\mathcal{S}_\pi)$ *and the boundary functions of* Q *and* P *satisfy the identities*

$$\begin{aligned} Q(\theta+i\pi)+Q(\theta-i\pi) &= 4\pi v(\theta)P(\theta)\,, \\ P(\theta+i\pi)+P(\theta-i\pi) &= 4\pi v(\theta)Q(\theta) \end{aligned} \tag{2.10}$$

for real θ.

Proof. Because of the relations $Q = E + K^2(I-K^2)^{-1}E$, $P = KQ$ all statements of the lemma except for the first part of (2.10) follow from Lemma 5 and the remark preceding it. Since $Q - E = KP$, Lemma 5 gives the identity

$$(Q-E)(\theta+i\pi)+(Q-E)(\theta-i\pi) = 4\pi v(\theta)P(\theta)\,.$$

But it follows from the definition (2.2) of E and the fact that u has period $2\pi i$ that $E(\theta+i\pi)+E(\theta-i\pi)=0$. Thus we obtain the desired identity for Q.

Using Lemma 6 and the fact that v is an entire function we can conclude that Q and P have analytic continuations to entire functions of θ; each use of the pair of identities allows us to widen by π the strip of analyticity. Here is the crucial identity relating these continuations from which our results will follow.

Proposition. *We have*

$$\begin{aligned} &Q(\theta+i\pi/2)Q(\theta-i\pi/2) - P(\theta+i\pi/2)P(\theta-i\pi/2) \\ &\quad = E(\theta+i\pi/2)\,E(\theta-i\pi/2)\,. \end{aligned}$$

Proof. Set

$$S(\theta) := Q(\theta+i\pi/2)\,Q(\theta-i\pi/2) - P(\theta+i\pi/2)\,P(\theta-i\pi/2)\,.$$

Then

$$S(\theta+i\pi/2) = Q(\theta+i\pi)\,Q(\theta) - P(\theta+i\pi)\,P(\theta)\,,$$

$$S(\theta-i\pi/2) = Q(\theta-i\pi)\,Q(\theta) - P(\theta-i\pi)\,P(\theta)\,,$$

and so by (2.10),

$$S(\theta+i\pi/2)+S(\theta-i\pi/2) = 4\pi v(\theta)[P(\theta)\,Q(\theta) - Q(\theta)\,P(\theta)] = 0\,.$$

It follows that $S(\theta)$ extends to an entire function of period $2\pi i$ whose values at $\theta \pm i\pi/2$ are negatives of each other. Therefore $1 - S(\theta)/E(\theta+i\pi/2)\,E(\theta-i\pi/2)$ extends to an entire function of period πi whose values at $\theta \pm i\pi/2$ are equal. (We used again the fact that $E(\theta-i\pi) = -E(\theta+i\pi)$.) To show that this is 0 (this is equivalent to the claimed identity) it suffices, by Liouville's theorem, to show that it is bounded and that it tends to 0 as $\theta \to +\infty$ through real values. For a πi-periodic function it suffices to show that these properties hold in the strip $\mathcal{S}_{\pi/2}$. They do hold there because for θ in this strip $\theta \pm i\pi/2$ lie in the strip $\mathcal{S}_\pi$, for which we have the conclusions of Lemma 6.

III. Proof of the Conjectures

We have to show that if $\varepsilon := -\log R_+$ and if η is defined by

$$\eta(\theta) = 2\int_{-\infty}^{\infty} \frac{e^{-\varepsilon(\theta')}}{\cosh(\theta-\theta')}\, d\theta' \tag{3.1}$$

(recall that we have taken $\lambda = 1$), then we have the two identities

$$2\,u(\theta) - \varepsilon(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\log(1+\eta^2(\theta'))}{\cosh(\theta-\theta')}\, d\theta' , \tag{3.2}$$

$$R_-(\theta) = \frac{1}{\pi} R_+(\theta) \int_{-\infty}^{\infty} \frac{\arctan \eta(\theta')}{\cosh^2(\theta-\theta')}\, d\theta' . \tag{3.3}$$

We shall assume that we have a function η satisfying (3.1)–(3.3), and formally apply the first parts of Lemmas 2 and 3 to obtain the three identities (3.4)–(3.6) below. From these we shall see that η must have a certain representation in terms of Q and P. Then, using the identities of the proposition, we shall show that (3.4)–(3.6) in fact hold if η is defined this way. Finally, using the second parts of Lemmas 2 and 3, we show that (3.1)–(3.3) hold. (Of course the first step is unnecessary for the proof, but it provides motivation for the eventual definition of η.)

