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Random words, Toeplitz determinants and integrable systems: II

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Abstract

This paper connects the analysis of the length of the longest weakly increasing subsequence of inhomogeneous random words to a Riemnn-Hilbert problem and an associated system of integrable partial differential equations. In particular, we show that the Poissonization of the distribution function of this length can be identified as the Jimbo-Miwa-Ueno tau function.

1. Introduction

This paper, a continuation of [16], connects the analysis of the length of the longest weakly increasing subsequence of inhomogeneous random words to a Riemann–Hilbert problem and an associated system of integrable PDEs. That such a connection exists is not so surprising given the fundamental work of Baik, Deift and Johansson [3] connecting the related problem involving random permutations to a Riemann–Hilbert problem. For the reader's convenience we first summarize some of the results of [16] before presenting our new results.

A word is a string of symbols, called *letters*, which belong to an ordered alphabet $\mathcal{A}$ of fixed size k. The set of all such words of length N, $\mathcal{W}(\mathcal{A}, N)$, forms the sample space in our statistical analysis. We equip the space $\mathcal{W}(\mathcal{A}, N)$ with a natural *inhomogeneous* measure by assigning to each letter $i \in \mathcal{A}$ a probability p_i and defining the probability measure on words by the product measure. We also order the p_i so that

$$p_1 \geq p_2 \geq \cdots \geq p_k$$

and decompose our alphabet $\mathcal{A}$ into subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_M$, $M \leq k$, such that $p_i = p_j$ if and only if i and j belong to the same $\mathcal{A}_\alpha$.

Let

$$w = \alpha_1\alpha_2 \cdots \alpha_N \in \mathcal{W}, \quad \alpha_i \in \mathcal{A},$$

be a word. A *weakly increasing subsequence* of the word w is a subsequence $\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_m}$ such that $i_1 < i_2 < \cdots < i_m$ and $\alpha_{i_1} \leq \alpha_{i_2} \leq \cdots \leq \alpha_{i_m}$. The positive integer m is called the *length* of this weakly increasing subsequence.

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For each word $w \in \mathcal{W}$ we define $\ell_N(w)$ to equal the *length of the longest weakly increasing subsequence* in w.[1] The function

$$\ell_N : \mathcal{W}(\mathcal{A}, N) \mapsto \mathbf{R}$$

is the principal random variable in our analysis, and the corresponding distribution function,

$$F_N(n) := \mathrm{Prob}(\ell_N(w) \le n),$$

is our principal object. (Prob is the inhomogeneous measure on random words; it depends upon N and the probabilities p_i.)

To formulate the basic result of [16], define

$$k_\alpha = |\mathcal{A}_\alpha|,$$

where

$$\mathcal{A} = \bigcup_{\alpha=1}^{M} \mathcal{A}_\alpha$$

is the decomposition of the alphabet $\mathcal{A}$ introduced above, then

$$\lim_{N\to\infty} \mathrm{Prob}\left(\frac{\ell_N - Np_1}{\sqrt{Np_1}} \le s\right) = (2\pi)^{-(k-1)/2} \prod_\alpha (1!2!\cdots(k_\alpha - 1)!)^{-1} \times \int \cdots \int_{\substack{\xi_i \in \Xi \\ \xi_1 \le s}} \prod_\alpha \Delta_\alpha(\xi)^2 \,\mathrm{e}^{-\sum \xi_i^2/2} \delta\left(\sum \sqrt{p_i}\xi_i\right) \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k,$$

where $\Delta_\alpha(\xi)$ is the Vandermonde determinant of those ξ_i with $i \in \mathcal{A}_\alpha$, and Ξ denotes the set of those ξ_i that $\xi_{i+1} \le \xi_i$ whenever i and $i+1$ belong to the same $\mathcal{A}_\alpha$.

This result has the following random matrix interpretation. The limiting distribution function (as $N \to \infty$) for the appropriately centered and normalized random variable ℓ_N is related to the distribution function for the eigenvalues ξ_i in the *direct sum* of mutually independent $k_\alpha \times k_\alpha$ Gaussian unitary ensembles,[2] conditional on the eigenvalues ξ_i satisfying $\sum \sqrt{p_i}\xi_i = 0$. In the case when one letter occurs with greater probability than the others, this result implies that the limiting distribution of $(\ell_N - Np_1)/\sqrt{N}$ is Gaussian with variance equal to $p_1(1-p_1)$. In the case when all the probabilities are distinct, we proved the refined asymptotic result

$$E(\ell_N) = Np_1 + \sum_{j>1} \frac{p_j}{p_1 - p_j} + \mathrm{O}\left(\frac{1}{\sqrt{N}}\right), \quad N \to \infty.$$

The derivation of the above asymptotic formulae follows from a direct asymptotic analysis of the right-hand side of the basic combinatorial equation,

$$\mathrm{Prob}(\ell_N(w) \le n) = \sum_{\substack{\lambda \vdash N \\ \lambda_1 \le n}} s_\lambda(p) f^\lambda.$$

Here $\lambda \vdash N$ denotes a partition of N, $s_\lambda(p)$ is the Schur function of shape λ evaluated at $p := (p_1, p_2, \ldots, p_k, 0, 0, \ldots)$, and f^λ equals the number of standard Young tableaux of shape λ, see, e.g. [27]. After [16] was written,

[1] There may be many subsequences of w that have the identical length $\ell_N(w)$.

[2] A basic reference for random matrices is Mehta's book [23].

Stanley [28] showed that the measure $\text{Prob}(\{\lambda\}) := s_\lambda(p) f^\lambda$ also underlies the analysis of certain (generalized) riffle shuffles of Bayer and Diaconis [5]. Stanley relates this measure to quasisymmetric functions and does not require that p have finite support. (Many of our results generalize to the case when p does not have finite support, but we do not consider this here.) The measure considered here and in [28] is a specialization of the Schur measure $\text{Prob}(\{\lambda\}) := s_\lambda(x) s_\lambda(y)$ [25]. For the Schur measure, Okounkov [25] has shown that the associated correlation functions satisfy an infinite hierarchy of PDEs; namely, the Toda lattice hierarchy of Ueno and Takasaki [32]. Similar results were also obtained by Adler and van Moerbeke [1,33].

Gessel's theorem [13] (see also [16,30]) implies that the (exponential) generating function of $\text{Prob}(\ell_N \leq n)$ is a Toeplitz determinant [3]

$$G_I(n; \{p_i\}, t) := \sum_{N=0}^{\infty} \text{Prob}(\ell_N(w) \leq n) \frac{t^N}{N!} = D_n(f_I), \tag{1.1}$$

where

$$f_I(z) = \mathrm{e}^{t/z} \prod_{j=1}^{k} (1 + p_j z).$$

Probabilistically, $G_I(n; \{p_i\}, t)$ is the *Poissonization* of ℓ_N. Similar Poissonizations have proved crucial in the analysis of the length of the longest increasing subsequences in random permutations [2,3,20] (see also [21,22,30] and references therein).

In the present paper we use (1.1) to express $G_I(n; \{p_i\}, t)$ in terms of the solution of a certain integrable system of nonlinear PDEs. Indeed, we show that $G_I(n; \{p_i\}, t)$ can be identified as the *Jimbo–Miwa–Ueno* [18,19] *τ-function* corresponding to the (generalized) Schlesinger isomonodromy deformation equations of the 2×2 matrix linear ODE which has $M + 1$ simple poles in the finite complex plane and one Poincaré index 1 irregular singular point at infinity. Recall that the number M is the total number of the subsets $\mathcal{A}_\alpha \subset \mathcal{A}$. The poles are located at 0 and $-p_{i_\alpha}$ ($i_\alpha = \max \mathcal{A}_\alpha$). The integers k_α appear as the formal monodromy exponents at the respective points $-p_{i_\alpha}$. We also evaluate the remaining monodromy data and formulate a 2×2 matrix Riemann–Hilbert problem which provides yet another analytic representation for the function $G_I(n; \{p_i\}, t)$. Similar to the problems considered in [3,6], the Riemann–Hilbert representation of $G_I(n; \{p_i\}, t)$ can be used for the further asymptotic analysis of the random variable $\ell_N(w)$ via the Deift and Zhou [11] method. In the *homogeneous* case, i.e. when $M = 1$, the system of Schlesinger equations we obtain reduces to a special case of Painlevé V equation. This result was obtained earlier in [30]. The exact formulation of the results indicated above is presented in Theorem 1 in Section 4.

