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Asymptotics of a Class of Solutions to the Cylindrical Toda Equations

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Abstract: The small t asymptotics of a class of solutions to the $2D$ cylindrical Toda equations is computed. The solutions, $q_k(t)$, have the representation

$$q_k(t) = \log\,\det\,(I - \lambda\,K_k) - \log\,\det\,(I - \lambda\,K_{k-1}),$$

where K_k are integral operators. This class includes the n-periodic cylindrical Toda equations. For $n = 2$ our results reduce to the previously computed asymptotics of the $2D$ radial sinh-Gordon equation and for $n = 3$ (and with an additional symmetry constraint) they reduce to earlier results for the radial Bullough-Dodd equation. Both of these special cases are examples of Painlevé III and have arisen in various applications. The asymptotics of $q_k(t)$ are derived by computing the small t asymptotics

$$\det\,(I - \lambda\,K_k) \sim b_k\,\left(\frac{t}{n}\right)^{a_k},$$

where explicit formulas are given for the quantities a_k and b_k. The method consists of showing that the resolvent operator of K_k has an approximation in terms of resolvents of certain Wiener-Hopf operators, for which there are explicit integral formulas.

1. Introduction

We consider here solutions of the cylindrical Toda equations

$$q_k''(t) + t^{-1}q_k'(t) = 4\,(e^{q_k(t)-q_{k-1}(t)} - e^{q_{k+1}(t)-q_k(t)}), \qquad k \in \mathbf{Z}, \tag{1.1}$$

satisfying the periodicity conditions $q_{k+n} = q_k$. The integer n is arbitrary but fixed. It follows from results in [9] that solutions valid for all $t > 0$ are given by

$$q_k(t) = \log\,\det\,(I - \lambda\,K_k) - \log\,\det\,(I - \lambda\,K_{k-1}), \tag{1.2}$$

where K_k is the integral operator on $\mathbf{R}^+$ with kernel

$$\sum_\omega \omega^k\, c_\omega\, \frac{e^{-t[(1-\omega)u+(1-\omega^{-1})u^{-1}]}}{-\omega u+v}, \tag{1.3}$$

ω running over the n^{th} roots of unity other than 1.

In the case $n = 2$ we have $q_{k+1} = -q_k$ and (1.1) becomes, with q equal to either q_k,

$$q''(t) + t^{-1}q'(t) = 8\, \sinh 2q(t),$$

which can be reduced to a particular case of the Painlevé III equation. The connection with Fredholm determinants was discovered by McCoy, Tracy and Wu [6], and in the same paper the asymptotics as $t \to 0$ of these solutions $q(t)$ were determined. (Note that all asymptotics as $t \to \infty$ are trivial.) The asymptotics as $t \to 0$ of $\det(I - \lambda^2 K_0^2) = \det(I-\lambda\, K_0)\, \det(I+\lambda\, K_0)$ were determined in [7]. (See also [2], where the asymptotics were found for a family of kernels including this one as a special case.) The asymptotics of $\det(I - \lambda\, K_0)$ itself were stated without proof in [10].

A class of periodic cylindrical Toda equations arises in thermodynamic Bethe Ansatz considerations [3]. There the additional constraint $q_{-k-1} = -q_k$ is imposed. The solutions (1.2) satisfy this constraint as long as the coefficients c_ω satisfy $c_\omega = -\omega^3\, c_{\omega^{-1}}$. (This follows from the fact that $\det(I - K_k) = \det(I - K_{-k-2})$ in this case, which is proved by applying the change of variable $u \to u^{-1}$.) The case $n = 3$ of this gives the cylindrical Bullough-Dodd equation ($q = q_3$ now)

$$q''(t) + t^{-1}q'(t) = 4\,(e^{2q(t)} - e^{-q(t)}),$$

which can be reduced to another special case of Painlevé III. Asymptotics of a class of solutions to P_{III} including this one were announced in [5].

This paper is devoted to the determination of the asymptotics of the quantities $\det(I - \lambda\, K_k)$ in the general case, under the condition stated below. (In the final sections we shall compare our results in the cases $n = 2$ and 3 with those cited above.) We write K for K_0 and consider at first only the asymptotics of $\det(I-\lambda\, K)$. This is no loss of generality since K_k is obtained from K upon replacing the coefficients c_ω by $\omega^k\, c_\omega$. The problem reduces to the asymptotics of $\int_0^\infty R(u,u;\lambda)\, du$ as $t \to 0$, where $R(u,v;\lambda)$ is the resolvent kernel of K, the kernel of $K\,(I - \lambda\, K)^{-1}$. Using operator techniques, we show that $R(u,u;\lambda)$ is well-approximated on $[1,\infty]$ by the corresponding function when the exponentials in (1.3) are replaced by $e^{-t(1-\omega)u}$ and on $[0,1]$ by the corresponding function when the exponentials are replaced by $e^{-t(1-\omega^{-1})u^{-1}}$. (Actually the kernels have to be modified first by multiplying by factors $(u/v)^\beta$ with β depending on λ.) We shall show that after these replacements we obtain operators which can be transformed into Wiener-Hopf operators, whose resolvent kernels have explicit integral representations. By these means the problem becomes that of determining the asymptotics of certain integrals. This is achieved by contour-shifting, and we find in the end that as $t \to 0$,

$$\det(I - \lambda\, K) \sim b \left(\frac{t}{n}\right)^a \tag{1.4}$$

with a and b constants given explicitly in terms of certain zeros of the function

$$h(s) := \sin\, \pi s - \lambda\, \pi \sum_\omega c_\omega\, (-\omega)^{s-1}.$$

These are the values at λ of those zeros which equal $1, \cdots, n$ when $\lambda = 0$.

To state the result precisely, we denote by $\alpha_k = \alpha_k(\lambda)$ ($k \in \mathbf{Z}$) the zeros of this function indexed so that $\alpha_k(0) = k$. The zeros depend analytically on λ as long as they are unequal, and when $\lambda = 0$ they are the integers. We derive the asymptotics (1.4) under the assumption that there is a path in the complex plane $\mathbf{C}$ running from 0 to λ such that everywhere on the path

$$\Re\,\alpha_0 < \Re\,\alpha_1, \;\; \Re\,\alpha_0 < 1, \;\; \Re\,\alpha_1 > 0, \tag{1.5}$$

and no zero lies in the strip $\Re\,\alpha_0 < \Re\, s < \Re\,\alpha_1$. With this assumption the constants a and b are given by the formulas

$$a = \frac{1}{n}\sum_{\alpha} \alpha^2 - \frac{(n+1)(2n+1)}{6},$$

$$b = \frac{\prod_{|j|<n} G(\frac{j}{n}+1)^{n-|j|}}{\prod_{\alpha,\alpha'} G(\frac{\alpha-\alpha'}{n}+1)},$$

where α and α' run over the set $\{\alpha_1(\lambda), \cdots, \alpha_n(\lambda)\}$ and G denotes the Barnes G-function [1].

From these formulas we obtain the asymptotics of the solutions (1.2). The requirement now is that everywhere on a path from 0 to λ we have for all k,

$$\Re\,\alpha_k < \Re\,\alpha_{k+1}, \;\; k-1 < \Re\,\alpha_k < k+1.$$

If this holds then

$$q_k(t) = A\log\left(\frac{t}{n}\right) + \log B + o(1),$$

where for $k = 1, \cdots, n$ the constants A and B are given by

$$A = 2\,(\alpha_k - k), \;\; B = \prod_{1\le j<k} \frac{\Gamma(\frac{\alpha_j-\alpha_k}{n}+1)}{\Gamma(\frac{\alpha_k-\alpha_j}{n})} \prod_{k<j\le n} \frac{\Gamma(\frac{\alpha_j-\alpha_k}{n})}{\Gamma(\frac{\alpha_k-\alpha_j}{n}+1)},$$

and for other values of k are given by periodicity.

As for the correct range of validity of the formulas, we conjecture that it is enough that $\Re\,\alpha_k < \Re\,\alpha_{k+1}$ for all k for some path from 0 to λ, and that the extra condition $k-1 < \Re\,\alpha_k < k+1$ is automatically satisfied then. In the cases which we consider in detail this is so and we obtain the correct range of validity. Another way of stating the condition is as follows. Define Λ to be the complement of

$$\{\lambda : \Re\,\alpha_k = \Re\,\alpha_{k+1} \text{ for some } k\}. \tag{1.6}$$

Then the region of validity should be the connected component of Λ containing $\lambda = 0$. The region for which we prove the result is the largest connected subset of this set in which the extra condition holds.

Remark. It is shown in [9] that the more general class of kernels

$$\int_{\Omega} \omega^k \, \frac{e^{-t[(1-\omega)u+(1-\omega^{-1})u^{-1}]}}{-\omega u + v}\, d\rho(\omega),$$

gives a solution to the the cylindrical Toda equations by the same formulas. Here ρ can be any finite complex measure supported on a compact subset Ω of

$$\{\omega \in \mathbf{C} : \Re\,\omega < 1,\ \Re\,\omega^{-1} < 1\}.$$

This assures that the operator is trace class. In case Ω is the set of n^{th} roots of unity other than 1 the condition is satisfied and the solution will clearly be n-periodic. We shall actually do everything in the more general case and we find asymptotic formulas of the form (1.4) for the corresponding determinants, with the constants a and b being given by integral formulas involving the function $h(s)$ now defined by

$$h(s) := \sin\,\pi s - \lambda\,\pi \int_\Omega (-\omega)^{s-1}\,d\rho(\omega). \tag{1.7}$$

In the periodic case $h(s)$ is itself periodic and the integrals are expressible in terms of its zeros in a strip of width n. This is why the result there is so explicit. The requirements on the α_k stated above now refer to this function, and we assume throughout that they are satisfied.