Applying Lemmas 2 and 3 to (3.1)–(3.3) give

$$4\pi R_+(\theta) = \eta(\theta + i\pi/2) + \eta(\theta - i\pi/2) , \tag{3.4}$$

$$\log(1+\eta^2(\theta)) = 2u(\theta+i\pi/2) - \varepsilon(\theta+i\pi/2) + 2u(\theta-i\pi/2) - \varepsilon(\theta-i\pi/2) , \tag{3.5}$$

$$2i\frac{\eta'(\theta)}{1+\eta(\theta)^2} = \frac{R_-(\theta+i\pi/2)}{R_+(\theta+i\pi/2)} - \frac{R_-(\theta-i\pi/2)}{R_+(\theta-i\pi/2)} . \tag{3.6}$$

Exponentiating (3.5) and using the definition of ε give

$$1+\eta^2(\theta) = R_+(\theta+i\pi/2)\,R_+(\theta-i\pi/2)\,e^{2\,(u(\theta+i\pi/2)+u(\theta-i\pi/2))} . \tag{3.7}$$

By Lemma 1 and (2.2) this may be written

$$1+\eta^2(\theta) = \frac{(Q^2-P^2)(\theta+i\pi/2)\cdot(Q^2-P^2)(\theta-i\pi/2)}{E(\theta+i\pi/2)^2 E(\theta-i\pi/2)^2} . \tag{3.8}$$

Lemma 1 shows that (3.6) may be written

$$i\frac{\eta'(\theta)}{1+\eta^2(\theta)} = \frac{Q'P-P'Q}{Q^2-P^2}(\theta+i\pi/2) - \frac{Q'P-P'Q}{Q^2-P^2}(\theta-i\pi/2) .$$

Taking the logarithmic derivative of both sides of (3.8) and dividing by 2 gives

$$\eta(\theta)\frac{\eta'(\theta)}{1+\eta^2(\theta)} = \frac{Q'Q-P'P}{Q^2-P^2}(\theta+i\pi/2) + \frac{Q'Q-P'P}{Q^2-P^2}(\theta-i\pi/2)$$
$$-\frac{E'}{E}(\theta+i\pi/2) - \frac{E'}{E}(\theta-i\pi/2) .$$

We have a choice now of either adding or subtracting the last two displayed formulas. Choosing the former, we obtain

$$\frac{\eta'(\theta)}{\eta(\theta)-i}=\frac{Q'-P'}{Q-P}(\theta+i\pi/2)+\frac{Q'+P'}{Q+P}(\theta-i\pi/2)-\frac{E'}{E}(\theta+i\pi/2)-\frac{E'}{E}(\theta-i\pi/2)\,,$$

or equivalently

$$\frac{\eta'(\theta)}{\eta(\theta)-i}=\frac{d}{d\theta}\log(Q-P)(\theta+i\pi/2)+\frac{d}{d\theta}\log(Q+P)(\theta-i\pi/2)$$
$$-\frac{d}{d\theta}\log E(\theta+i\pi/2)E(\theta-i\pi/2)\,.$$

Integrating and exponentiating gives the desired formula

$$\eta(\theta)-i=-i\frac{(Q-P)(\theta+i\pi/2)\cdot(Q+P)(\theta-i\pi/2)}{E(\theta+i\pi/2)E(\theta-i\pi/2)}\,. \tag{3.9}$$

The reason the constant factor on the right must be $-i$ is that we want $\eta(+\infty)=0$. (See (3.8) and Lemma 6.) If we had subtracted before instead of adding we would have been led to the similar but apparently different formula

$$\eta(\theta)+i=i\frac{(Q+P)(\theta+i\pi/2)\cdot(Q-P)(\theta-i\pi/2)}{E(\theta+i\pi/2)E(\theta-i\pi/2)}\,. \tag{3.10}$$

We shall now show that if η is defined by (3.9) then (3.10) also holds, as do relations (3.4)–(3.6).