Our derivation of the differential equations for the function $G_I(n; \{p_i\}, t)$ follows a scheme well known in soliton theory (see e.g. [24]) called the *Zakharov–Shabat dressing method*. We are able to apply this scheme since there exists a matrix Riemann–Hilbert problem associated to any Toeplitz determinant as was shown by Deift [10]. For the reader's convenience this is derived in Section 2.

The basic idea of the Riemann–Hilbert approach to Toeplitz determinants suggested in [10] is a representation of a Toeplitz determinant $D_n(\phi)$ as a Fredholm determinant of an integral operator acting on $L_2(C)$, C = unit circle, and belonging to a special *integrable* class which admits a Riemann–Hilbert representation [15]. Borodin and Okounkov [8] (see also [4] for a simplified derivation and [22,9] for a particular case of ϕ) found a different Fredholm determinant representation for $D_n(\phi)$. The Fredholm operator in this representation acts on $l_2(\{n, n+1, \dots\})$ which makes the representation quite suitable for the analysis of the large n asymptotics of $D_n(\phi)$ (see [9,21,22]). Borodin

[3] If ϕ is a function on the unit circle with Fourier coefficients $\phi_j := 1/2\pi \int_{-\pi}^{\pi} \mathrm{e}^{-ij\theta} \phi(\mathrm{e}^{i\theta})\, \mathrm{d}\theta$ then $T_n(\phi)$ denotes the Toeplitz matrix $(\phi_{i-j})_{i,j=0,1,\dots,n-1}$ and $D_n(\phi)$ its determinant.

[7] subsequently observed that the discrete Fredholm representation of [8] involves a discrete analog of the integrable kernels and can be supplemented by a discrete analog of the Riemann–Hilbert problem. (This is similar to the *pure* soliton constructions in the theory of integrable PDEs [24].)

We conclude this introduction by noting that our derivation of integrable PDEs for the Toeplitz determinant $D_n(f_I)$ can be applied to any Toeplitz determinant whose symbol ϕ satisfies the condition,

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\phi(z) = \text{rational function of } z.$$

This is one place where the finite support of p is crucial. It is an interesting open problem, particularly in light of [28], to remove this restriction.

2. Fredholm determinant representation of the Toeplitz determinant and the Riemann–Hilbert problem

Let $\phi(z)$ be a continuous function on the unit circle $C = \{|z| = 1\}$ oriented in the counterclockwise direction. Let $n \in \mathbf{N}$ and denote by $K_n(\phi)$ the integral operator acting on $L_2(C)$ with kernel

$$K_n(z, z') = \frac{z^n(z')^{-n} - 1}{z - z'}\frac{1 - \phi(z')}{2\pi\mathrm{i}}. \tag{2.1}$$

It was shown in [10] that

$$D_n(\phi) = \det(1 - K_n(\phi)), \tag{2.2}$$

where the determinant on the right is a Fredholm determinant taken in $L_2(C)$. (Note that $K_n(z, z')$ has no singularity at $z = z'$.) Eq. (2.2) follows from the "geometric sum form" of the kernel K_n,

$$K_n(z, z') = \sum_{k=0}^{n-1} z^k \frac{1 - \phi(z')}{2\pi\mathrm{i}}(z')^{-k-1},$$

which shows that the Toeplitz matrix $T_n(\phi)$ is essentially the matrix representation of the operator $1 - K_n(\phi)$ in the basis $\{z^k\}_{-\infty<k<\infty}$. (For more details see [10].)

The integral operator $K_n(\phi)$ belongs to the class of *integrable Fredholm operators* [10,15,31], i.e., its kernel is of the form

$$K_n(z, z') = \frac{f^{\mathrm{T}}(z)g(z')}{z - z'},$$

where

$$f(z) = (f_1, f_2)^{\mathrm{T}} = (z^n, 1)^{\mathrm{T}} \tag{2.3}$$

and

$$g(z) = (g_1, g_2)^{\mathrm{T}} = (z^{-n}, -1)^{\mathrm{T}}\frac{1 - \phi(z)}{2\pi\mathrm{i}}. \tag{2.4}$$

We require, so that there is no singularity on the diagonal of the kernel,

$$f^{\mathrm{T}}(z)g(z) = 0. \tag{2.5}$$

An important property of these operators is that the resolvent $R_n = (1-K)^{-1} - 1$ also belongs to the same class (see again [10,15,31]). Precisely,

$$R_n(z, z') = \frac{F^{\mathrm{T}}(z)G(z')}{z - z'}, \tag{2.6}$$

where

$$F_j = (1-K_n)^{-1} f_j, \qquad G_j = (1-K_n^{\mathrm{T}})^{-1} g_j, \quad j = 1, 2.$$

The vector functions F and G can be in turn computed in terms of a certain matrix Riemann–Hilbert problem [15]. Indeed, let us define (cf. [10,15]) the 2×2 matrix valued function

$$Y(z) = I - \int_C F(z') g^{\mathrm{T}}(z') \frac{\mathrm{d}z'}{z' - z}, \quad z \notin C. \tag{2.7}$$

Let $Y_\pm(z)$ denote the boundary values of the function $Y(z)$ on the contour C,

$$Y_\pm(z) = \lim_{\substack{z' \to z \\ z' \in \pm \mathrm{side}}} Y_+(z').$$

From (2.7) it follows that

$$Y_+(z) - Y_-(z) = -2\pi \mathrm{i} F(z) g^{\mathrm{T}}(z) \tag{2.8}$$

and hence (recall (2.5))

$$Y_+(z) f(z) = Y_-(z) f(z).$$

Using this, the matrix identity

$$F(z') g^{\mathrm{T}}(z') f(z) = f^{\mathrm{T}}(z) g(z') F(z') \quad \text{(associativity of the matrix product)}$$

and (2.7), we have

$$Y_\pm(z) f(z) = f(z) - \int_C f^{\mathrm{T}}(z) g(z') F(z') \frac{\mathrm{d}z'}{z' - z} = f(z) + \int_C K(z, z') F(z')\, \mathrm{d}z'.$$

From the definition of F it follows that

$$F(z) = Y_\pm(z) f(z). \tag{2.9}$$

This and (2.8) imply the jump equation,

$$Y_-(z) = Y_+(z)(I + 2\pi \mathrm{i} f^{\mathrm{T}}(z) g(z)), \quad z \in C. \tag{2.10}$$

This equation, supplemented by the obvious analytic properties of the Cauchy integral in (2.7), shows that the function $Y(z)$ solves the following 2×2 matrix Riemann–Hilbert problem:

- $Y(z)$ is holomorphic for all $z \notin C$,
- $Y(\infty) = I$,
- $Y_-(z) = Y_+(z) H(z), \quad z \in C$,

where the jump matrix H is

$$H(z) = I + 2\pi \mathrm{i} f^{\mathrm{T}}(z) g(z), \tag{2.11}$$

$$H(z) = \begin{pmatrix} 2-\phi(z) & (\phi(z)-1)z^n \\ (1-\phi(z))z^{-n} & \phi(z) \end{pmatrix}. \tag{2.12}$$

These analytic properties determine Y uniquely. To see this, we first observe that $\det H(z) \equiv 1$ implies that the scalar function $\det Y(z)$ has no jump on C; hence, it is holomorphic and bounded on the whole complex plane. This together with the normalization condition at $z = \infty$ implies that $\det Y(z) \equiv 1$. Suppose that $\tilde{Y}(z)$ is another solution. Since both the functions $\tilde{Y}(z)$ and $Y(z)$ satisfy the same jump condition across the contour C, the matrix ratio $\tilde{Y}(z)Y^{-1}(z)$ has no jump across C. This means that $\tilde{Y}(z)Y^{-1}(z) \equiv$ constant, and from the condition at $z = \infty$ we actually have that $\tilde{Y}(z)Y^{-1}(z) \equiv I$. The uniqueness now follows.