2. The Approximating Operators

Recall that K is the operator on $L_2(\mathbf{R}^+)$ with kernel

$$K(u,v) = \int_\Omega \frac{e^{-t[(1-\omega)u+(1-\omega^{-1})u^{-1}]}}{-\omega u + v}\,d\rho(\omega).$$

We denote by $R(u,v;\lambda)$ the resolvent kernel of K, the kernel of $R_\lambda := K\,(I-\lambda\,K)^{-1}$. It is well-known that

$$-\frac{d}{d\lambda}\log\,\det\,(I-\lambda\,K) = \int_0^\infty R(u,u;\lambda)\,du,$$

the trace of the operator R_λ. Hence

$$\log\,\det\,(I-\lambda\,K) = -\int_0^\lambda \int_0^\infty R(u,u;\mu)\,du\,d\mu. \tag{2.1}$$

In this section we are going to find a good approximation to the integral $\int_0^\infty R(u,u;\lambda)\,du$ when λ satisfies the condition stated in the Introduction. (Afterwards we shall replace λ by μ and integrate with respect to μ over the path from 0 to λ throughout which (1.5) holds.) We begin with an observation. If we multiply the kernel $K(u,v)$ above by $(u/v)^\beta$ for any β then the resulting kernel still represents a bounded operator on $L_2(\mathbf{R}^+)$ because of the decay of the exponential factor at 0 and ∞ and, although the resolvent kernel changes, its value on the diagonal $u = v$ does not. We are going to find approximations to the resolvent kernels for these modified operators, and precisely which β we take depends on λ.

Here is how we choose it. It follows from our main assumption that for each λ there there exists $s_\lambda \in (0,1)$ such that the function $h(s)$ given by (1.7) has no zeros on the line $\Re\,s = s_\lambda$. (In fact the assumption guarantees that s_λ can be chosen to vary continuously with λ.) With this s_λ we set $\beta = \frac{1}{2} - s_\lambda$. Notice that $|\beta| < \frac{1}{2}$, a fact we shall need in order to apply our approximation argument.

We write the kernel as

$$K_t(u,v) = \left(\frac{u}{v}\right)^\beta \int_\Omega \frac{e^{-t[(1-\omega)u+(1-\omega^{-1})u^{-1}]}}{-\omega u+v}\, d\rho(\omega) \tag{2.2}$$

and denote the operator itself by K_t. We do not display the dependence on β, which is fixed for now, but use the subscript t to help the reader distinguish those operators that depend on t from those that don't. Both kinds will arise; the former have the subscript t and the latter will not.

The exponential in (2.2) is the product $e^{-t(1-\omega)u}\, e^{-t(1-\omega^{-1})u^{-1}}$. For $u \geq 1$ the second factor is uniformly close to 1 when t is small while for $u \leq 1$ the first factor is uniformly close to 1. This suggests that the operators $K_t^\pm$ with kernels

$$K_t^+(u,v) := \left(\frac{u}{v}\right)^\beta \int_\Omega \frac{e^{-t(1-\omega)u}}{-\omega u+v}\, d\rho(\omega), \quad K_t^-(u,v) := \left(\frac{u}{v}\right)^\beta \int_\Omega \frac{e^{-t(1-\omega^{-1})u^{-1}}}{-\omega u+v}\, d\rho(\omega),$$

should in some sense approximate K_t on $u \geq 1,\ u \leq 1$, respectively, and therefore the resolvent kernels of these operators should approximate the resolvent kernel of K_t on these intervals. We shall show that this is so, and that if $R_t^\pm(u,v;\lambda)$ denote the resolvent kernels of $K_t^\pm$, also on $\mathbf{R}^+$, then

$$\begin{aligned} \int_0^1 R(u,u;\lambda)\, du &= \int_0^1 R_t^-(u,u;\lambda)\, du + o(1), \\ \int_1^\infty R(u,u;\lambda)\, du &= \int_1^\infty R_t^+(u,u;\lambda)\, du + o(1) \end{aligned} \tag{2.3}$$

as $t \to 0$. We denote by P^+ multiplication by the characteristic function of $(1,\infty)$ and by P^- multiplication by the characteristic function of $(0,1)$. We shall use the notation $o_1(\lambda)$ to denote any family of operators whose trace norms are at most $|\lambda|$ times a function of t which is $o(1)$ as $t \to 0$. (The subscript 1 refers to the trace norm. We shall also use the obvious notation $o_1(1)$ later on.) The main approximation statement will be

$$P^\pm (I - \lambda K_t)^{-1} P^\pm = P^\pm (I - \lambda K_t^\pm)^{-1} P^\pm + o_1(\lambda). \tag{2.4}$$

Relations (2.3) with $\lambda = 1$ follow from this since it may be rewritten

$$P^\pm [(I - \lambda K_t)^{-1} - I]\, P^\pm = P^\pm [(I - \lambda K_t^\pm)^{-1} - I]\, P^\pm + o_1(\lambda),$$

and if we take the trace of both sides and divide by λ we obtain (2.3).

Here is an outline of the proof of (2.4). We use the matrix representations of our operators corresponding to the decomposition of $L_2(\mathbf{R}^+)$ as the direct sum of the spaces $L_2(0,1)$ and $L_2(1,\infty)$. Thus (the equal sign meaning "has matrix representation")

$$I - \lambda K_t = \begin{pmatrix} I - \lambda P^- K_t P^- & -\lambda P^- K_t P^+ \\ -\lambda P^+ K_t P^- & I - \lambda P^+ K_t P^+ \end{pmatrix}.$$

Because the nondiagonal corners of the matrix have the mutually orthogonal projections $P^\pm$ occurring as they do we will be able, with error $o_1(\lambda)$, to replace the operator K_t appearing there by the operator K_0 obtained from it by setting $t = 0$. Thus K_0 has kernel

$$K_0(u,v) = \left(\frac{u}{v}\right)^\beta \int_\Omega \frac{1}{-\omega u + v}\, d\rho(\omega). \tag{2.5}$$

(Note the lack of consistency with the notation K_k in the introduction; this should cause no confusion.) If the diagonal entries $I - \lambda\, P^\pm K_t P^\pm$ are invertible, we can write the resulting matrix as the product of

$$\begin{pmatrix} I - \lambda\, P^- K_t P^- & 0 \\ 0 & I - \lambda\, P^+ K_t P^+ \end{pmatrix}$$

on the left and

$$\begin{pmatrix} I & -\lambda\,(I - \lambda\, P^- K_t P^-)^{-1} P^- K_0 P^+ \\ -\lambda\,(I - \lambda\, P^+ K_t^+ P^+)^{-1} P^+ K_0 P^- & I \end{pmatrix}$$

on the right. Next, because of our assumption we shall be able to show that the operators $I - \lambda\, P^\pm K_t P^\pm$ are uniformly invertible for small t (i.e., the operator norms of their inverses are bounded) and that their inverses converge strongly to $(I - \lambda\, P^\pm K_0 P^\pm)^{-1}$ as $t \to 0$. (Recall that A_t is said to converge strongly to A as $t \to 0$ if $A_t f \to Af$ for all f in the underlying space.) This is actually the crux of the proof. After that it follows, because $P^\pm K_0 P^\mp$ are trace class, that with error $o_1(\lambda)$ we can replace $(I - \lambda\, P^\pm K_t P^\pm)^{-1}$ by $(I - \lambda\, P^\pm K_0 P^\pm)^{-1}$ in the nondiagonal entries. Thus, if we define

$$\mathcal{M} := \begin{pmatrix} I & -\lambda\,(I - \lambda\, P^- K_0 P^-)^{-1} P^- K_0 P^+ \\ -\lambda\,(I - \lambda\, P^+ K_0^+ P^+)^{-1} P^+ K_0 P^- & I \end{pmatrix},$$

we will have shown

$$I - \lambda\, K_t = \begin{pmatrix} I - \lambda\, P^- K_t P^- & 0 \\ 0 & I - \lambda\, P^+ K_t P^+ \end{pmatrix} \mathcal{M} + o_1(\lambda).$$

From this, using the uniform invertibility of the $I - \lambda\, P^\pm K_t P^\pm$ again and the invertibility of the constant matrix $\mathcal{M}$ (which we have to prove) we deduce

$$(I - \lambda\, K_t)^{-1} = \mathcal{M}^{-1} \begin{pmatrix} (I - \lambda\, P^- K_t P^-)^{-1} & 0 \\ 0 & (I - \lambda\, P^+ K_t P^+)^{-1} \end{pmatrix} + o_1(\lambda).$$

Now (here is the trick), applying an analogous procedure to the operator family $I - \lambda\, K_t^+$ gives

$$(I - \lambda\, K_t^+)^{-1} = \mathcal{M}^{-1} \begin{pmatrix} (I - \lambda\, P^- K_0 P^-)^{-1} & 0 \\ 0 & (I - \lambda\, P^+ K_t P^+)^{-1} \end{pmatrix} + o_1(\lambda).$$

It is clear that the lower-right entries of the two matrix products are the same, and this is exactly the statement

$$P^+ (I - \lambda\, K_t)^{-1}\, P^+ = P^+ (I - \lambda\, K_t^+)^{-1}\, P^+ + o_1(\lambda),$$

which is half of (2.4). The other half is obtained similarly.

Carrying out the details of the proof of this will require, first, some general facts about families of operators on a Hilbert space.

Fact 1. If A_t converges strongly to an invertible operator A as $t \to 0$ and if the A_t are uniformly invertible then A_t^{-1} also converges strongly to A^{-1}.

This follows from the assumptions and the identity $A_t^{-1} - A^{-1} = A_t^{-1}(A - A_t)A^{-1}$. The next fact says that strong convergence can sometimes be converted into trace norm convergence.

Fact 2. If $A_t \to A$ strongly and $B_t \to B$ in trace norm then $A_t B_t \to AB$ in trace norm.

This is a variant of Proposition 2.1 of [8]. There the families of operators depended on a parameter $n \in \mathbf{Z}^+$ rather than $t \in \mathbf{R}^+$, a matter of no importance since we may consider general sequences $t_n \to 0$. Also, instead of a sequence of operators converging in trace norm to B there was the single trace class operator B. The apparently more general result follows trivially from this special case.

Fact 3. Suppose A_t and A are as in Fact 1 and that B_t are trace class operators converging in trace norm to B. Assume also that $A+B$ is invertible. Then the $A_t + B_t$ are uniformly invertible for sufficiently small t, and if $B_t = o_1(1)$ then $(A_t + B_t)^{-1} = A_t^{-1} + o_1(1)$.

Proof. Write $A_t + B_t = A_t\,(I + A_t^{-1}B_t)$. By Fact 1 $A_t^{-1} \to A^{-1}$ strongly, and so by Fact 2 (with A_t repaced by A_t^{-1}) we deduce $I + A_t^{-1}B_t = I + A^{-1}B + o_1(1) = (I + o_1(1))(I + A^{-1}B)$, since clearly $(I + o_1(1))^{-1} = I + o_1(1)$. Both statements now follow. □

In our derivation of (2.4) we have to know that the $I - P^{\pm}K_t^{\pm}P^{\pm}$ are uniformly invertible for small t. We shall deduce this from known facts about uniform invertibility of truncated Wiener-Hopf operators, which we now describe. The proofs can be found in [4].