First, the statement that the right side of (3.10) minus the right side of (3.9) is equal to $2i$ follows from the proposition. Thus (3.9) and (3.10) are completely equivalent.

Second, taking the product of (3.9) and (3.10) gives (3.8) and hence (3.5).

Third, reversing the argument that showed (3.8) and (3.6) imply (3.9) we see that (3.9) and (3.8), which we now have, imply (3.6).

Finally, to obtain (3.4) we use (3.10) to express $\eta(\theta+i\pi/2)$ and (3.9) to express $\eta(\theta-i\pi/2)$ and find that their sum equals

$$i\frac{(Q-P)(\theta)\,[(Q+P)(\theta+i\pi)+(Q+P)(\theta-i\pi)]}{E(\theta)E(\theta+i\pi)}\,.$$

(We used yet again the fact that $E(\theta-i\pi)=-E(\theta+i\pi)$.) By (2.10) this is equal to

$$\frac{4\pi i\,v(\theta)}{E(\theta)E(\theta+i\pi)}[Q(\theta)^2-P(\theta)^2]\,,$$

and by (2.4) and the definitions of $E(\theta)$ and $v(\theta)$ this equals $4\pi R_+(\theta)$.

So (3.4)–(3.6) are established. Now we show that they imply (3.1)–(3.3). By Lemmas 2 and 3 this will be true if the functions

$$R_+,\ \log(1+\eta^2),\ \arctan\eta,\ \eta'/(1+\eta^2) \tag{3.11}$$

belong to W and the functions

$$\eta,\ 2u-\varepsilon,\ R_-/R_+ \tag{3.12}$$

belong to $A(\mathcal{S}_{\pi/2})$.

Our assumption has been that $\lambda = 1$ satisfies inequality (2.1). This identity is still satisfied if u is increased or, equivalently, if E is decreased. It follows that we could, in our representation (2.3) of the kernel of K, have replaced E by δE for any $\delta \in [0, 1]$. It is clear that all quantities in (3.1)–(3.3) would then be real-analytic functions of δ for $\delta \in [0, 1]$. So if the relations hold for sufficiently small δ they hold for all, including $\delta = 1$. If we retrace the steps leading to bounds for the functions (2.9) we find that they tend to 0 as $\delta \to 0$. (In fact, they are $O(\delta^2)$.) Hence we may assume that the bounds for these functions are as small as we like. In fact we assume that

$$\left|\frac{Q(\theta)}{E(\theta)} - 1\right| < \frac{1}{4} \quad \text{and} \quad \left|\frac{P(\theta)}{E(\theta)}\right| < \frac{1}{4} \qquad (\theta \in \mathscr{S}_\pi)\,. \tag{3.13}$$

We take in succession the items we have to verify. Recall that a function belongs to W if it and its derivative belong to L_2. This is what we shall show for the functions (3.11). First, from (2.2) and (2.4) we have

$$R_+(\theta) = \frac{Q(\theta)^2 - P(\theta)^2}{E(\theta)^2} e^{-2u(\theta)}\,.$$

This first factor is bounded and analytic in $\mathscr{S}_\pi$ and so is bounded and has bounded derivative on $\mathbf{R}$. The second factor is in L_2 and so is its derivative. Hence $R_+ \in W$.