Since Y is the unique solution of the Riemann–Hilbert problem, one can now reconstruct the resolvent R using (2.9) and the similarly derived identity

$$G(z) = (Y_{\pm}^{\mathrm{T}})^{-1}(z) g(z) \tag{2.13}$$

for G. We shall refer to this Riemann–Hilbert problem as the Y-RH problem.

Following [10] the Y-RH problem can be transformed to an equivalent Riemann–Hilbert problem which is directly connected with the polynomials on the circle C orthogonal with respect to the (generally complex) weight $\phi(\mathrm{e}^{\mathrm{i}\theta})$. To this end we first note that since the entries of f are polynomials in z, (2.9) implies that F is an entire function of z. Since $Y(z) \to I$ as $z \to \infty$, it follows in fact that F is polynomial,

$$F(z) = \begin{pmatrix} P_n(z) \\ Q_{n-1}(z) \end{pmatrix}, \quad P_n(z) = z^n + \cdots, \quad Q_{n-1}(z) = q_{n-1} z^{n-1} + \cdots \tag{2.14}$$

for some constant q_{n-1}. On the other hand, denoting by Y_j the jth column of the matrix Y, we obtain from the jump equation (2.10) (or, more precisely, from the equation $Y_+ = Y_- H^{-1}$) that

$$\begin{aligned} Y_{1+}(z) = Y_-(z) \begin{pmatrix} \phi(z) \\ (\phi(z)-1)z^{-n} \end{pmatrix} &= -z^{-n} Y_-(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \phi(z) z^{-n} Y_-(z) \begin{pmatrix} z^n \\ 1 \end{pmatrix} \\ &= -z^{-n} Y_{2-}(z) + \phi(z) z^{-n} Y_-(z) f(z) \\ &= -z^{-n} Y_{2-}(z) + \phi(z) z^{-n} F(z). \end{aligned} \tag{2.15}$$

Define

$$J(z) = \begin{cases} -Y_1(z), & |z| < 1, \\ z^{-n} Y_2(z), & |z| > 1, \end{cases}$$

and consider the 2×2 matrix function

$$Z(z) = \sigma_3 (F(z), J(z)) \sigma_3, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.16}$$

The function Z is analytic outside of C, and it has the following asymptotic behavior as $z \to \infty$:

$$Z(z) = \left(I + \mathrm{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \tag{2.17}$$

For the jump relation on the contour C we have from (2.15),

$$\begin{aligned} Z_+(z) &= \sigma_3(F(z), -Y_{1+}(z))\sigma_3 \\ &= \sigma_3(F(z), z^{-n}Y_{2-}(z) - \phi(z)z^{-n}F(z))\sigma_3 = \sigma_3(F(z), z^{-n}Y_{2-}(z))\begin{pmatrix} 1 & -\phi(z)z^{-n} \\ 0 & 1 \end{pmatrix}\sigma_3 \\ &= \sigma_3(F(z), z^{-n}Y_{2-}(z))\sigma_3\sigma_3\begin{pmatrix} 1 & -\phi(z)z^{-n} \\ 0 & 1 \end{pmatrix}\sigma_3 = Z_-(z)\begin{pmatrix} 1 & \phi(z)z^{-n} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Summarizing the analytic properties of Z we conclude that it solves the following Riemann–Hilbert problem:

- Z is holomorphic for all $z \notin C$,
- $Z(z)z^{-n\sigma_3} \to I, \quad z \to \infty$,
- $Z_+(z) = Z_-(z)S(z), \quad z \in C$,

where the jump matrix $S(z)$ is

$$S(z) = \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}. \tag{2.18}$$

We shall refer to this Riemann–Hilbert problem as Z-RH problem. As in the Y-RH problem, the solution of the Z-RH problem is unique. Indeed, assuming that $\tilde{Z}$ is another solution, we introduce the matrix ratio $X := \tilde{Z}Z^{-1}$. By the same reasoning as in the case of the Y-RH problem, we conclude that X is entire. Since

$$X(z) = (\tilde{Z}(z)z^{-n\sigma_3})(z^{n\sigma_3}Z^{-1}(z)) \to I \quad \text{as } z \to \infty,$$

it follows that $X \equiv I$ and hence, that Z is unique. We note that Y (and hence the resolvent R) can be reconstructed from Z using (2.16). It also should be pointed out that the existence of the solution of the Z-RH problem (as well as of the Y-RH problem) is equivalent to the nondegeneracy of the Toeplitz matrix $T_n(\phi)$, i.e. to the inequality

$$D_n(\phi) \neq 0,$$

which we always assume.

Remark. There is a more direct and elegant way to pass to the Z-RH problem which was pointed out by the referee of this paper. One first notes that the jump matrix H admits the factorization,

$$H(z) = \begin{pmatrix} z^n & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z^{-n} & 0 \\ -1 & z^n \end{pmatrix},$$

which then suggests the definition

$$\tilde{Y}(z) = \begin{cases} Y(z)\begin{pmatrix} z^n & -1 \\ 1 & 0 \end{pmatrix}, & |z| < 1, \\ Y(z)\begin{pmatrix} z^n & 0 \\ 1 & z^{-n} \end{pmatrix} \equiv Y(z)\begin{pmatrix} 1 & 0 \\ z^{-n} & 1 \end{pmatrix}z^{n\sigma_3}, & |z| > 1, \end{cases} \tag{2.19}$$

so that the function $\tilde{Y}$ would satisfy the Riemann–Hilbert problem,

- $\tilde{Y}$ is holomorphic for all $z \notin C$,
- $\tilde{Y}(z)z^{-n\sigma_3} \to I, \quad z \to \infty$,

- $\tilde{Y}_-(z) = \tilde{Y}_+(z)\begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}, \quad z \in C.$

The function $Z(z)$ is related to $\tilde{Y}(z)$ by

$$Z(z) = \sigma_3 \tilde{Y}(z) \sigma_3. \tag{2.20}$$

We conclude this section by summarizing the relation of the Z-RH problem to the orthogonal polynomials on C with respect to the (generally complex) weight ϕ. This relation is due to Deift [3] (see also [10]). [4] Let $\{P_k(z)\}_{k=0,1,\dots}$ denote the system of the monic polynomials defined by

$$P_k(z) = z^k + \cdots, \qquad \int_C P_n(z)\bar{P}_m(z)\phi(z)\frac{\mathrm{d}z}{\mathrm{i}z} = h_n\delta_{nm}, \quad n \geq m,$$

where bar denotes complex conjugation. Similarly, introduce a second system of polynomials, $\{P_k^*(z)\}_{k=0,1,\dots}$, by replacing ϕ with $\bar{\phi}$ in the definition of P_n. Suppose now that

$$D_k(\phi) \neq 0, \quad k = 1, \dots, n+1. \tag{2.21}$$

Then (see [29]) both the sets of polynomials $\{P_k\}_{k=0,1,\dots,n}$ and $\{P_k^*\}_{k=0,1,\dots,n}$ exist, and the normalization constants h_k and h_k^*, $k = 1, \dots, n$, are all nonzero. In fact, we have the explicit representations

$$P_n(z) = \frac{D_{n+1}(\phi|z)}{D_n(\phi)}, \qquad h_n = 2\pi\frac{D_{n+1}(\phi)}{D_n(\phi)}, \qquad P_n^*(z) = \frac{D_{n+1}(\bar{\phi}|z)}{D_n(\bar{\phi})}, \qquad h_n^* = 2\pi\frac{D_{n+1}(\bar{\phi})}{D_n(\bar{\phi})}, \tag{2.22}$$