The Wiener-Hopf operator W associated with a function $k \in L_1(\mathbf{R})$ is the operator on $L_2(\mathbf{R}^+)$ with kernel $k(x-y)$. Introduce the Fourier transform of k,

$$\hat{k}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\, k(x)\, dx.$$

This is a continuous function on $\mathbf{R}$ tending to 0 as $\xi \to \pm\infty$. A necessary and sufficient condition that $I - W$ be invertible is that $1 - \hat{k}(\xi) \neq 0$ for all ξ, and

$$\arg\,(1 - \hat{k}(\xi))\Big|_{-\infty}^{\infty} = 0.$$

The truncated Wiener-Hopf operators are the operators $P_\alpha W P_\alpha$, where P_α denotes multiplication by the characteristic function of $(0,\alpha)$. Clearly these operators converge strongly to W as $\alpha \to \infty$. The important fact is that if W is invertible, in other words if the conditions on $\hat{k}$ stated above hold, then the operators $I - P_\alpha W P_\alpha$ are uniformly invertible for sufficiently large α. We mention also that the operator with kernel $k(x-y)$ on the whole line $\mathbf{R}$ is invertible if and only if the first condtion alone is satisfied, that $1 - \hat{k}(\xi) \neq 0$ for all ξ.

From this we can deduce information about kernels $k(u,v)$ which are homogeneous of degree -1 since the variable change $u = e^x$ transforms this kernel into a convolution kernel. More precisely, denote by U the unitary operator from $L_2(1,\infty)$ to $L_2(0,\infty)$ given by $Uf(x) = e^{x/2}f(e^x)$. Then if T denotes the operator on $L_2(1,\infty)$ with kernel $k(u,v)$, the operator UTU^{-1} is the operator on $L_2(0,\infty)$ with kernel

$$e^{x/2}\,e^{y/2}\,k(e^x,e^y) = e^{(y-x)/2}\,k(1,e^{y-x}),$$

where we used the homogeneity of $k(u,v)$. Notice also that if P_t^+ denotes multiplication by the characteristic function of $(1,t^{-1})$ then $UP_t^+U^{-1}$ is the projection operator $P_{\log t^{-1}}$ of the last paragraph. After making an obvious change of variable in computing the Fourier transform of $e^{-x/2}\,k(1,e^{-x})$ we deduce

Fact 4. Assume that $\int_0^\infty v^{-1/2}\,|k(1,v)|\,dv < \infty$, denote by T the operator with kernel $k(u,v)$ on $L_2(1,\infty)$ with kernel $k(u,v)$, and by $M(s)$ the Mellin transform

$$M(s) := \int_0^\infty v^{s-1}\,k(1,v)\,dv.$$

Then a necessary and sufficient condition that $I - T$ be invertible is that

$$1 - M(s) \neq 0 \text{ for } \Re\, s = \tfrac{1}{2}, \quad \arg\,(1 - M(s))\Big|_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} = 0. \tag{2.6}$$

If this holds then the operators $I - P_t^+TP_t^+$ are uniformly invertible for sufficiently small t.

Remark. If we use the variable change $u = e^{-x}$ instead of $u = e^x$ then our operator T acts on $L_2(0,1)$ and we find (again using homogeneity) that the condition for invertibility of $I - T$ is exactly the same as before, and that if P_t^- denotes multiplication by the characteristic function of $(t,1)$ then this condition implies the uniform invertibility of $I - P_t^-TP_t^-$ for sufficiently small t. Also (transferring to this context the last sentence of the discussion of Wiener-Hopf operators), the same condition implies the invertibility of the operator $I - T$ on $L_2(0,\infty)$.

We apply this to the kernel λK_0, which is homogeneous of degree -1. The relevant Mellin transform is found to be

$$M(s) = \lambda\,\frac{\pi}{\sin \pi s}\,\int_\Omega (-\omega)^{s-\beta-1}\,d\rho(\omega),$$

so that $1 - M(s) = h(s - \beta)$, where h is given by (1.7). If we recall that $\beta = \frac{1}{2} - s_\lambda$ we see that the conditions (2.6) are met, the first immediately from the definition of s_λ and the second because the index (the variation of the argument, which is necessarily an integer) is a continuous function of λ, locally constant and clearly equal to 0 when $\lambda = 0$. So we know that the operators $I - \lambda\,K_0$ and $I - \lambda\,P^\pm K_0 P^\pm$ are invertible and the operator families $I - \lambda\,P_t^\pm K_0 P_t^\pm$ are uniformly invertible for small t.

We introduce one last piece of notation. We denote by $K^\pm$ the operators on $L_2(\mathbf{R}^+)$ with kernels

$$K^+(u,v) = \left(\frac{u}{v}\right)^\beta \int_\Omega \frac{e^{-(1-\omega)\,u}}{-\omega u + v}\,d\rho(\omega), \qquad K^-(u,v) = \left(\frac{u}{v}\right)^\beta \int_\Omega \frac{e^{-(1-\omega^{-1})\,u^{-1}}}{-\omega u + v}\,d\rho(\omega).$$

Notice that rescaling $K^{\pm}(u,v)$ under the variable change $u \to t^{\pm 1}u$ gives $K_t^{\pm}(u,v)$.

Next we derive various trace class properties of our operators which will be needed. For these we shall use an estimate for the trace norm of an operator on $L_2(\mathbf{R}^+)$ with kernel of the form

$$\int_{\Omega} \frac{q_1(\omega,u)\,q_2(\omega,v)}{-\omega u+v}\,d\rho(\omega), \tag{2.7}$$

which is a special case of the sublemma in the appendix of [9]. We denote by $\phi(s)$ any positive function on $\mathbf{R}^+$, by Φ its Laplace transform, and by Ψ the Laplace transform of $\phi(s)^{-1}$. Then there is a constant m depending on only on Ω such that the trace norm of the operator with kernel given above is at most m^{-1} times the square root of

$$\int_0^{\infty}\int_{\Omega}|q_1(\omega,u)|^2\,\Phi(mu)\,d\rho(\omega)\,du\cdot\int_0^{\infty}\int_{\Omega}|q_2(\omega,u)|^2\,\Psi(mu)\,d\rho(\omega)\,du. \tag{2.8}$$

From this will follow our first lemma. In the proof we denote by χ^+ the characteristic function of $(1,\infty)$ and by χ^- the characteristic function of $(0,1)$, so $P^{\pm}$ is multiplication by $\chi^{\pm}$.

Lemma 1. *The operators (independent of t)*

$$P^+K_0P^-,\ \ P^+K^+,\ \ P^-(K^+-K_0)$$

are trace class. The operators (depending on t)

$$P^+(K_t-K_0)P^-,\ \ P^-(K_t^+-K_0),\ \ P^+(K_t-K_t^+)$$

are $o_1(1)$ as $t\to 0$. The statements also hold if all superscripts + and − are interchanged.

Proof. All the operators have kernel of the form (2.7). We list the operators below, together with the corresponding functions $q_1(\omega,u)$ and $q_2(\omega,u)$.

Operator	$q_1(\omega,u)$	$q_2(\omega,u)$
$P^+K_0P^-$	$\chi_{(1,\infty)}(u)\,u^{\beta}$	$\chi_{(0,1)}(u)\,u^{-\beta}$
P^+K^+	$\chi_{(1,\infty)}(u)\,e^{-(1-\omega)\,u}\,u^{\beta}$	$u^{-\beta}$
$P^-(K^+-K_0)$	$\chi_{(0,1)}(u)\,(e^{-(1-\omega)\,u}-1)\,u^{\beta}$	$u^{-\beta}$
$P^+(K_t-K_0)P^-$	$\chi_{(1,\infty)}(u)\,(e^{-t\,[(1-\omega)\,u+(1-\omega^{-1})\,u^{-1}]}-1)\,u^{\beta}$	$\chi_{(0,1)}(u)\,u^{-\beta}$
$P^-(K_t^+-K_0)$	$\chi_{(0,1)}(u)\,(e^{-t(1-\omega)\,u}-1)\,u^{\beta}$	$u^{-\beta}$
$P^+(K_t-K_t^+)$	$\chi_{(1,\infty)}(u)\,e^{-t(1-\omega)\,u}\,(e^{-t(1-\omega^{-1})\,u^{-1}}-1)\,u^{\beta}$	$u^{-\beta}$.

For each of these operators we take two numbers $p,q\in(-1,1)$ and define $\phi(s)=s^p$ for $s\le 1$ and $\phi(s)=s^q$ for $s\ge 1$. We easily see that

$$\Phi(s)=\begin{cases}O(u^{-p-1})\ \text{for } u\ge 1\\[2ex] O(u^{-q-1})\ \text{for } u\le 1,\end{cases}\qquad \Psi(s)=\begin{cases}O(u^{p-1})\ \text{for } u\ge 1\\[2ex] O(u^{q-1})\ \text{for } u\le 1.\end{cases}$$

For each operator one can find p and q such that both integrals in (2.8) are finite, and any integral depending on t is $o(1)$ as $t\to 0$. In fact, as the reader can check, we may take for all the operators any $q\in(2\beta,2\beta+2)$, for the first and fourth operators any $p>2\beta$ and for the other four any $p\in(2\beta-2,2\beta)$. This takes care of the six displayed operators. For the other six we use the fact that the substitutions $u\to u^{-1},\ \omega\to\omega^{-1}$ yield operators of the same form with β replaced with $-\beta$ and with the superscripts interchanged.

As a preliminary to the next lemma we show that certain modified Laplace tranforms are bounded operators on $L_2(\mathbf{R}^+)$.

Lemma 2. *The integral operator on* $L_2(\mathbf{R}^+)$ *with kernel* $(uv)^{-\beta}\, e^{-uv}$ *is bounded if* $\beta < \frac{1}{2}$.