The next three are not obvious. If we use the representation (2.3) for the kernel of K we see that for any f we have

$$\frac{Kf}{E}(\theta + i\pi/2) - \frac{Kf}{E}(\theta - i\pi/2) = -2i \int_{-\infty}^{\infty} \frac{E(\theta')f(\theta')e^\theta}{e^{2\theta} + e^{2\theta'}}\, d\theta' = O(\operatorname{sech}\theta)$$

if, say, $f \in L_\infty$. This holds for $|\Im\,\theta|$ strictly less than $\pi/4$. Applying this to $f = Q$ and $f = P$ we deduce that

$$\frac{P}{E}(\theta + i\pi/2) - \frac{P}{E}(\theta - i\pi/2) = O(\operatorname{sech}\theta)\,,$$

$$\frac{Q}{E}(\theta + i\pi/2) - \frac{Q}{E}(\theta - i\pi/2) = O(\operatorname{sech}\theta)$$

as long as $|\Im\,\theta| < \pi/8$, for example. Now adding (3.9) and (3.10) gives the representation

$$\eta(\theta) = i\,\frac{Q(\theta - i\pi/2)P(\theta + i\pi/2) - Q(\theta + i\pi/2)P(\theta - i\pi/2)}{E(\theta - i\pi/2)E(\theta + i\pi/2)}\,. \tag{3.14}$$

By the above and (3.13), this is

$$\left[\frac{Q}{E}(\theta + i\pi/2) + O(\operatorname{sech}\theta)\right] \frac{P}{E}(\theta + i\pi/2) - \frac{Q}{E}(\theta + i\pi/2)$$

$$\times \left[\frac{P}{E}(\theta + i\pi/2) + O(\operatorname{sech}\theta)\right] = O(\operatorname{sech}\theta)\,.$$

(Observe that if $\theta \in \mathscr{S}_{\pi/8}$ then $\theta \pm i\pi/2 \in \mathscr{S}_\pi$.) Since this bound on η holds in a full strip, the same bound (with different constants) holds on $\mathbf{R}$ for each of the derivatives of η. Since η is real-valued on $\mathbf{R}$ (this follows from (3.14) and the fact that Q, P, and E are real-valued on $\mathbf{R}$) we deduce that the last three functions in (3.11), and all their derivatives, belong to $L_2(\mathbf{R})$. Hence they all belong to W.

Now we show that the functions (3.11) all belong to $A(\mathscr{S}_{\pi/2})$. First, by (2.4),

$$\frac{R_-}{R_+} = 2\frac{Q'P - P'Q}{Q^2 - P^2} = 2\frac{Q'P - P'Q}{Q^2}\,\frac{1}{1-(P/Q)^2}\,.$$

By (3.13) P/Q is bounded in the larger strip $\mathscr{S}_\pi$ by 1/3 and so it follows that the last factor above is bounded. It also follows that $P/Q \in A(\mathscr{S}_\pi)$ from which it follows that $(P/Q)' \in A(\mathscr{S}_{\pi/2})$. Hence $R_-/R_+ \in A(\mathscr{S}_{\pi/2})$.

Next, we use (2.4) again to write

$$2u - \varepsilon = 2u + \log\frac{Q^2 - P^2}{2e^\theta} = \log\frac{Q^2 - P^2}{E^2}\,.$$

By (3.13) we have $|1 - (Q \pm P)/E| < 1/2$ in $\mathscr{S}_{\pi/2}$ and we deduce as above that $2u - \varepsilon \in A(\mathscr{S}_{\pi/2})$.

Finally since $1 - Q/E$ and P/E belong to $A(\mathscr{S}_\pi)$, the functions

$$1 - \frac{Q(\theta \pm i\pi/2)}{E(\theta \pm i\pi/2)}, \qquad \frac{P(\theta \pm i\pi/2)}{E(\theta \pm i\pi/2)}$$

belong to $A(\mathscr{S}_{\pi/2})$. It follows from this and the representation (3.14) that $\eta \in A(\mathscr{S}_{\pi/2})$.