where $D_{n+1}(\phi|z)$ denotes the Toeplitz determinant $D_{n+1}(\phi)$ whose last row is replaced by the row $(1, z, z^2, \dots, z^n)$. If we define

$$Q_k = -\frac{2\pi}{\bar{h}_k^*}\bar{P}_k^*\left(\frac{1}{\bar{z}}\right)z^k, \tag{2.23}$$

and

$$Z(z) = \begin{pmatrix} P_n(z) & \dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_C P_n(z')(z')^{-n}\phi(z')\frac{\mathrm{d}z'}{z'-z} \\ Q_{n-1}(z) & \dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_C Q_{n-1}(z')(z')^{-n}\phi(z')\frac{\mathrm{d}z'}{z'-z} \end{pmatrix}, \tag{2.24}$$

then it is a calculation to show that this Z defines a (unique) solution of the Z-RH problem (cf. [3,6,12]). Indeed, the analyticity in $\mathbf{C} \setminus C$ and the jump condition follow from the basic properties of Cauchy integral, and the asymptotic condition at $z = \infty$ is equivalent to the fact that the polynomials P_n and P_{n-1}^* are monic orthogonal polynomials with the weights $\phi(z)\,\mathrm{d}z$ and $\bar{\phi}(z)\,\mathrm{d}z$, respectively.

[4] The Z-RH problem is the analog for polynomials on the circle of the Riemann–Hilbert problem derived in [12] for polynomials which are orthogonal with respect to an exponential weight on the line (see also [6]).

3. Toeplitz determinants as integrable systems

3.1. Universal recursion relation

In this section ϕ will be an arbitrary continuous function with Fourier coefficients ϕ_j. We assume that the associated Toeplitz matrix $T_n(\phi)$ is invertible. Then the corresponding matrix RH problem is uniquely solvable, and the following equation connects the Toeplitz determinant $D_n(\phi)$ with the solution Z of the Riemann–Hilbert problem,

$$\frac{D_{n+1}}{D_n} = Z_{12}(0), \tag{3.1}$$

where Z_{ij}, $i, j = 1, 2$, denotes the entries of matrix Z. Indeed, using (2.22) we have that

$$\frac{D_{n+1}}{D_n} = \frac{1}{2\pi} h_n,$$

On the other hand, (2.24) gives

$$Z_{12}(0) = \frac{1}{2\pi} h_n,$$

and (3.1) follows.

Remark. One can prove (3.1) using only the connection with the integrable operator $K_n(\phi)$ introduced in (2.1). To see this first note

$$K_{n+1}(z, z') = \frac{(z/z')^{n+1} - 1}{z - z'} \frac{1 - \phi(z')}{2\pi \mathrm{i}} = \frac{1}{z'} \sum_{k=0}^{n} \left(\frac{z}{z'}\right)^k \frac{1 - \phi(z')}{2\pi \mathrm{i}} = K_n(z, z') + f_1(z) g_1(z') \frac{1}{z'},$$

where f and g are defined in (2.3) and (2.4), respectively. Attaching to the functions f, g superscript "n" to denote their n dependence, we can rewrite the last equation as an operator equation

$$K_{n+1} = K_n + f_1^n \otimes g_1^{n+1}, \tag{3.2}$$

where the symbol $a \otimes b$ denotes the integral operator with kernel $a(z)b(z')$. Recalling the definition of F, (2.6), it follows from (3.2) (cf. [15,31]) that

$$\begin{aligned} \det(1 - K_{n+1}) &= \det(1 - K_n)\det(1 - [(1 - K_n)^{-1} f_1^n] \otimes g_1^{n+1}) = \det(1 - K_n)\det(1 - F_1^n \otimes g_1^{n+1}) \\ &= \det(1 - K_n)(1 - \operatorname{trace} F_1^n \otimes g_1^{n+1}) = \det(1 - K_n) \left(1 - \int_C F_1(z) g_1(z) \frac{\mathrm{d}z}{z}\right), \end{aligned}$$

where $F_1 := F_1^n$ and $g_1 := g_1^n$. Thus (see also (2.2))

$$\frac{D_{n+1}}{D_n} = \frac{\det K_{n+1}}{\det K_n} = 1 - \int_C F_1(z) g_1(z) \frac{\mathrm{d}z}{z}. \tag{3.3}$$

Recalling (2.7), we rewrite (3.3) as

$$\frac{D_{n+1}}{D_n} = Y_{11}(0),$$

which together with (2.16) yields (3.1).

3.2. Differentiation formulae

Here we restrict to the symbol

$$\phi(z) = \mathrm{e}^{tz} \prod_{\alpha=1}^{M} \left(\frac{z - r_\alpha}{z} \right)^{k_\alpha}, \tag{3.4}$$

where $r_\alpha := -p_{i_\alpha}$, and we recall (see Section 1) that $i_\alpha = \max \mathcal{A}_\alpha$, $k_\alpha = |\mathcal{A}_\alpha|$, and

$$\mathcal{A} = \bigcup_{\alpha=1}^{M} \mathcal{A}_\alpha$$

is the decomposition of the alphabet $\mathcal{A}$ into subsets $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M$ such that $p_i = p_j$ if and only if i and j belong the same $\mathcal{A}_\alpha$. We also recall that

$$\sum_{\alpha=1}^{M} k_\alpha = k,$$

and

$$1 > p_1 \geq p_2 \geq \cdots \geq p_k > 0, \quad \sum_{j=1}^{k} p_j = 1 \tag{3.5}$$

denote the probabilities assigned to the letters $i = 1, 2, \dots, k$, in the alphabet $\mathcal{A}$. Note that from the probabilistic conditions (3.5) it follows that

$$-1 < r_\alpha < 0, \quad \alpha = 1, \dots, M, \quad r_\alpha \neq r_\beta, \ \alpha \neq \beta, \tag{3.6}$$

and

$$\sum_{\alpha=1}^{M} k_\alpha r_\alpha = -1. \tag{3.7}$$

The symbols f_I and ϕ are related by

$$f_I(z) = \phi\left(\frac{1}{z}\right),$$

and therefore; the corresponding Toeplitz matrices are mutually transpose. Thus

$$G_I(n; \{p_i\}, t) = D_n(\phi).$$

In what follows, we will write $T_n(t)$, $K_n(t)$ and $D_n(t)$ for $T_n(\phi)$, $K_n(\phi)$ and $D_n(\phi)$, respectively, or $T_n(\{p_i\}, t)$, $K_n(\{p_i\}, t)$ and $D_n(\{p_i\}, t)$ if the dependence on $p_1, \dots, p_k$ is of interest.

We shall derive the differential formulae for the Toeplitz determinant $D_n(t)$ with respect to the variables t and r_α, $\alpha = 1, \dots, M$ assuming that the latter are subject to restriction (3.6) only, i.e. we will only assume that

$$-1 < r_\alpha < 0, \quad \alpha = 1, \dots, M, \quad r_\alpha \neq r_\beta, \ \alpha \neq \beta.$$

The integers k_α will be kept constant. This means that *when vary the points r_α we do not assume restriction* (3.7) *to hold*. We will begin with the t-derivative.