Proof. The mapping $f(v) \to v^{-1} f(v^{-1})$ is unitary, so we may replace the kernel by $u^{-\beta} v^{\beta-1}\, e^{-u/v}$. Under the unitary mapping $f(u) \to e^{x/2}\, f(e^x)$ this becomes the kernel $e^{(\frac{1}{2}-\beta)(x-y)} e^{-e^{x-y}}$ on $(-\infty, \infty)$. Thus the operator becomes convolution by the L_1 function $k(x) = e^{(\frac{1}{2}-\beta)x} e^{-e^x}$ and so is bounded. $\square$

Lemma 3. *The operators* $I - \lambda\, K^{\pm}$ *are invertible.*

Proof. We consider K^+, which we can write as $P^- K_0 + P^-(K^+ - K_0) + P^+ K^+$. By the first part of Lemma 1 the second and third summands are trace class, therefore certainly compact. Our assumption implies that $I - \lambda\, P^- K_0$ is invertible. Hence $I - \lambda\, K^+$ is the sum of an invertible operator and a compact operator, and so it follows from general theory that it will be invertible if 0 is not an eigenvalue. In other words it suffices to prove that $\lambda\, K^+ f = f$ for $f \in L_2(\mathbf{R}^+)$ implies $f = 0$.

For any $\omega \in \mathbf{C} \backslash \mathbf{R}^+$ and any $f \in L_2(\mathbf{R}^+)$ we have for $x > 0$,

$$\int_0^\infty e^{-xu} u^{-\beta}\, du \int_0^\infty \left(\frac{u}{v}\right)^\beta \frac{e^{-(1-\omega)\, u}}{-\omega u + v}\, f(v)\, dv$$
$$= \int_0^\infty \frac{dy}{x + 1 - \omega(y+1)} \int_0^\infty e^{-yv}\, v^{-\beta} f(v)\, dv.$$

This can be seen for $\Re\, \omega < 0$ by using the integral reresentation

$$\frac{1}{-\omega u + v} = \int_0^\infty e^{-y(-\omega u + v)}\, dy$$

in the integral on the left above and interchanging the order of integration. The identity follows for all $\omega \in \Omega$ since both sides are analytic functions of ω in this domain. Now suppose that $\lambda\, K^+ f = f$. Then if we integrate both sides of the identity with respect to $d\rho(\omega)$ and multiply by λ, the left side becomes the Laplace transform of $u^{-\beta}\, f(u)$, which we denote by $g(x)$, and the the right side becomes

$$\lambda \int_0^\infty \int_\Omega \frac{d\rho(\omega)}{x + 1 - \omega(y+1)}\, g(y)\, dy.$$

Using the fact $y^{-\beta} g(y) \in L_2$, which we know by the previous lemma, we see that the integral is a bounded function of x. If we recall the definition (2.5) then we see that the identity becomes

$$g(x) = \lambda \int_0^\infty \left(\frac{x+1}{y+1}\right)^\beta K_0(y+1, x+1)\, g(y)\, dy \quad (x \geq 0),$$

or

$$x^{-\beta}\, g(x-1) = \lambda \int_1^\infty K_0(y, x)\, g(y-1)\, y^{-\beta}\, dy \quad (x \geq 1).$$

Now we know that $x^{-\beta}\, g(x)$ is in $L_2(0, \infty)$ and that $g(x)$ is bounded. It follows that $x^{-\beta}\, g(x - 1)$ belongs to $L_2(1, \infty)$. The right side above is the operator with kernel

$K_0(y,x)$ acting on this function. Thus (if $f \not\equiv 0$) the operator $\lambda\, P^+ K_0' P^+$ has 1 as an eigenvalue, where $'$ denotes transpose, so $I - \lambda\, P^+ K_0' P^+$ is not invertible. But this implies $I - \lambda\, P^+ K_0 P^+$ is not invertible, whereas we know that it is. This contradiction establishes the lemma.

Lemma 4. *The operators $I - \lambda\, P^\pm K_t^\pm P^\pm$ are uniformly invertible for sufficiently small t.*

Proof. We consider $I - \lambda\, P^+ K_t^+ P^+$ and for this it is enough to show that the $I - \lambda P^+ K_t^+$ are uniformly invertible. The kernel of $P^+ K_t^+$ is $\chi_{(1,\infty)}(u)\, K_t^+(u,v)$ and the substitution $u \to t^{-1} u$ allows us to consider instead the operator with kernel $\chi_{(t,\infty)}(u)\, K^+(u,v)$. We write this (not displaying the variables u and v) as

$$\chi_{(t,1)}\, K_0 + \chi_{(1,\infty)}\, K^+ + \chi_{(0,1)}\, (K^+ - K_0) + \chi_{(0,t)}\, (K_0 - K^+).$$

Recalling the definitions of our various projection operators we see that the first kernel corresponds to the operator $P_t^- K_0$, and we know that the $I - \lambda\, P_t^- K_0$ are uniformly invertible for sufficiently small t. The second and third summands correspond to the operators $P^+ K^+$ and $P^+ (K^+ - K_0)$, which we know by Lemma 1 to be trace class. The last summand corresponds to the operator $P^- (K_0 - K^+)$, which we know by Lemma 1 to be trace class, left-multiplied by multiplication by $\chi_{(0,t)}$, which converges strongly to 0. An application of Fact 2 shows that this last operator is $o_1(1)$. The strong limit of the sum of the four operators is, of course, K^+ and we know by Lemma 2 that $I - \lambda\, K^+$ is invertible. Hence we can apply Fact 3 to deduce the result.

We can now fill in the details of the proof of (2.4) outlined earlier. Thus we begin with the representation

$$I - \lambda\, K_t = \begin{pmatrix} I - \lambda\, P^- K_t P^- & -\lambda\, P^- K_t P^+ \\ -\lambda\, P^+ K_t P^- & I - \lambda\, P^+ K_t P^+ \end{pmatrix}.$$

Applying Lemma 1 to the nondiagonal entries we deduce

$$I - \lambda\, K_t = \begin{pmatrix} I - \lambda\, P^- K_t^- P^- & -\lambda\, P^- K_0 P^+ \\ -\lambda\, P^+ K_0 P^- & I - \lambda\, P^+ K_t^+ P^+ \end{pmatrix} + o_1(\lambda).$$

Lemma 3 tells us in particular that the diagonal entries of this matrix are invertible for small t so we may factor out

$$\begin{pmatrix} I - \lambda\, P^- K_t P^- & 0 \\ 0 & I - \lambda\, P^+ K_t P^+ \end{pmatrix}$$

on the left, leaving

$$\begin{pmatrix} I & -\lambda\,(I - \lambda\, P^- K_t^- P^-)^{-1} P^- K_0 P^+ \\ -\lambda\,(I - \lambda\, P^+ K_t^+ P^+)^{-1} P^+ K_0 P^- & I \end{pmatrix}$$

on the right. Next we combine the uniform invertibility of the $I - \lambda\, P^\pm K_t^\pm P^\pm$ proved in Lemma 3 with Fact 1 to deduce that the inverses of these operators converge strongly to $(I - \lambda\, P^{\pm} K_0^{\pm} P^{\pm})^{-1}$. Since $P^{\pm} K_0 P^{\mp}$ are trace class, by Lemma 1, we deduce by Fact 2 that the matrix above is $\mathcal{M} + o_1(\lambda)$, where $\mathcal{M}$ is the matrix obtained by replacing $K_t^{\pm}$ by K_0. Thus

$$I - \lambda\, K_t = \begin{pmatrix} I - \lambda\, P^{-} K_t P^{-} & 0 \\ 0 & I - \lambda\, P^{+} K_t P^{+} \end{pmatrix} \mathcal{M} + o_1(\lambda).$$

Now we have to know that $\mathcal{M}$ is invertible, and we see this as follows. If, instead of the operator $I - \lambda\, K_t$ which depends on t, we had started with the operator $I - \lambda K_0$ then we would have obtained the exact representation

$$I - \lambda\, K_0 = \begin{pmatrix} I - \lambda\, P^{-} K_0 P^{-} & 0 \\ 0 & I - \lambda\, P^{+} K_0 P^{+} \end{pmatrix} \mathcal{M}.$$

Since both $I - \lambda\, K_0$ and the matrix on the left are invertible, by our assumption, we deduce that $\mathcal{M}$ is invertible.

Next we go through a similar process starting with the operator $I - \lambda\, K_t^{+}$ rather than $I - \lambda\, K$. Using the fact that $P^{-} K_t^{+} = P^{-} K_0 + o_1(1)$, which we know by Lemma 1, we obtain in this case

$$I - \lambda\, K_t^{+} = \begin{pmatrix} I - \lambda\, P^{-} K_0 P^{-} & 0 \\ 0 & I - \lambda\, P^{+} K_t P^{+} \end{pmatrix} \mathcal{M} + o_1(\lambda).$$

From these matrix representations and the facts that $\mathcal{M}$ and $I - \lambda\, P^{-} K_0 P^{-}$ are invertible and $I - P^{\pm} K_t^{\pm} P^{\pm}$ uniformly invertible we deduce, using Fact 3 with $o_1(1)$ replaced by $o_1(\lambda)$,

$$(I - \lambda\, K_t)^{-1} = \mathcal{M}^{-1} \begin{pmatrix} (I - \lambda\, P^{-} K_t P^{-})^{-1} & 0 \\ 0 & (I - \lambda\, P^{+} K_t P^{+})^{-1} \end{pmatrix} + o_1(\lambda),$$

$$(I - \lambda\, K_t^{+})^{-1} = \mathcal{M}^{-1} \begin{pmatrix} (I - \lambda\, P^{-} K_0 P^{-})^{-1} & 0 \\ 0 & (I - \lambda\, P^{+} K_t P^{+})^{-1} \end{pmatrix} + o_1(\lambda).$$

Comparing lower-right entries of the matrices gives

$$P^{+} (I - K)^{-1} P^{+} = P^{+} (I - K_t^{+})^{-1} P^{+} + o_1(\lambda),$$

which is half of (2.4). The other half is obtained similarly.

Remark. To apply (2.3) to (2.1) we need something extra, e.g., that (2.3) holds uniformly for these λ. With a little care our argument gives this also, but we spare the reader the details.

3. The Resolvents of $K^{\pm}$

We are going to find integral representations for the integrals on the right side of (2.3), and we consider $\int_1^\infty R_t^+(u,u;\lambda)\,du$ first. The substitution $u \to u/t$ shows that this equals $\int_t^\infty R^+(u,u;\lambda)\,du$, where $R^+(u,v;\lambda)$ is the resolvent kernel of the operator K^+. For this we require only that Ω be a compact subset of

$$\{\omega \in \mathbf{C}:\ \Re\,\omega < 1,\ \omega \notin \mathbf{R}^+\}, \tag{3.1}$$

since the term $1-\omega^{-1}$ does not appear in the exponent in the kernel of K^+. The derivation will involve an initial step which is valid only when Ω is contained in the left half-plane so we assume this to begin with. We shall also assume that λ is so small that $h(s) \neq 0$ for $\Re\,s = \frac{1}{2}$, so that with the notation of the last section we may take $s_\lambda = \frac{1}{2},\ \beta = 0$. Eventually these two assumptions will be removed by an analytic continuation argument.