Thus we have proved the conjectures in the case where $u(\theta)$ has the special form (2.8). For a general u, more precisely for any u which is continuous and bounded below, we can find a sequence of u_n of the special type such that e^{-u_n} converge boundedly and locally uniformly to e^{-u}. (This will be demonstrated in the Appendix.) This is enough to deduce the result for u from the results for the u_n.

IV. Appendix

We give details here of certain matters postponed from the previous sections. First we recall some facts about the Fourier transform, which we denote, as usual, by a circumflex:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi\theta} f(\theta)\, d\theta\,.$$

If $\hat{f} \in L_1$, in other words, if $f \in W$, we have the Fourier inversion formula

$$f(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi\theta}\hat{f}(\xi)\, d\xi\,,$$

and $f(\pm\infty) = 0$. If $\|\;\|_p$ denotes the norm in the space L_p, then by the inversion formula we have the inequality

$$\|f\|_\infty \leqq \frac{1}{2\pi}\|\hat{f}\|_1\,. \tag{4.1}$$

Parseval's identity reads

$$\|\hat{f}\|_2 = \frac{1}{\sqrt{2\pi}}\|f\|_2\,,$$

and we have the general formula $\widehat{f'}(\xi) = i\xi\hat{f}(\xi)$.

Proof that $f, f' \in L_2$ implies $f \in W$. It suffices to show that if we write $\hat{f}(\xi)$ as $[\hat{f}(\xi)(\xi+i)](\xi+i)^{-1}$ then both factors on the right belong to L_2. The second factor surely does, and the square of the absolute value of the first factor equals

$$|\hat{f}(\xi)|^2(\xi^2+1) = |\widehat{f'}(\xi)|^2 + |\hat{f}(\xi)|^2 .$$

The right side belongs to L_1 by Parseval's identity and the assumption $f, f' \in L_2$.

Proof of Lemma 2. It is clear that g, when defined by (2.6), is analytic in the strip. If we write $g_y(\theta) := g(\theta + iy)$ for real θ then

$$\widehat{g_y}(\xi) = \hat{f}(\xi)\frac{e^{-y\xi}}{2\cosh \pi\xi/2} . \tag{4.2}$$

The first factor belongs to L_1, by assumption, and the second factor is bounded by 1 for $|y| < \pi/2$. In particular $g_y \in W$ for each y so $g(\pm\infty + iy) = 0$. From

$$\int_{-\infty}^{\infty} |\widehat{g_y}(\xi)|\, d\xi \leqq \int_{-\infty}^{\infty} |\hat{f}(\xi)|\, d\xi$$

and (4.1) we see that g_y is uniformly bounded for $|y| < \pi/2$. Finally, as $y \to \pm\pi/2$ the second factor in (4.2) converges pointwise and boundedly. This implies (by the Lebesgue dominated convergence theorem) that $\widehat{g_y}(\xi)$ converges in L_1 and so (by (4.1) again) $g_y(\theta)$ converges uniformly in θ. This implies that $g(\theta)$ extends continuously to the closure of $\mathscr{S}_{\pi/2}$. Thus $g \in A(\mathscr{S}_{\pi/2})$. The sum of the two limiting values of the second factor in (4.2) is equal to 1, and this gives (2.7).

To prove the converse, let h denote the difference between the two sides of (2.6). Then, using our assumption and what we have already shown, $h \in A(\mathscr{S}_{\pi/2})$ and its boundary functions satisfy

$$h(\theta + i\pi/2) + h(\theta - i\pi/2) = 0$$

for real θ. It follows that h extends to a $2\pi i$-periodic entire function. This function is bounded, and so must be a constant, and the constant must be 0 because $h(+\infty) = 0$.