Since $\partial\phi/\partial t = z\phi$,

$$\frac{\partial}{\partial t}K_n(z,z') = \frac{(z/z')^n-1}{z-z'}(-z')\frac{\phi(z')}{2\pi\mathrm{i}} = \frac{1-\phi(z')}{2\pi\mathrm{i}} + z\frac{(z/z')^{n-1}-1}{z-z'}\frac{1-\phi(z')}{2\pi\mathrm{i}} - \frac{(z/z')^n-1}{z-z'}\frac{z'}{2\pi\mathrm{i}}. \tag{3.8}$$

Let Λ be the integral operator with the kernel

$$\Lambda(z,z') = \frac{(z/z')^n-1}{z-z'}\frac{z'}{2\pi\mathrm{i}}.$$

Consider the operator product ΛK_n:

$$\begin{aligned}(\Lambda K_n)(z,z') &= \frac{1-\phi(z')}{2\pi\mathrm{i}}\int_C \frac{(z/w)^n-1}{z-w}w\frac{(w/z')^n-1}{w-z'}\frac{\mathrm{d}w}{2\pi\mathrm{i}} = \frac{1-\phi(z')}{2\pi\mathrm{i}}\int_C\sum_{j,l=0}^{n-1}\left(\frac{z}{w}\right)^l\left(\frac{w}{z'}\right)^j(z')^{-1}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\\ &= \frac{1-\phi(z')}{2\pi\mathrm{i}}\sum_{j-l=-1} z^l(z')^{-j-1} = \frac{1-\phi(z')}{2\pi\mathrm{i}}\sum_{j=0}^{n-2} z^{j+1}(z')^{-j-1} = \frac{1-\phi(z')}{2\pi\mathrm{i}}z\frac{(z/z')^{n-1}-1}{z-z'}.\end{aligned}$$

Recalling the definitions of f and g, (2.3) and (2.4), (3.8) can be written compactly as

$$\frac{\partial}{\partial t}K_n = -f_2\otimes g_2 - \Lambda(1-K_n). \tag{3.9}$$

From this formula we see that (cf. the derivation of (3.3))

$$\frac{\partial}{\partial t}\log D_n(t) = -\mathrm{trace}\left((1-K_n)^{-1}\frac{\partial}{\partial t}K_n\right) = \mathrm{trace}\, F_2\otimes g_2 + \mathrm{trace}\,\Lambda = \int_C F_2(z)g_2(z)\,\mathrm{d}z, \tag{3.10}$$

where we used the fact that

$$\mathrm{trace}\,\Lambda = \frac{n}{2\pi\mathrm{i}}\int_C \mathrm{d}z = 0.$$

Recalling (2.7) we convert (3.10) into the identity

$$\frac{\partial}{\partial t}\log D_n(t) = -\mathrm{res}_{z=\infty}(Y_{22}(z)),$$

which in terms of the Z-function is

$$\frac{\partial}{\partial t}\log D_n(t) = -\mathrm{res}_{z=\infty}(z^n Z_{22}(z)),$$

or equivalently,

$$\frac{\partial}{\partial t}\log D_n(t) = (\Gamma_1)_{22}, \tag{3.11}$$

where the matrix $\Gamma_1 = \Gamma_1(\{p_i\},t)$ is defined by the expansion,

$$Z(z) = \left(I + \sum_{j=1}^{\infty}\frac{\Gamma_j}{z^j}\right)z^{n\sigma_3}, \quad |z|>1. \tag{3.12}$$

Remark. In the basis $\{z^n\}_{n=-\infty}^{\infty}$, (3.9) coincides with (3.22) of [30].

Eq. (3.11) is the *t-differentiation formula*, i.e. it gives an expression of the t-derivative of $\log D_n$ in terms of the solution Z of the Riemann–Hilbert problem. We shall proceed now with the derivation of the r_α-differentiation formula. Since (we recall that r_α are assumed independent and that the k_α are kept constant)

$$\frac{\partial}{\partial r_\alpha}\phi = -\frac{k_\alpha}{z-r_\alpha}\phi,$$

the r_α-analog of (3.8) reads

$$\frac{\partial}{\partial r_\alpha}K_n(z,z') = k_\alpha \frac{(z/z')^n-1}{z-z'}\frac{1}{z'-r_\alpha}\frac{\phi(z')}{2\pi \mathrm{i}} = \frac{k_\alpha}{2\pi \mathrm{i}}\frac{(z/z')^n-1}{z-z'}\frac{1}{z'-r_\alpha} - k_\alpha \frac{(z/z')^n-1}{z-z'}\frac{1}{z'-r_\alpha}\frac{1-\phi(z')}{2\pi \mathrm{i}}. \tag{3.13}$$

Introducing the integral operator Λ_α with the kernel,

$$\Lambda_\alpha(z,z') = \frac{k_\alpha}{2\pi \mathrm{i}}\frac{(z/z')^n-1}{z-z'}\frac{1}{z'-r_\alpha},$$

we consider again the operator product $\Lambda_\alpha K_n$. The residue type calculations, similar to the ones used in the t-case, yield the equation

$$(\Lambda_\alpha K_n)(z,z') = \frac{k_\alpha}{2\pi \mathrm{i}}(1-\phi(z'))\left[\frac{(z/z')^n-1}{(z'-z)(r_\alpha-z)} + \frac{(r_\alpha/z')^n-1}{(r_\alpha-z)(r_\alpha-z')}\right],$$

which in turn implies that (3.13) can be rewritten as

$$\frac{\partial}{\partial r_\alpha}K_n(z,z') = \frac{k_\alpha}{2\pi \mathrm{i}}\frac{1-\phi(z')}{(r_\alpha-z)(r_\alpha-z')}\left[\left(\frac{r_\alpha}{z'}\right)^n - \left(\frac{z}{z'}\right)^n\right] + [\Lambda_\alpha(1-K_n)](z,z').$$

With the help of the vector functions,

$$\tilde{f}(z) := \frac{1}{z-r_\alpha}f(z), \qquad \tilde{g}(z) := \frac{1}{z-r_\alpha}g(z),$$

the last equation can be transformed into the following compact form (cf. (3.9)):

$$\frac{\partial}{\partial r_\alpha}K_n = k_\alpha r_\alpha^n \tilde{f}_2 \otimes \tilde{g}_1 - k_\alpha \tilde{f}_1 \otimes \tilde{g}_1 + \Lambda_\alpha(1-K_n). \tag{3.14}$$

Let the vector function $\tilde{F}(z) = (\tilde{F}_1(z), \tilde{F}_2(z))^{\mathrm{T}}$ be defined by the equation

$$\tilde{F}_j := (1-K_n)^{-1}\tilde{f}_j, \quad j = 1,2.$$

We observe that

$$\tilde{F}(z) = \frac{1}{z-r_\alpha}Y^{-1}(r_\alpha)F(z), \tag{3.15}$$

where the matrix function $Y(z)$ is the solution of the Y-RH problem corresponding to $D_n(t)$. Indeed by the definition of the vector function $F(z)$ (see (2.6)) its component $F_j(z)$ satisfies the integral equation,

$$F_j(z) - \int_C K_n(z,z')F_j(z')\,\mathrm{d}z' = f_j(z). \tag{3.16}$$

Dividing both sides of this equation by $(z-r_\alpha)$, using the formula

$$K_n(z,z') = \frac{f^{\mathrm{T}}(z)g(z')}{z-z'},$$

and simple algebra, we can rewrite (3.16) as an equation for the ratio $F_j/(z-r_\alpha)$:

$$\left[\frac{F_j(z)}{z-r_\alpha}\right] - \int_C K_n(z,z')\left[\frac{F_j(z')}{z'-r_\alpha}\right]\mathrm{d}z' + \int_C \tilde{f}^{\mathrm{T}}(z)g(z')\left[\frac{F_j(z')}{z'-r_\alpha}\right]\mathrm{d}z' = \tilde{f}_j(z).$$

By applying the operator $(1-K_n)^{-1}$ to the both sides of this equation it can be transformed into the relation,

$$\left[\frac{F_j(z)}{z-r_\alpha}\right] + \int_C \tilde{F}^{\mathrm{T}}(z)g(z')\left[\frac{F_j(z')}{z'-r_\alpha}\right]\mathrm{d}z' = \tilde{F}_j(z),$$

or

$$\frac{1}{z-r_\alpha}F_j(z) + \sum_{i=1}^{2}\tilde{F}_i(z)\int_C g_i(z')F_j(z')\frac{\mathrm{d}z'}{z'-r_\alpha} = \tilde{F}_j(z).$$

The last equation in turn can be viewed as the linear algebraic system for the vector $\tilde{F}(z)$,

$$\tilde{F}_j(z) - \sum_{i=1}^{2}A_{ji}\tilde{F}_i(z) = \frac{1}{z-r_\alpha}F_j(z), \quad j=1,2, \tag{3.17}$$

where the matrix A is given by the formula,

$$A_{ji} = \int_C F_j(z')g_i(z')\frac{\mathrm{d}z'}{z'-r_\alpha}.$$

Eq. (3.15) follows directly from (3.17) in virtue of definition (2.7) of the matrix function $Y(z)$.