Because $\beta = 0$ the kernel of K^+ is

$$K^+(u,v) = \int_\Omega \frac{e^{-t(1-\omega)u}}{-\omega u + v}\,d\rho(\omega). \tag{3.2}$$

If we set

$$A(u,x) := \int_\Omega e^{-(1-\omega)u}\,e^{\omega u x}\,d\rho(\omega), \quad B(x,u) := e^{-ux},$$

then

$$K^+(u,v) = \int_0^\infty A(u,x)\,B(x,v)\,dx.$$

Lemma 2 of the preceding section tells us that $B(u,x)$ is the kernel of a bounded operator from $L_2(\mathbf{R}^+)$ to $L_2(\mathbf{R}^+)$ and, with our assumption on Ω, that the same is true of $A(x,u)$. The above shows that $K^+ = AB$, and the operator BA has kernel

$$\int_0^\infty B(x,u)\,A(u,y)\,du = \int_\Omega \frac{d\rho(\omega)}{x - \omega y + 1 - \omega} = \int_\Omega \frac{d\rho(\omega)}{(x+1) - \omega(y+1)}.$$

We use the general fact $AB(I-\lambda AB)^{-1} = A(I-\lambda BA)^{-1}B$ to deduce that $R^+(u,u)$ is given by an inner product,

$$R^+(u,u) = \Big((I-\lambda BA)^{-1}B(\,\cdot\,,u),\, A(u,\,\cdot\,)\Big). \tag{3.3}$$

We begin by computing

$$f := (I-\lambda BA)^{-1}B(\,\cdot\,,u).$$

Thus we want to solve

$$f(x) - \lambda \int_0^\infty \int_\Omega \frac{d\rho(\omega)}{(x+1)-\omega(y+1)}\,f(y)\,dy = e^{-ux} \qquad (x \geq 0),$$

or

$$f(x-1) - \lambda \int_1^\infty \int_\Omega \frac{d\rho(\omega)}{x - \omega y}\,f(y-1)\,dy = e^{-u(x-1)} \qquad (x \geq 1).$$

The substitution $x \to e^x$ brings this to the form of a Wiener-Hopf equation, so we can use the factorization method to find the solution.

We begin by decreeing that the last identity holds for all $x \geq 0$, in other words we define f on $(-1,0)$ by the identity. Then we define

$$F_-(s) := \int_1^\infty x^{s-1}\, f(x-1)\, dx, \qquad F_+(s) := \int_0^1 x^{s-1}\, f(x-1)\, dx.$$

These belong to the Hardy spaces $H_2(\Re s < \frac{1}{2}),\ H_2(\Re s > \frac{1}{2})$, respectively. We take Mellin transforms of both sides of the equation, and find that for $\Re s = \frac{1}{2}$,

$$F_-(s) + F_+(s) - \frac{\lambda\,\pi}{\sin\,\pi s}\int_\Omega (-\omega)^{s-1}\, d\rho(\omega)\, F_-(s) = e^u\, u^{-s}\,\Gamma(s).$$

(The exponential in the integral is made definite by taking $|\arg(-\omega)| < \pi$.) We write this as

$$H(s)\, F_-(s) + F_+(s) = e^u\, u^{-s}\,\Gamma(s), \tag{3.4}$$

where

$$H(s) := \frac{h(s)}{\sin\,\pi s} = 1 - \frac{\lambda\,\pi}{\sin\,\pi s}\int_\Omega (-\omega)^{s-1}\, d\rho(\omega). \tag{3.5}$$

This function is bounded and analytic in each vertical strip of the complex s-plane, away from the zeros of $\sin\,\pi s$, $H(s) - 1 \to 0$ exponentially as $\Im s \to \pm\infty$ and

$$\arg\,H(s)\Big|_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} = 0.$$

Thus there is be a representation

$$H(s) = \frac{H_-(s)}{H_+(s)},$$

where $H_-(s)^{\pm 1}$ are bounded and analytic in $\Re s \le \frac{1}{2} + \delta$ for some $\delta > 0$ and $H_+(s)^{\pm 1}$ are bounded and analytic in $\Re s \ge \frac{1}{2} - \delta$. We multiply (3.4) by $H_+(s)$ and use the decomposition $F = F_- + F_+$ of an arbitrary function in $L_2(\Re s = \frac{1}{2})$ into boundary functions of functions in $H_2(\Re s < \frac{1}{2})$ and $H_2(\Re s > \frac{1}{2})$ to write the result as

$$H_-(s)\, F_-(s) - e^u \left(u^{-s}\,\Gamma(s)\, H_+(s)\right)_- = -H_+(s)\, F_+(s) + e^u \left(u^{-s}\,\Gamma(s)\, H_+(s)\right)_+.$$

The two sides are boundary functions of functions in $H_2(\Re s < \frac{1}{2})$ and $H_2(\Re s > \frac{1}{2})$, respectively, so they both vanish. This gives the representation

$$F_-(s) = \frac{e^u}{H_-(s)}\left(u^{-s}\,\Gamma(s)\, H_+(s)\right)_-. \tag{3.6}$$

Now (see (3.3)) we have to multiply $f(x)$ by $A(u, x)$ and integrate with respect to x over $(0, \infty)$. This is

$$\int_\Omega d\rho(\omega)\int_0^\infty f(x)\, e^{-(1-\omega)u}\, e^{\omega u x}\, dx = e^{-u}\int_\Omega d\rho(\omega)\int_0^\infty f(x-1)\,\chi_{(1,\infty)}(x)\, e^{\omega u x}\, dx.$$

The Mellin transform of $f(x-1)\,\chi_{(1,\infty)}(x)$ equals $F_-(s)$ and the Mellin transform of $e^{\omega u x}$ equals $(-\omega u)^{-s}\,\Gamma(s)$, so Parseval's formula for Mellin transforms shows that the above equals

$$e^{-u}\int\int_\Omega (-\omega u)^{s-1}\, d\rho(\omega)\,\Gamma(1-s)\, F_-(s)\frac{ds}{2\pi i},$$

the outer integration taken over $\Re\, s = \frac{1}{2}$. (All vertical integrals are taken in the direction from $-i\infty$ to $i\infty$.) Next we recall (3.6) and use the integral representation of the operator $G \to G_-$ to write the above as

$$\int \int_\Omega (-\omega u)^{s-1}\, d\rho(\omega)\, \frac{\Gamma(1-s)}{H_-(s)}\, \frac{ds}{2\pi i} \int \frac{u^{-s'}\,\Gamma(s')H_+(s')}{s'-s}\, \frac{ds'}{2\pi i},$$

the inner integral taken over $\Re\, s' = \frac{1}{2} + \delta$. Alternatively, this may be written

$$\int \int_\Omega (-\omega)^{s-1}\, d\rho(\omega)\, \frac{\Gamma(1-s)}{H_-(s)}\, \frac{ds}{2\pi i} \int \frac{u^{-s'-1}\,\Gamma(s'+s)H_+(s'+s)}{s'}\, \frac{ds'}{2\pi i},$$

where now the inner integral is taken over $\Re\, s' = \delta$. The integrands of these integrals vanish exponentially at infinity, and u occurs to the power $-s'-1$, which has real part $-\delta - 1$. Thus we may integrate with respect to u from t to ∞ under the integral signs and deduce that

$$\int_t^\infty R^+(u,u;\lambda)\, du$$
$$= \int \int_\Omega (-\omega)^{s-1}\, d\rho(\omega)\, \frac{\Gamma(1-s)}{H_-(s)}\, \frac{ds}{2\pi i} \int \frac{t^{-s'}\,\Gamma(s'+s)H_+(s'+s)}{s'^2}\, \frac{ds'}{2\pi i}.$$

It follows from (3.5), and the gamma function representation of the last factor there, that

$$\int_\Omega (-\omega)^{s-1}\, d\rho(\omega)\, \Gamma(1-s) = \lambda^{-1}\,(1 - H(s))\,\Gamma(s)^{-1}.$$

Thus we have shown (reverting to the resolvent R_t^+) that

$$\lambda \int_1^\infty R_t^+(u,u;\lambda)\, du$$
$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} (H_-(s)^{-1} - H_+(s)^{-1})\,\Gamma(s)^{-1}\, \frac{ds}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{t^{-s'}\,\Gamma(s'+s)H_+(s'+s)}{s'^2}\, \frac{ds'}{2\pi i}.$$

This was proved if λ is sufficiently small and if Ω lies in the left half-plane. Let us remove the latter condition first. For any η define the measure ρ_η by $\rho_\eta(E) = \rho(E - \eta)$. This has support $\Omega + \eta$. For all η in a neighborhood in **C** of $[-1,\, 0]$ the set $\Omega + \eta$ is contained in the region (3.1). For η near -1 the set will also lie in the left half-plane. If λ is small enough the condition on the zeros will be satisfied for the measures ρ_η for all η in a neighborhood of $[-1,\, 0]$. For such λ we know that the above formula holds for η in a neighborhood of -1. But both sides are analytic functions of η in our neighborhood. Thus the formula must hold for $\eta = 0$ also, which is what we wanted to show.

To remove the condition that λ be small we must modify the formula to read

$$\lambda \int_1^\infty R_t^+(u,u;\lambda)\, du$$
$$= \int_{s_\lambda-i\infty}^{s_\lambda+i\infty} (H_-(s)^{-1} - H_+(s)^{-1})\,\Gamma(s)^{-1}\, \frac{ds}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{t^{-s'}\,\Gamma(s'+s)H_+(s'+s)}{s'^2}\, \frac{ds'}{2\pi i}, \tag{3.7}$$

where $s_\lambda \in (0,1)$ is as in the previous section, a continuously varying function of λ such that $1 - H(s)$ is nonzero on the line $\Re s = s_\lambda$. Both sides of (3.7) are analytic functions of λ for λ in a neighborhood of our path, they agree near $\lambda = 0$, so they agree everywhere on the path.