Proof of Lemma 3. In this case the Fourier transform of g_y equals

$$\hat{f}(\xi)\,\xi\,\frac{e^{-y\xi}}{2\sinh \pi\xi/2} = \hat{f}(\xi)(\xi+i)\frac{\xi\, e^{-y\xi}}{2(\xi+i)\sinh \pi\xi/2} .$$

Now $\hat{f}(\xi)(\xi+i) = -i\widehat{f'}(\xi) + i\hat{f}(\xi) \in L_1$, the last factor on the right is uniformly bounded for $|y| < \pi/2$ and it converges pointwise as $y \to \pm\pi/2$. We deduce as before that $g \in A(\mathscr{S}_{\pi/2})$. The difference of the limiting values of the last factor on the left side at $y = \pm\pi/2$ is -1 and so the difference of the limiting values of g has Fourier transform $-\hat{f}(\xi)\xi = \widehat{if'}$. This establishes the first part of the lemma and the second follows just as before.

Proof of Lemma 4. For fixed θ' the factor $(e^{\theta} + e^{\theta'})^{-1}$ in the integral defining $g(\theta)$ is analytic in θ. For θ in any subset of $\mathscr{S}_\pi$ of the form

$$\Re\theta \geqq \theta_0, \qquad |\Im\theta| \leqq \pi - \delta \quad (\delta > 0)$$

this factor is bounded uniformly in θ' and tends to 0 pointwise as $\Re\theta \to +\infty$. Since $f \in L_1$ this is enough to conclude that g is analytic in $\mathscr{S}_\pi$ and tends to 0 in this strip as $\Re\theta \to +\infty$ and $\Im\theta$ is fixed.

It remains to prove boundedness of g and continuity near the boundary of $\mathscr{S}_\pi$, and for this we use the function h. In the lower part of the strip, $-\pi < y < 0$, we set $z = -e^{\theta+iy}$ with θ real, so that $\Im z > 0$, and we can write

$$g(\theta+iy) = \int_0^\infty \frac{h(x)}{x-z}\,dx = \int_{-\infty}^\infty \frac{h(x)}{x-z}\,dx\,.$$

Using the Fourier inversion formula and interchanging the order of integration, we find the representation

$$g(\theta+iy) = i\int_0^\infty e^{iz\xi}\,\hat{h}(\xi)\,d\xi\,.$$

Since $\hat{h} \in L_1$ the integral is bounded uniformly for $\Im z > 0$ and we deduce as in Lemmas 2 and 3 that it extends continuously to $\Im z = 0$. Thus $g(\theta+iy)$ is bounded for $0 \leqq y < \pi$ and extends continuously to $y = \pi$. A similar argument holds for the upper half of $\mathscr{S}_\pi$, and so $g \in A(\mathscr{S}_\pi)$.

Extension to general u. We begin with an approximation fact, reminiscent of the Weierstrass approximation theorem. Recall the notation $C_0(\mathbf{R})$ for the space of continuous functions f on $\mathbf{R}$ satisfying $f(\pm\infty) = 0$, which is a Banach space under the norm

$$\|f\| := \sup\{|f(x)| : x \in \mathbf{R}\}\,.$$

The fact is that for each $\delta > 0$ the finite linear combinations of the functions

$$\sinh^k\theta\, e^{-\delta\sinh^2\theta} \quad (k = 0, 1, \ldots)$$

are dense in $C_0(\mathbf{R})$; in other words, for any $f \in C_0(\mathbf{R})$ and any $\delta' > 0$ there exists a finite linear combination

$$p(\theta) = \sum_{k=0}^N a_k \sinh^k\theta$$

such that

$$|p(\theta)\,e^{-\delta\sinh^2\theta} - f(\theta)| < \delta' \tag{4.3}$$

for all θ. This is true because the change of variable $t = \sinh\theta$ converts it to the statement that the finite linear combinations of the functions $t^k\,e^{-\delta t^2}$ are dense in $C_0(\mathbf{R})$. And this in turn is true because if it weren't then the Hahn–Banach theorem and Riesz representation theorem ([7], Theorems 5.19 and 6.19) would imply that there is a function of bounded variation (= signed measure) μ on $\mathbf{R}$, not identically zero, such that $\int_{-\infty}^\infty t^k\,e^{-\delta t^2}d\mu(t) = 0$ for all $k \geqq 0$. But then the entire function $F(z) := \int_{-\infty}^\infty e^{izt}\,e^{-\delta t^2}d\mu(t)$ would satisfy $F^{(k)}(0) = 0$ for $k \geqq 0$ and so $F \equiv 0$. This in turn implies $e^{-\delta t^2}d\mu(t) \equiv 0$ and so $\mu(t) \equiv 0$, a contradiction.