We now able to finish the derivation of the r_α-differentiation formula for the Toeplitz determinant $D_n(t)$. In fact from (3.14) it follows that (cf. the derivation of (3.10))

$$\begin{aligned}\frac{\partial}{\partial r_\alpha}\log D_n(t) &= -\mathrm{trace}\left((1-K_n)^{-1}\frac{\partial}{\partial r_\alpha}K_n\right) = -k_\alpha r_\alpha^n\,\mathrm{trace}\,\tilde{F}_2\otimes\tilde{g}_1 + k_\alpha\,\mathrm{trace}\,\tilde{F}_1\otimes\tilde{g}_1 - \mathrm{trace}\,\Lambda_\alpha\\ &= -k_\alpha r_\alpha^n\int_C \tilde{F}_2(z)g_1(z)\frac{\mathrm{d}z}{z-r_\alpha} + k_\alpha\int_C\tilde{F}_1(z)g_1(z)\frac{\mathrm{d}z}{z-r_\alpha},\end{aligned} \tag{3.18}$$

where, similar to the t-derivative case, we used the fact that

$$\mathrm{trace}\,\Lambda_\alpha = \frac{nk_\alpha}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}z}{z(z-r_\alpha)} = 0.$$

Using now (3.15) and the fact that $\det Y(z)\equiv 1$ we derive from (3.18) that

$$\begin{aligned}\frac{\partial}{\partial r_\alpha}\log D_n(t) &= k_\alpha r_\alpha^n Y_{21}(r_\alpha)\int_C F_1(z)g_1(z)\frac{\mathrm{d}z}{(z-r_\alpha)^2} - k_\alpha r_\alpha^n Y_{11}(r_\alpha)\int_C F_2(z)g_1(z)\frac{\mathrm{d}z}{(z-r_\alpha)^2}\\ &\quad + k_\alpha Y_{22}(r_\alpha)\int_C F_1(z)g_1(z)\frac{\mathrm{d}z}{(z-r_\alpha)^2} - k_\alpha Y_{12}(r_\alpha)\int_C F_2(z)g_1(z)\frac{\mathrm{d}z}{(z-r_\alpha)^2},\end{aligned}$$

or

$$\frac{\partial}{\partial r_\alpha}\log D_n(t) = -k_\alpha r_\alpha^n Y_{21}(r_\alpha)Y'_{11}(r_\alpha) + k_\alpha r_\alpha^n Y_{11}(r_\alpha)Y'_{21}(r_\alpha) - k_\alpha Y_{22}(r_\alpha)Y'_{11}(r_\alpha) + k_\alpha Y_{12}(r_\alpha)Y'_{21}(r_\alpha), \tag{3.19}$$

where we use the notation,

$$Y'_{ij}(r_\alpha) := \left.\frac{\partial Y_{ij}(z)}{\partial z}\right|_{z=r_\alpha}, \quad i,j=1,2.$$

Eq. (3.19) can be also rewritten as

$$\frac{\partial}{\partial r_\alpha}\log D_n(t) = -k_\alpha(r_\alpha^n Y_{21}(r_\alpha) + Y_{22}(r_\alpha))Y'_{11}(r_\alpha) + k_\alpha(r_\alpha^n Y_{11}(r_\alpha) + Y_{12}(r_\alpha))Y'_{21}(r_\alpha), \tag{3.20}$$

which in turn can be transformed into an expression of the $\partial \log D_n(t)/\partial r_\alpha$ in terms of the Z-function. Indeed recalling formulae (2.3), (2.9) and (2.16), we see that inside the unit circle C the following equation takes place:

$$Z(z) = \begin{pmatrix} z^n Y_{11}(z) + Y_{12}(z) & Y_{11}(z) \\ -z^n Y_{21}(z) - Y_{22}(z) & -Y_{21}(z) \end{pmatrix}, \quad |z| < 1,$$

so that (3.20) can be converted into the r_α-differentiation formula,

$$\frac{\partial}{\partial r_\alpha}\log D_n(t) = -k_\alpha(Z_{11}(r_\alpha)Z'_{22}(r_\alpha) - Z_{21}(r_\alpha)Z'_{12}(r_\alpha)). \tag{3.21}$$

3.3. Schlesinger equations

In this section we show that $D_n(t)$ is the Jimbo–Miwa–Ueno τ-function of the generalized Schlesinger system of nonlinear differential equations describing the isomonodromy deformations of the 2×2 matrix linear ODE which has $M + 1$ simple poles in the finite complex plane and one Poincaré index 1 irregular singular point at infinity. We will also evaluate the relevant monodromy data that single out the $D_n(t)$ from all the other solutions of the Schlesinger system. In the uniform case, when all p_i are equal, the system reduces to the particular case of Painlevé V equation, i.e. we are back to the uniform result of [30].

Define

$$\Phi^0(z) = \mathrm{e}^{(tz/2)\sigma_3}\begin{pmatrix} 1 & 0 \\ 0 & z^n\psi^{-1}(z) \end{pmatrix}, \tag{3.22}$$

where

$$\psi(z) = \prod_{\alpha=1}^{M}\left(\frac{z - r_\alpha}{z}\right)^{k_\alpha}. \tag{3.23}$$

We note that that the product

$$\mathrm{e}^{tz}\psi(z) := \phi(z)$$

is our symbol, i.e. the function defined in (3.4). We also note that Φ^0 is analytic and invertible in $\mathbf{C}\setminus\{0, r_1, \ldots, r_M\}$, and that it satisfies the linear differential equations

$$\Phi^0_z(z) = \Omega(z)\Phi^0(z), \tag{3.24}$$

$$\Phi^0_t(z) = \tfrac{1}{2}z\sigma_3\Phi^0(z), \tag{3.25}$$

$$\Phi^0_{r_\alpha}(z) = \frac{k_\alpha}{z - r_\alpha}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Phi^0(z), \tag{3.26}$$

where Ω is the rational matrix function,

$$\Omega(z) = \frac{t}{2}\sigma_3 + \frac{n+k}{z}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \sum_{\alpha=1}^{M}\frac{k_\alpha}{z - r_\alpha}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.27}$$

(Subscripts on Φ^0 denote differentiation.) Introduce

$$\Phi(z) = Z(z)\Phi^0(z), \tag{3.28}$$

where Z is the solution of the Z-RH problem corresponding to our symbol ϕ, and consider the logarithmic derivative

$$B(z) := \Phi_z(z)\Phi^{-1}(z). \tag{3.29}$$

The key observation is that B is continuous across the contour C. Indeed, the Z-jump matrix S (see (2.18)) admits the following factorization:

$$S(z) = \Phi^0(z)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(\Phi^0(z))^{-1}, \tag{3.30}$$

so that the Φ-jump matrix does not depend on z. In fact we have

$$\Phi_+(z) = \Phi_-(z)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad z \in C. \tag{3.31}$$

This implies that

$$B_+(z) = B_-(z), \quad z \in C,$$

and hence the function $B(z)$ is an analytic function on $\mathbf{C} \setminus \{0, r_1, \ldots, r_M\}$. We also recall that the only conditions which we impose on the points r_α are the inequalities (3.6), i.e.,

$$-1 < r_\alpha < 0, \quad \alpha = 1, \ldots, M, \quad r_\alpha \neq r_\beta, \ \alpha \neq \beta. \tag{3.32}$$

We now calculate the principal part of B at each of its singular points. Since Z is holomorphic and invertible inside of C, it follows from:

$$B(z) = Z(z)\Omega(z)Z^{-1}(z) + Z_z(z)Z^{-1}(z) \tag{3.33}$$

that in a neighborhood of $z = r_\alpha$,

$$B(z) = -\frac{k_\alpha}{z - r_\alpha} Z(r_\alpha)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Z^{-1}(r_\alpha) + \sum_{j=0}^{\infty} b_j^\alpha (z - r_\alpha)^j. \tag{3.34}$$