As for $R^-(u,u;\lambda)$ on $(0,1)$ the change of variable $u \to u^{-1}$ transforms the kernel $K^-(u,v)$ into

$$\widetilde{K}^+(u,v) := \int_\Omega \frac{e^{-t(1-\omega^{-1})u}}{-\omega v + u}\, d\rho(\omega) = \int_{\Omega^{-1}} \frac{e^{-t(1-\omega)u}}{-\omega u + v}\,(-\omega)\, d\rho(\omega^{-1}).$$

Therefore

$$R^-(u,u;\lambda) = u^{-2}\widetilde{R}^+(u^{-1},u^{-1};\lambda),$$

where $\widetilde{R}^+$ is the resolvent kernel for $\widetilde{K}^+$, and so

$$\int_0^1 R^-(u,u;\lambda)\, du = \int_1^\infty \widetilde{R}^+(u,u;\lambda)\, du.$$

It is easy to see that replacing K^+ by $\widetilde{K}^+$ replaces $H(s)$ by $H(1-s)$ and so the integral is equal to (3.7) with $H(s)$ replaced by $H(1-s)$ and s_λ replaced by $1 - s_\lambda$.

Now that we have these explicit representations it is obvious what we do: in the inner integral in (3.7) we move the line of s'-integration from $\Re s' = \delta$ to $\Re s' = -\delta$. We can do this if δ is small enough. The residue at the double pole at $s' = 0$ contributes

$$-\int (H(s)^{-1} - 1)\,\frac{ds}{2\pi i}\,\log t + \int (H_-(s)^{-1} - H_+(s)^{-1})\,\frac{1}{\Gamma(s)}(\Gamma(s)\,H_+(s))'\,\frac{ds}{2\pi i}, \quad (3.8)$$

the integrations taken over $\Re s = s_\lambda$, and the error term is $O(t^\delta)$. We do the same with $H(s)$ replaced by $H(1-s)$ and add, and so we have obtained the asymptotics of $\lambda \int_0^\infty R(u,u;\lambda)\, du$.

4. Asymptotics of det $(I - K)$—The Periodic case

The formula for $H(s)$ is now

$$H(s) = 1 - \frac{\lambda\,\pi}{\sin\,\pi s}\sum_\Omega c_\omega\,(-\omega)^{s-1},$$

ω running over the n^{th} roots of unity other than 1. If in our sums we set $-\omega = e^{\frac{\pi i}{n}(2j-n)}$ $(j = 1, \cdots, n-1)$, then $|\arg(-\omega)| = |\frac{\pi}{n}(2j-n)| < \pi$ as required. If we also set $z = e^{\frac{\pi i}{n}s}$, then the above may be written as

$$\sin\pi s\, H(s) = \sin\,\pi s + \lambda\pi\sum_{j=1}^{n-1}\omega^{-1}c_\omega\, z^{2j-n},$$

or

$$2i\,\sin\pi s\, H(s) = z^{-n}\left[z^{2n} - 1 + 2i\lambda\pi\sum_{j=1}^{n-1}\omega^{-1}c_\omega\, z^{2j}\right].$$

Recall that $\sin \pi s\, H(s) = h(s)$. The expression in brackets above is a polynomial of degree n in z^2 and its zeros are the quantities by $e^{\frac{2\pi i}{n}\alpha}$, where α runs through the zeros $\alpha_k = \alpha_k(\lambda)$ ($k = 1, \cdots, n$). With this notation the right side above is equal to

$$z^{-n} \prod_\alpha (z^2 - e^{\frac{2\pi i}{n}\alpha}) = \prod_\alpha (z\, e^{-\frac{\pi i}{n}\alpha} - z^{-1}\, e^{\frac{\pi i}{n}\alpha}) \prod_\alpha e^{\frac{\pi i}{n}\alpha}. \tag{4.1}$$

Here and below the index α runs over the set $\{\alpha_1, \cdots, \alpha_n\}$. The last product, a square root of the product of the roots of the polynomial, equals ± 1 or $\pm i$. This product equals $e^{\frac{\pi i}{n}(1+\cdots+n)} = i^{n+1}$ for $\lambda = 0$ and so for all λ. Recalling that $z = e^{\frac{\pi i}{n}s}$ we see that we have obtained the representation

$$H(s) = \frac{(-1)^n\, 2^{n-1}}{\sin \pi s} \prod_\alpha \sin \frac{\pi}{n}(s - \alpha).$$

We now evaluate the integrals in (3.8). For λ sufficiently small again, the α_k will all lie in the strip $\frac{1}{2} < \Re\, s < n+\frac{1}{2}$. We may assume this since the usual analytic continuation will give the general case.

To evaluate the first integral in (3.8) we consider

$$\int (H(s)^{-1} - 1)\, s\, \frac{ds}{2\pi i}$$

taken over the infinite rectangle which is the contour running from $n+\frac{1}{2}-i\infty$ to $n+\frac{1}{2}+i\infty$ and then from $\frac{1}{2} + i\infty$ to $\frac{1}{2} - i\infty$. On the one hand this equals n times the first integral in (3.8), and on the other hand it equals the sum of the residues at the poles between the two lines. Thus we have shown that

$$\int (H(s)^{-1} - 1)\, \frac{ds}{2\pi i} = \frac{1}{n} \sum_\alpha \alpha\, H'(\alpha)^{-1}. \tag{4.2}$$

For the second integral in (3.8) we have to write down the explicit expression for the factors $H_\pm(s)$. These are given by

$$H_+(s) = \frac{\prod_\alpha \Gamma(\frac{s-\alpha}{n} + 1)}{\Gamma(s)}\, n^s, \quad H_-(s) = \frac{(-1)^n\, 2^{n-1}\, \Gamma(1-s)}{\prod_\alpha \Gamma(-\frac{s-\alpha}{n})}\, n^s.$$

It is readily verified that $H(s) = H_-(s)/H_+(s)$ and that $H_-(s)^{\pm 1}$ and $H_+(s)^{\pm 1}$ are are bounded and analytic in $\Re\, s \le \frac{1}{2} + \delta$ and $\Re\, s \ge \frac{1}{2} - \delta$, respectively, for small λ. Thus, they are the correct factors. The second integral in (3.8) may be written

$$\int (H(s)^{-1} - 1)\, \frac{(\Gamma(s)\, H_+(s))'}{\Gamma(s)\, H_+(s)}\, \frac{ds}{2\pi i},$$

and by the above expression for $H_+(s)$ this equals

$$\int (H(s)^{-1} - 1) \left[\frac{1}{n} \sum_{\alpha'} \frac{\Gamma'(\frac{s-\alpha'}{n} + 1)}{\Gamma(\frac{s-\alpha'}{n} + 1)} + \log\, n\right] \frac{ds}{2\pi i},$$

where in the sum α' also runs over the set $\{\alpha_1, \cdots, \alpha_n\}$. The contribution of the term $\log n$ is exactly $\log n$ times (4.2). To evaluate the rest of this integral we use the characteristic property of the Barnes G-function, $G(z+1) = \Gamma(z)\, G(z)$. Putting z equal to $\frac{s-\alpha'}{n} + 1$ and taking logarithmic derivatives gives

$$\frac{G'(\frac{s-\alpha'}{n}+2)}{G(\frac{s-\alpha'}{n}+2)} - \frac{G'(\frac{s-\alpha'}{n}+1)}{G(\frac{s-\alpha'}{n}+1)} = \frac{\Gamma'(\frac{s-\alpha'}{n}+1)}{\Gamma(\frac{s-\alpha'}{n}+1)}.$$

We integrate

$$(H(s)^{-1}-1) \sum_{\alpha'} \frac{G'(\frac{s-\alpha'}{n}+1)}{G(\frac{s-\alpha'}{n}+1)}$$

over the same infinite rectangle as before. (This is justified by the fact that $H(s)^{-1}-1$ vanishes exponentially at ∞ in vertical strips while $G'(z)/G(z)$ grows like $z \log z$.) By the above relation the result is exactly the integral we want, and so computing residues gives the formula

$$\frac{1}{n}\int (H(s)^{-1}-1) \sum_{\alpha'} \frac{\Gamma'(\frac{s-\alpha'}{n}+1)}{\Gamma(\frac{s-\alpha'}{n}+1)} \frac{ds}{2\pi i} = \frac{1}{n}\sum_{\alpha,\alpha'} \frac{G'(\frac{\alpha-\alpha'}{n}+1)}{G(\frac{\alpha-\alpha'}{n}+1)} H'(\alpha)^{-1}.$$

Thus, we have shown that

$$\lambda \int_1^\infty R_t^+(u,u,\lambda)du = a^+(\lambda)\, \log\left(\frac{t}{n}\right) + b^+(\lambda) + O(t^\delta),$$

where

$$a^+(\lambda) = -\frac{1}{n}\sum_\alpha \alpha\, H'(\alpha)^{-1}, \;\; b^+(\lambda) = \frac{1}{n}\sum_{\alpha,\alpha'} \frac{G'(\frac{\alpha-\alpha'}{n}+1)}{G(\frac{\alpha-\alpha'}{n}+1)} H'(\alpha)^{-1}. \tag{4.3}$$

We must add to this $\lambda \int_0^1 R_t^-(u,u,\lambda)\, du$ which, as was mentioned earlier, is obtained by replacing $H(s)$ by $H(1-s)$. The zeros of this function which lie near $1, \cdots, n$ for small λ are $n-\alpha+1$. Hence (4.3) is replaced by

$$a^-(\lambda) = \frac{1}{n}\sum_\alpha (n-\alpha+1)\, H'(\alpha)^{-1},$$

$$b^-(\lambda) = -\frac{1}{n}\sum_{\alpha,\alpha'} \frac{G'(\frac{\alpha'-\alpha}{n}+1)}{G(\frac{\alpha'-\alpha}{n}+1)} H'(\alpha)^{-1} = -\frac{1}{n}\sum_{\alpha,\alpha'} \frac{G'(\frac{\alpha-\alpha'}{n}+1)}{G(\frac{\alpha-\alpha'}{n}+1)} H'(\alpha')^{-1}.$$

Here we have used the periodicity of H and the fact $\frac{d}{ds}H(1-s) = -H'(1-s)$. Adding and using (2.3), we see that

$$\lambda \int_0^\infty R(u,u;\lambda)\, du = a(\lambda)\, \log\left(\frac{t}{n}\right) + b(\lambda) + O(t^\delta) \tag{4.4}$$

where $a(\lambda) = a^+(\lambda) + a^-(\lambda),\; b(\lambda) = b^+(\lambda) + b^-(\lambda)$.