Here is how to construct the sequence u_n described at the end of Sect. III. We may clearly assume that u is uniformly positive, i.e. that for some $\alpha > 0$ we have $u(\theta) \geqq \alpha$ for all θ. Let n be given and define

$$w := \min(u, n)\,.$$

Then find $p(\theta)$, a linear combination of the powers of sinh θ, such that (4.3) holds with

$$f(\theta) = \sqrt{w(\theta)}\, e^{-n^{-1}\sinh^2\theta}$$

and $\delta = \delta' = n^{-1}$. The inequality may be rewritten

$$|p(\theta) - \sqrt{w(\theta)}| < n^{-1} e^{n^{-1}\sinh^2\theta}.$$

It is an easy exercise to deduce from this that for sufficiently large n,

$$|p(\theta)^2 - u(\theta)| < 6n^{-1/2} \quad \text{if } u(\theta) < n \text{ and } \sinh^2\theta < n.$$

(We use here the facts that u is uniformly positive and that $e < 3$.) The function $p(\theta)^2$ is our $u_n(\theta)$.

We now deduce the identities (3.1)–(3.3) for general u. Denote by $R_{n\pm}$ the $R_\pm$ functions associated with the functions u_n. If we can show that $R_{n\pm}(\theta) \to R_\pm(\theta)$ boundedly and pointwise then (3.1)–(3.3) for u will follow from the corresponding identities for the u_n, since by the dominated convergence theorem we could take the limits as $n \to \infty$ under the integral signs. The function $R_+(\theta)$ is given by the series

$$\sum_{m=0}^{\infty} \lambda^{2m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K(\theta,\theta_1)\cdots K(\theta_{2m},\theta)\, d\theta_1 \cdots d\theta_{2m}, \tag{4.4}$$

where $K(\theta,\theta')$ is given by (1.1). It follows from the fact

$$\int_{-\infty}^{\infty} \frac{d\theta}{\cosh\theta/2} = 2\pi \tag{4.5}$$

that the series converges uniformly in θ when λ satisfies (2.1). By a real-analyticity argument already used we may assume that u is uniformly positive, so that by the previous construction $u_n \geqq 0$ for all n, and that $\lambda < 1/2\pi$. Denote by $K_n(\theta,\theta')$ the kernel corresponding to u_n so that $R_{n+}(\theta)$ is given by the series

$$\sum_{m=0}^{\infty} \lambda^{2m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K_n(\theta,\theta_1)\cdots K_n(\theta_{2m},\theta)\, d\theta_1 \cdots d\theta_{2m}. \tag{4.6}$$

It follows from (4.5) and the inequality $e^{-u_n} \leqq 1$ that the integral in the m^{th} term of (4.6) is at most $(2\pi)^{2m}$ for all n and so, since $\lambda < 1/2\pi$, the series converges uniformly in n. Thus we may take the limit as $n \to \infty$ under the summation sign. Next, each integrand $K_n(\theta,\theta_1)\cdots K_n(\theta_{2m},\theta)$ is uniformly bounded by $K(\theta,\theta_1)\cdots K(\theta_{2m},\theta)$, which has finite integral over $\mathbf{R}^{2m}$, and so we may take each limit as $n \to \infty$ under the integral sign (again by the dominated convergence theorem). The result is the series (4.4), and this gives

$$\lim_{n\to\infty} R_{n+}(\theta) = R_+(\theta).$$

Since $R_{n+}(\theta)$ is at most the sum of the series (4.4) corresponding to $u = 0$ we have established that $R_{n+}(\theta) \to R_+(\theta)$ boundedly and pointwise. A similar argument applies to $R_{n-}(\theta)$, and the proof is complete.

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