Likewise in a neighborhood of $z = 0$, (3.33) and (3.27) imply that

$$B(z) = \frac{n + k}{z} Z(0)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Z^{-1}(0) + \sum_{j=0}^{\infty} b_j^0 z^j. \tag{3.35}$$

Finally, from (3.33) and the Laurent expansion (3.12) we obtain the power series of B at ∞,

$$B(z) = \frac{t}{2}\sigma_3 + \sum_{j=1}^{\infty} b_j^\infty z^{-j}. \tag{3.36}$$

Eqs. (3.34),(3.45) and (3.46) imply that B is a rational function,

$$B(z) = \frac{t}{2}\sigma_3 + \frac{B_0}{z} + \sum_{\alpha=1}^{M} \frac{B_\alpha}{z - r_\alpha}, \tag{3.37}$$

with the matrix residues given by

$$B_0 = Z(0)\begin{pmatrix} 0 & 0 \\ 0 & n+k \end{pmatrix} Z^{-1}(0), \tag{3.38}$$

$$B_\alpha = -Z(r_\alpha)\begin{pmatrix} 0 & 0 \\ 0 & k_\alpha \end{pmatrix} Z^{-1}(r_\alpha), \quad \alpha = 1, \dots, M. \tag{3.39}$$

Thus from (3.29) we conclude that Φ satisfies the linear differential equation,

$$\Phi_z(z) = B(z)\Phi(z), \tag{3.40}$$

with the coefficient matrix B determined by (3.37)–(3.39).

Remark. In soliton theory (see [24]), the method that we used to derive (3.40) is called *Zakharov–Shabat dressing* of the vacuum equation (3.24). We also note that, as is common in the analysis of soliton equations, we have moved the exponential factor $e^{\pm z/2}$ to the asymptotic condition at $z = \infty$.

Let us now dress the t-vacuum equation (3.25), i.e. consider the t-logarithmic derivative of Φ

$$V(z) := \Phi_t(z)\Phi^{-1}(z). \tag{3.41}$$

The Φ-jump matrix (3.31) does not depend on t as well. Hence

$$V_+(z) = V_-(z), \quad z \in C,$$

and V is analytic on $\mathbf{C} \setminus \{0, r_1, \dots, r_m\}$. In fact, since

$$V(z) = Z(z)\tfrac{1}{2}(z)\sigma_3 Z^{-1}(z) + Z_t(z)Z^{-1}(z) \tag{3.42}$$

(cf. (3.33)) and Z is holomorphic at the points $\{0, r_1, \dots, r_M\}$, we conclude that V is entire. Moreover, from the expansion (3.12) we have that

$$V(z) = \frac{z}{2}\sigma_3 + \frac{1}{2}[\sigma_3, \Gamma_1] + \sum_{j=1}^{\infty} v_j z^{-j}, \quad |z| > 1,$$

and hence

$$V(z) = \tfrac{1}{2}z\sigma_3 + \tfrac{1}{2}[\sigma_3, \Gamma_1]. \tag{3.43}$$

($[L, M] := LM - ML$.)

This in turn yields the t-equation for Φ,

$$\Phi_t(z) = V(z)\Phi(z), \tag{3.44}$$

where the coefficient matrix V is defined by the equations,

$$V(z) = \tfrac{1}{2}z\sigma_3 + V_0, \tag{3.45}$$

$$V_0 = \tfrac{1}{2}[\sigma_3, \Gamma_1], \tag{3.46}$$

$$\Gamma_1 = -\mathrm{res}_{z=\infty}(Z(z)z^{-n\sigma_3}) \tag{3.47}$$

(see also (3.12)).

Eqs. (3.40) and (3.44) form an overdetermined system for the function Φ in the variables z and t. From the compatibility condition,

$$\Phi_{zt} = \Phi_{tz},$$

we derive the following equation for the coefficient matrices B and V:

$$B_t(z) - V_z(z) = [V(z), B(z)], \tag{3.48}$$

or, taking into account (3.45),

$$B_t(z) - \tfrac{1}{2}\sigma_3 = \tfrac{1}{2}z[\sigma_3, B(z)] + [V_0, B(z)]. \tag{3.49}$$

Since this equation is satisfied identically in z, a comparison of the principal parts of both the sides at $z = 0, r_1, \dots, r_M$, then leads to the differential relations,

$$\frac{\partial B_\alpha}{\partial t} = [\tfrac{1}{2}(r_\alpha)\sigma_3 + V_0, B_\alpha], \quad \alpha = 0, 1, \dots, M. \tag{3.50}$$

(It is notationally convenient to define $r_0 = 0$.)

The important point now is that the matrix V_0 can be expressed in terms of the matrices B_α, so that relations (3.50) form a closed system of nonlinear ODEs for the matrix residues B_α. In fact, expanding both sides of (3.33) in a Laurent series at $z = \infty$, using (3.12)–(3.37), and equating the terms of order z^{-1} we have

$$\begin{aligned}\sum_{\alpha=0}^{m} B_\alpha &= \frac{t}{2}[\Gamma_1, \sigma_3] + (n+k)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} - \sum_{\alpha=1}^{m} k_\alpha \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + n\sigma_3 \\ &= \frac{t}{2}[\Gamma_1, \sigma_3] + n\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + n\sigma_3 = \frac{t}{2}[\Gamma_1, \sigma_3] + \frac{n}{2}\sigma_3 + \frac{n}{2}I.\end{aligned} \tag{3.51}$$

Comparing the last equation with (3.46) we obtain

$$V_0 = \frac{1}{t}\sum_{\alpha=0}^{m} B_\alpha - \frac{n}{2t}\sigma_3 - \frac{n}{2t}I, \tag{3.52}$$

so that (3.50) becomes

$$\frac{\partial B_\alpha}{\partial t} = \frac{n - tr_\alpha}{2t}[B_\alpha, \sigma_3] + \sum_{\gamma=0}^{m} \frac{[B_\gamma, B_\alpha]}{t}, \quad \alpha = 0, 1, \dots, M. \tag{3.53}$$

If we vary the points r_α, then we obtain M additional linear differential equations for $\Phi(z) = \Phi(z, t, r_1, \dots, r_M)$,

$$\Phi_{r_\alpha}(z) = -\frac{B_\alpha}{z - r_\alpha}\Phi(z), \quad \alpha = 1, \dots, M. \tag{3.54}$$

Indeed, introducing the r_α-logarithmic derivative,

$$U_\alpha(z) := \Phi_{r_\alpha}(z)\Phi^{-1}(z),$$

and using exactly the same line of arguments as before, we conclude that U_α is analytic on $\mathbf{C} \setminus \{0, r_1, \dots, r_m\}$. Simultaneously, the r_α-vacuum equation (3.26) implies the identity,

$$U_\alpha(z) = \frac{k_\alpha}{z - r_\alpha} Z(z)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} Z^{-1}(z) + Z_{r_\alpha}(z)Z^{-1}(z) \tag{3.55}$$

(cf. (3.33) and (3.42)) which indicates that the only singularity of U_α is a simple pole at $z = r_\alpha$ with

$$\frac{k_\alpha}{z-r_\alpha} Z(r_\alpha) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Z^{-1}(r_\alpha) \equiv -\frac{B_\alpha}{z-r_\alpha}$$

(see also (3.39)) as the corresponding principal part. Moreover, taking into account that the asymptotics of $Z(z)$ as $z \to \infty$ does not depend on r_α we conclude that

$$U_\alpha(z) \to 0, \quad z \to \infty,$$

and hence

$$U_\alpha(z) = -\frac{B_\alpha}{z-r_\alpha}.$$

Eq. (3.54) now follows.