Now to obtain the asymptotics of $\log\,\det(I-K)$ we must replace λ by μ, multiply the above by $-d\mu/\mu$ and integrate from 0 to λ. (Notice the factor λ on the left side of (4.4) and recall the minus sign in (2.1).) We obtain from (3.5) that for a zero $\alpha(\lambda)$ of H

we have $\lambda\, d\alpha/d\lambda = H'(\alpha)^{-1}$. Thus the coefficient $a(\lambda)$ in (4.4) may be written (after replacing λ by μ and thinking of α as $\alpha(\mu)$)

$$-\frac{1}{n}\sum_{\alpha}(2\alpha - n - 1)\,\mu\, d\alpha/d\mu.$$

Multiplying by $-d\mu/\mu$ and integrating gives

$$\frac{1}{n}\sum_{\alpha}(\alpha^2 - (n+1)\,\alpha)\Big|_{\mu=0}^{\mu=\lambda} = \frac{1}{n}\sum_{\alpha}\,\alpha^2\,\Big|_{\mu=0}^{\mu=\lambda}$$

since, as we have already seen, $\sum\alpha$ is independent of λ. Similarly $b(\mu)$ may be written

$$\frac{1}{n}\sum_{\alpha,\alpha'}\frac{G'(\frac{\alpha-\alpha'}{n}+1)}{G(\frac{\alpha-\alpha'}{n}+1)}\mu\, d(\alpha-\alpha')/d\mu,$$

and multiplying by $-d\mu/\mu$ and integrating gives

$$-\sum_{\alpha,\alpha'}\log\, G(\frac{\alpha-\alpha'}{n}+1)\Big|_{\mu=0}^{\mu=\lambda}.$$

If we recall that when $\mu = 0$ the zeros are $1,\cdots,n$ we see that the formulas for the constants in (1.4) are the ones stated in the introduction.

To obtain the asymptotics of the $q_k(t)$ we must consider $\det(I - \lambda K_k)$ instead of $\det(I - \lambda K)$. This amounts to replacing the coefficients c_ω by $\omega^k\, c_\omega$, and this in turn amounts to replacing $H(s)$ by $H(s+k)$. The zeros of this function modulo n are $\alpha_1(\lambda) - k,\cdots,\alpha_n(\lambda) - k$. But these are not the zeros which are to replace the $\alpha,\ \alpha'$ in our formulas for a and b since they do not arise from the zeros whose values are $1,\cdots,n$ when $\lambda = 0$. Rather, the replacements must be

$$\alpha_{k+1}(\lambda) - k,\cdots,\alpha_n(\lambda) - k,\ \alpha_1(\lambda) + n - k,\cdots,\alpha_k(\lambda) + n - k,$$

which are the zeros with this property. Thus, for the asymptotics of $q_k(t)$ we make these replacements in our formulas and the corresponding replacements with $k-1$ instead of k, subtract, and take logarithms. The result is found, after some computation and the use of the functional equation for the G-function, to be the asymptotics stated in the introduction.

5. Asymptotics of $\det(I - K)$ – The Nonperiodic Case

The coefficients $a(\lambda) = a^+(\lambda) + a^-(\lambda),\ b(\lambda) = b^+(\lambda) + b^-(\lambda)$ of the last section were in general given by integral formulas. They were

$$a^+(\lambda) = -\int(H(s)^{-1} - 1)\,\frac{ds}{2\pi i}, \tag{5.1}$$

$$b^+(\lambda) - a^+(\lambda)\,\log n = \int(H_-(s)^{-1} - H_+(s)^{-1})\,\frac{1}{\Gamma(s)}(\Gamma(s)\,H_+(s))'\,\frac{ds}{2\pi i}, \tag{5.2}$$

with the formulas for $a^-(\lambda)$, and $b^-(\lambda)$ obtained by replacing $H(s)$ by $H(1-s)$. The integrations may be taken over $\Re\, s = \frac{1}{2}$ if λ is small enough and, as usual, this is no loss of generality. To find a and b we integrate $a(\mu)$ and $b(\mu)$, respectively, with respect to $-d\mu/\mu$ over a path from 0 to λ.

Write

$$\varphi(s) := \frac{\pi}{\sin\, \pi s} \int_\Omega (-\omega)^{s-1}\, d\rho(\omega),$$

so that $H(s) = 1 - \lambda\, \varphi(s)$. Making this replacement in the integrand in (5.1) and integrating gives

$$-\int_0^\lambda \Big(\frac{1}{1-\mu\varphi(s)} - 1 \Big) \frac{d\mu}{\mu} = -\int_0^\lambda \frac{\varphi(s)}{1-\mu\varphi(s)}\, d\mu = \log\,(1-\lambda\varphi(s)),$$

so the contribution to the coefficient of log t is

$$-\int \log\,(1-\lambda\varphi(s)) \frac{ds}{2\pi i}$$

over $\Re\, s = \frac{1}{2}$. Replacing $H(s)$ by $H(1-s)$ gives the same contribution since we may make the substitution $s \to 1-s$. Therefore

$$a = -2 \int \log\,(1-\lambda\varphi(s)) \frac{ds}{2\pi i}. \tag{5.3}$$

For general λ the integration is to be on $\Re\, s = s_\lambda$.

Now we go to (5.2), which may be written

$$b^+(\lambda) - a^+(\lambda)\, \log n = \int \frac{\Gamma'(s)}{\Gamma(s)} \Big(\frac{1}{H(s)} - 1 \Big) \frac{ds}{2\pi i} + \int \Big(\frac{H_+'(s)}{H_-(s)} - \frac{H_+'(s)}{H_+(s)} \Big) \frac{ds}{2\pi i}. \tag{5.4}$$

By a computation like the earlier one we see that the first integral becomes after the μ-integration

$$\int \frac{\Gamma'(s)}{\Gamma(s)}\, \log\,(1-\lambda\varphi(s))\, \frac{ds}{2\pi i}.$$

Then we replace s by $1-s$, make the substitution $s \to 1-s$, and add. We see that the contribution of the first integral in (5.4) equals

$$\int \Big(\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} \Big)\, \log\,(1-\lambda\varphi(s))\, \frac{ds}{2\pi i}. \tag{5.5}$$

Finally, we look at the second integral in (5.4), which equals

$$\int \frac{H_+'(s)}{H_-(s)}\, \frac{ds}{2\pi i} = \int \frac{H_+'(s)}{H_+(s)} H(s)^{-1}\, \frac{ds}{2\pi i} = \int (\log\, H_+)'(s)\, H(s)^{-1}\, \frac{ds}{2\pi i}.$$

Replacing $H(s)$ by $H(1-s)$ replaces $H_+(s)$ by $1/H_-(1-s)$, so after making the substitution $s \to 1-s$ and adding we get

$$\int (\log\, H_+\, H_-)'(s)\, H(s)^{-1}\, \frac{ds}{2\pi i}. \tag{5.6}$$

Recall that for $\Re\, s = \frac{1}{2}$,

$$\log\, H_\pm(s) = \mp\frac{1}{2}\,\log\, H(s) + \int \frac{\log\, H(s')}{s'-s}\,\frac{ds'}{2\pi i},$$

where the integral is the Hilbert transform, a principal value integral over $\Re\, s' = \frac{1}{2}$. So

$$\log\, H_+(s)\, H_-(s) = \int \frac{\log\, H(s')}{s'-s}\,\frac{ds'}{\pi i}.$$

Since the Hilbert transform commutes with differentiation we get

$$(\log\, H_+\, H_-)'(s) = \frac{1}{\pi i}\int \frac{H'(s')}{H(s')}\,\frac{ds'}{s'-s},$$

and so (5.6) equals

$$-\frac{1}{2\pi^2}\int\int \frac{H'(s')}{H(s')}\,H(s)^{-1}\,\frac{ds'}{s'-s}\,ds.$$

The μ-integration gives

$$-\int_0^\lambda \frac{-\mu\,\varphi'(s')}{1-\mu\varphi(s')}\,\frac{1}{1-\mu\varphi(s)}\,\frac{d\mu}{\mu} = \varphi'(s')\int_0^\lambda \frac{1}{(1-\mu\varphi(s'))\,(1-\mu\varphi(s))}\,d\mu$$

$$= \varphi'(s')\,\frac{1}{\varphi(s')-\varphi(s)}\,\log\,\frac{1-\lambda\varphi(s)}{1-\lambda\varphi(s')},$$

and so the contribution of the second integral in (5.4) is

$$-\frac{1}{2\pi^2}\int\int \frac{\varphi'(s')}{\varphi(s')-\varphi(s)}\,\log\,\frac{1-\lambda\,\varphi(s)}{1-\lambda\,\varphi(s')}\,\frac{ds'}{s'-s}\,ds. \tag{5.7}$$

Thus $b - a\,\log n$ equals the sum of (5.5) and (5.7). As usual, for general λ the integrals are taken over $\Re\, s,\ s' = s_\lambda$.

Remark. The double integral (5.7) is exactly the constant in the known asymptotics for the determinants of the truncated Wiener-Hopf operators associated with φ (specifically, $\varphi(\frac{1}{2}+i\xi)$ is the Fourier transform of the convolving kernel), and (5.3) is (minus twice) the leading coefficient in the asymptotics. One can see by the argument of Section 2 how both these things arise and conclude also that (5.7) equals $\det\,\mathcal{M}^{-1}$. The extra ingredient here is therefore the integral (5.5).

6. The Case $n = 2$

In this case the only root is $\omega = -1$ and we may take c_{-1} equal to 1 since it occurs only in the product λc_{-1}. Thus the kernel of K_0 is

$$\frac{e^{-2t(u+u^{-1})}}{u+v},$$

and the equation (for either q_k) is

$$q''(t) + t^{-1}q'(t) = 8\,\sinh 2q(t). \tag{6.1}$$

We have in this case $h(s) = \sin \pi s - \pi\lambda$, the zeros are given by

$$\alpha_0 = \frac{1}{\pi} \arcsin \pi\lambda = \frac{1}{\pi i} \log(\pi i \lambda + \sqrt{1-\pi^2\lambda^2}), \quad \alpha_1 = 1 - \alpha_0,$$

and $\alpha_{k+2} = \alpha_k + 2$. The square root is that branch which is positive for $\lambda = 0$ and the logarithm that branch which is 0 there. From this it is easy to see that the set (1.6) consists of the rays $(-\infty, -1/\pi]$ and $[1/\pi, \infty)$ and Λ, the proposed region of validity of our formulas, is the complex plane cut along these rays. If we note that the function $\pi i\lambda + \sqrt{1-\pi^2\lambda^2}$ maps Λ onto the right half-plane, we see that $|\Re\, \alpha_k(\lambda) - k| < \frac{1}{2}$ for all $\lambda \in \Lambda$ and so the "extra" condition on the α is satisfied. The range of validity is therefore all of Λ. Using the formulas stated in the introduction we find that $\det(I - \lambda K_0) \sim b\,(t/2)^a$ with

$$a = \alpha_0^2 + \alpha_0, \quad b = \frac{G(\frac{1}{2})\,G(\frac{3}{2})}{G(\frac{1}{2}+\alpha_0)\,G(\frac{3}{2}-\alpha_0)} = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\alpha_0)}\,\frac{G(\frac{1}{2})^2}{G(\frac{1}{2}+\alpha_0)\,G(\frac{1}{2}-\alpha_0)}.$$

For $\det(I + \lambda K_0)$ we replace λ by $-\lambda$, which amounts to replacing α_0 by $-\alpha_0$. If we multiply the two results together we recover the asymptotics for $\det(I - \lambda^2 K_0^2)$ determined in [7] and [2].

For q_0 we have the asymptotics $A\log(t/2) + \log B + o(1)$, where

$$A = 2\,\alpha_0, \quad B = \frac{\Gamma(\frac{1}{2}-\alpha_0)}{\Gamma(\frac{1}{2}+\alpha_0)},$$

in agreement with [6]. This is the solution of (6.1) which is asymptotic to $-2\lambda K_0(4t)$ as $t \to \infty$, where this K_0 is the Bessel function.

For $\lambda \notin \Lambda$ the asymptotics are different. For $\lambda > 1/\pi$, $e^{q_0(t)}$ has an infinite sequence of zeros as $t \to 0$ and for $\lambda < -1/\pi$, it has an infinite sequence of poles; this follows from the fact that as $t \to 0$ the spectrum of K_0 fills up the interval $[0, \pi]$. A heuristic derivation of the asymptotics for λ on the cut is given in [6]. In the next section we present a similar derivation for some cases of $n = 3$.

7. $n = 3$ and Cylindrical Bullough-Dodd

The cylindrical Bullough-Dodd equation

$$q''(t) + t^{-1}q'(t) = 4e^{2q} - 4e^{-q}, \tag{7.1}$$

arises in the special case of $n = 3$, where $c_\omega = -\omega^3\, c_{\omega^{-1}}$. Then $q_1 = 0$, $q_2 = -q_3$ and (7.1) is satisfied by $q = q_3$. If we set $\zeta := e^{2\pi i/3}$, then c_ζ may be chosen arbitrarily. If we choose it to be $\zeta(1-\zeta)$ then $c_{\zeta^2} = \zeta^2(1-\zeta^2)$, $c_{-1} = 0$ and

$$h(s) = \sin \pi s + 2\pi\sqrt{3}\,\lambda \sin(\pi(s+2)/3).$$

Again λ is the one free parameter. The zeros are given by

$$\alpha_0 = \frac{1}{4} - \frac{3}{2\pi} \arcsin\left(\frac{1}{2} + \frac{\lambda}{2\lambda_c}\right), \quad \alpha_1 = 1, \quad \alpha_2 = 2 - \alpha_0,$$

where $\lambda_c = 1/(2\sqrt{3}\pi)$. Now Λ is the complement of the union of cuts

$$(-\infty,\ -3\lambda_c] \cup [\lambda_c,\ \infty).$$

For $\lambda \in \Lambda$ the zeros satisfy

$$\Re\,\alpha_0 \in (-\frac{1}{2},\ 1), \quad \Re\,\alpha_1 = 1, \quad \Re\,\alpha_2 \in (1,\ \frac{5}{2}),$$

and so the extra condition is again automatically satisfied and our formulas hold for all $\lambda \in \Lambda$. If we write

$$q(t) = A\log(\frac{1}{t}) - \log B + o(1) \tag{7.2}$$

then the connection formulas give in this case

$$A = -2\alpha, \quad B = 3^{-A}\,\frac{\Gamma\left(\frac{\alpha+2}{3}\right)\Gamma\left(\frac{2\alpha+1}{3}\right)}{\Gamma\left(\frac{1-\alpha}{3}\right)\Gamma\left(\frac{2-2\alpha}{3}\right)},$$

where we wrote α for α_0. For large t,

$$q(t) \sim 6\lambda K_0\left(2\sqrt{3}t\right).$$

Asymptotics at the critical value λ_c. This section and the following ones are heuristic. Using the differential equation (7.1) one can determine the correction terms to (7.2):

$$q(t) = A\log(\frac{1}{t}) + \log B + \frac{B^2}{(1-A)^2}\,t^{2-2A} - \frac{4}{B(2+A)^2}\,t^{2+A} + \frac{B^4}{2(1-A)^4}\,t^{4-4A} + \ldots . \tag{7.3}$$

This is valid for $\lambda \in \Lambda$. To understand the higher order terms in more detail it is convenient to define

$$w(t) = \exp\left(-q(t)\right).$$

where w satisfies the equation

$$w'' = \frac{1}{w}(w')^2 - \frac{1}{t}w' + 4w^2 - \frac{4}{w}. \tag{7.4}$$

The asymptotics we proved become the statement

$$w(t) = Bt^A\left(1 + o(1)\right).$$

Using (7.4) to calculate the higher order terms in the small t expansion for w we find

$$\begin{aligned} w(t)=Bt^A\Big(1 &- \frac{t^{2-2A}}{B^2(1-A)^2} + \frac{4B}{(2+A)^2}t^{2+A} + \frac{12B^2}{(2+A)^4}t^{4+2A} + \cdots \\ &+ \frac{(j+1)\,2^j B^j}{(2+A)^{2j}}\,t^{2j+jA} + \cdots \\ &+ \frac{24(A^2-2A-2)}{(2+A)^2(1-A)^2(4-A)^2B}t^{4-A} + \frac{4}{(1-A)^4(4-A)^2B^3}t^{6-3A} + \cdots\Big). \end{aligned} \tag{7.5}$$

In contrast to (7.3) the terms t^{2m-2mA} only appear for $m = 1$ in the above expansion.

As λ varies from 0 to λ_c α varies from 0 to $-\frac{1}{2}$, so A varies from 0 to 1 and B from 1 to ∞. Observe that the first two terms in (7.5) are of the same order in t as $t \to 0$ (and

$A \to 1$) whereas the others are of lower order. This suggests that when $\lambda = \lambda_c$ we have $w(t) \sim t\Omega_1$ as $t \to 0$, where

$$\Omega_1 := \lim_{\alpha \to -\frac{1}{2}} B\left(1 - \frac{t^{2-2A}}{B^2(1-A)^2}\right) = 2\log(1/t) - \frac{4}{3}\log 2 + 2\log 3 - 2\gamma. \qquad (7.6)$$

We now use the differential equation (7.4) to find the higher order terms, which are polynomial in t and Ω_1. (The only property of Ω_1 used in the formal expansion is $d\Omega_1/dt = -2/t$.) The expansion is

$$w(t) = t\Omega_1 + \frac{4}{9}t^4\left(\Omega_1^2 + \frac{4}{3}\Omega_1 + \frac{8}{9}\right) + \frac{4}{2187}t^7\left(81\Omega_1^3 + 216\Omega_1^2 + 240\Omega_1 + 80\right) + O(t^{10}\Omega_1^4). \qquad (7.7)$$

Thus (as for the $n = 2$ analogue [6]) if one were to alter the constant appearing in (7.6) then the solution of (7.4) whose asymptotics is (7.7) would not match onto the solution that approaches 1 as $t \to \infty$.

These asymptotics at $\lambda = \lambda_c$ were checked by numerically solving (7.4) in both a forward and backward integration. There was agreement to nine decimal places at $t = 1/4$.

Asymptotics at the critical value $-3\lambda_c$. We proceed as above and examine all terms that would be of the same order of magnitude as $\lambda \to -3\lambda_c$, when $\alpha \to 1$ and $A \to -2$. These are the terms of the geometric series, those involving the powers t^{2j+jA}. Summing the series we see that we must compute

$$\lim_{\alpha \to 1} \frac{B\, t^A}{\left(1 - 2Bt^{2+A}/(2+A)^2\right)^2} = \frac{1}{2t^2\left(\log t - \log 3 + \gamma\right)^2}.$$

Defining

$$\Omega_2 = \log t - \log 3 + \gamma$$

we thus see that at $\lambda = -3\lambda_c$,

$$w(t) \sim \frac{1}{2t^2\Omega_2^2}. \qquad (7.8)$$

To compute higher order terms it is convenient to look at $v(t) = 1/w(t)$. Using the differential equation and only the property $d\Omega_2/dt = 1/t$ of Ω_2 we find

$$v(t) = 2t^2\Omega_2^2 + \frac{t^8}{9}\left(8\Omega_2^6 - \frac{32}{3}\Omega_2^5 + \frac{76}{9}\Omega_2^4 - \frac{40}{9}\Omega_2^3 + \frac{40}{27}\Omega_2^2 - \frac{20}{81}\Omega_2\right) + O(t^{14}\Omega_2^{10}). \qquad (7.9)$$

Asymptotics for $\lambda > \lambda_c$. Think of λ as being on the lower part of the cut $[\lambda_c, \infty)$. Then

$$\alpha = -\frac{1}{2} - \frac{3}{2}i\mu,$$

where

$$\mu := \frac{1}{\pi}\operatorname{arccosh}\left(\frac{1}{2} + \frac{\lambda}{2\lambda_c}\right), \quad (\mu > 0).$$

Thus $A = 1 + 3i\mu$. Here again the first two terms in (7.5) are of the same order as $t \to 0$ whereas the others are of lower order, and we obtain

$$w(t) = Bt^A - \frac{t^{1-2A}}{B(1-A)^2} + O(t^4).$$

Substituting in the values of A and B in terms of μ we find

$$w(t) = \left(\frac{2t}{3\mu}\right) \Im \left[e^{3i\mu \log(t/3)} \frac{\Gamma(1/2 - i\mu/2)\Gamma(-i\mu)}{\Gamma(1/2 + i\mu/2)\Gamma(i\mu)} \right] + O(t^4).$$

If we had taken λ to be on the upper part of the cut then we would have replaced μ by $-\mu$. The result would have been precisely the same.

Remark. In [5] a method was described to find connection formulas for solutions of a class of equations including (7.1). Away from the critical values the short-range asymptotics stated there correspond to the first two terms in (7.5). As for the asymptotics at the critical values, our formulas agree with [5] at $\lambda = -3\lambda_c$ but at $\lambda = \lambda_c$ we differ by a factor of 2.

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