The compatibility conditions of Eqs. (3.54) with (3.40) lead to the nonlinear r-differential equations for the matrices $B_\alpha = B_\alpha(t, r_1, \dots, r_M)$,

$$\frac{\partial B_\alpha}{\partial r_\gamma} = \frac{[B_\alpha, B_\gamma]}{r_\alpha - r_\gamma}, \quad \alpha \neq \gamma = 1, \dots, M, \tag{3.56}$$

$$\frac{\partial B_0}{\partial r_\alpha} = \frac{[B_0, B_\alpha]}{r_0 - r_\alpha}, \quad \alpha = 1, \dots, M, \tag{3.57}$$

$$\frac{\partial B_\alpha}{\partial r_\alpha} = \sum_{\gamma \neq \alpha} \frac{[B_\gamma, B_\alpha]}{r_\gamma - r_\alpha}, \quad \alpha = 1, \dots, M, \tag{3.58}$$

which supplement t-equation (3.53).

The total system ((3.53)–(3.58)) of nonlinear PDEs is the (generalized) system of Schlesinger equations which describes the isomonodromy deformations (see e.g. [18,19]) of the coefficients of the 2×2 system of linear ODEs having $M + 1$ regular singularities at the points $z = r_\alpha$, $\alpha = 0, \dots, M$ and an irregular singular point of Poincaré index 1 at infinity (see (3.37) and (3.40)),

$$\frac{\mathrm{d}\Phi(z)}{\mathrm{d}z} = B(z)\Phi(z), \quad B(z) = \frac{t}{2}\sigma_3 + \sum_{\alpha=0}^{M} \frac{B_\alpha}{z - r_\alpha}. \tag{3.59}$$

The monodromy data of Eq. (3.59) which single out the solution of ((3.53),(3.56),(3.57) and (3.58)), which we are interested in, coincide, after the proper normalization, with the data of the Z-RH problem. More precisely, let us denote $\Phi^\infty(z)$ the analytic continuation of $\Phi(z)$ from $|z| > 1$ to the whole complex z-plane. Then, the Z-RH problem and Eq. (3.28) imply the following representations of the function $\Phi^\infty(z)$ in the neighborhoods of its singular points:

$$\Phi^\infty(z) = \hat{\Phi}_\alpha(z) \begin{pmatrix} 1 & 0 \\ 0 & (z-r_\alpha)^{-k_\alpha} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad z \in U_{r_\alpha}, \tag{3.60}$$

$$\Phi^\infty(z) = \hat{\Phi}_0(z) \begin{pmatrix} 1 & 0 \\ 0 & z^{n+k} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad z \in U_0, \tag{3.61}$$

$$\Phi^\infty(z) = \hat{\Phi}_\infty(z)\, \mathrm{e}^{(tz/2)\sigma_3} \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in U_\infty, \ \ \hat{\Phi}_\infty(\infty) = I. \tag{3.62}$$

Here $\hat{\Phi}_\alpha(z)$, $\hat{\Phi}_0(z)$, and $\hat{\Phi}_\infty(z)$ denote the matrix functions which are holomorphic and invertible in the neighborhoods U_{r_α}, U_0, and U_∞, respectively. Formulae (3.60),(3.61) and (3.62) allow us to identify the diagonal matrices,

$$E_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & -k_\alpha \end{pmatrix}, \qquad E_0 = \begin{pmatrix} 0 & 0 \\ 0 & n+k \end{pmatrix}, \quad \text{and} \quad E_\infty = \begin{pmatrix} -n & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.63}$$

as the formal monodromy exponents (cf. [18,19]) of $\Phi^\infty(z)$ at the points r_α, 0, and ∞, respectively. The corresponding connection matrices, i.e. the matrices C_α in the representations,

$$\Phi^\infty(z) = \hat{\Phi}_\alpha(z)(z - r_\alpha)^{E_\alpha} C_\alpha, \quad \alpha = 0, \dots, M,$$

all are given by

$$C_\alpha = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \alpha = 0, \dots, M. \tag{3.64}$$

Since the numbers k_α, k, and n are integers, all the monodromy matrices of $\Phi^\infty(z)$ are trivial. There are also no Stokes' matrices at the irregular singular point $z = \infty$ since the asymptotic series (3.12), as a Laurent series, converges in a disk centered at infinity. Therefore the complete monodromy data of the linear system (3.59) for our random word problem, consists of (i) (3.63), the formal monodromy exponents at the singular points, and (ii) (3.64), the corresponding connection matrices.

3.4. Toeplitz determinant as a τ-function

In this section we shall derive the exact formulae for the logarithmic derivatives of the Toeplitz determinant $D_n(t, r_1, \dots, r_M)$ in terms of the matrices B_α which, as we saw in the previous section, satisfy the Schlesinger equations ((3.53)–(3.58)). To this end we will exploit (3.11) and (3.21) whose right-hand sides we will express via B_α using a technique similar to the one that led to (3.52). We begin with (3.11).

Eq. (3.52) was obtained by expanding both sides of (3.33) about ∞ and then equating the terms of order z^{-1}. Let us now analyze the terms of order z^{-2}. From (3.27) it follows that

$$\Omega(z) = \frac{t}{2}\sigma_3 + \frac{n}{z}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{z^2}\sum_{\alpha=1}^{M} r_\alpha k_\alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{O}\left(\frac{1}{z^3}\right) := \frac{t}{2}\sigma_3 + \frac{\Omega_1}{z} + \frac{\Omega_2}{z^2} + \mathrm{O}\left(\frac{1}{z^3}\right).$$

Combining this with expansion (3.12) of Z, we get the following expression for the order z^{-2} term of the right-hand side of (3.33):

$$\tfrac{1}{2}t[\sigma_3, \Gamma_1]\Gamma_1 + \tfrac{1}{2}t[\Gamma_2, \sigma_3] + [\Gamma_1, \Omega_1] + \Omega_2 - \Gamma_1 + \tfrac{1}{2}n[\Gamma_1, \sigma_3].$$

The order z^{-2} term of the left-hand side follows directly from (3.37):

$$\sum_{\alpha=1}^{M} r_\alpha B_\alpha.$$

Equating the two expressions we arrive at

$$\sum_{\alpha=1}^{M} r_\alpha B_\alpha = \frac{t}{2}[\sigma_3, \Gamma_1]\Gamma_1 + \frac{t}{2}[\Gamma_2, \sigma_3] + [\Gamma_1, \Omega_1] + \Omega_2 - \Gamma_1 + \frac{n}{2}[\Gamma_1, \sigma_3]. \tag{3.65}$$

This equation together with (3.51) determines Γ_1 in terms of the matrices B_α. Indeed, (3.51) gives the off diagonal part of Γ_1. Using that for L diagonal

$$\mathrm{diag}[P, L] = 0,$$

we have from (3.65) that

$$\mathrm{diag}\,\Gamma_1 = -\mathrm{diag}\sum_{\alpha=1}^{m} r_\alpha B_\alpha + \frac{t}{2}\mathrm{diag}([\sigma_3, \Gamma_1]\Gamma_1) + \Omega_2. \tag{3.66}$$

Using the identity that for any 2×2 matrix P,

$$\mathrm{diag}([\sigma_3, P]P) = -\tfrac{1}{2}[\sigma_3, P]^2\sigma_3,$$

we obtain from (3.51) and (3.66) the final expression for the diagonal part of Γ_1,

$$\begin{aligned}
\mathrm{diag}\,\Gamma_1 &= -\mathrm{diag}\sum_{\alpha=1}^{M} r_\alpha B_\alpha - \frac{1}{t}\left(\sum_{\alpha=0}^{M} B_\alpha - \frac{n}{2}\sigma_3 - \frac{n}{2}I\right)\left(\sum_{\alpha=0}^{M} B_\alpha - \frac{n}{2}\sigma_3 - \frac{n}{2}I\right)\sigma_3 + \Omega_2 \\
&= -\mathrm{diag}\sum_{\alpha=1}^{M} r_\alpha B_\alpha - \frac{1}{t}\mathrm{diag}\left(\left(\sum_{\alpha=0}^{M} B_\alpha - \frac{n}{2}\sigma_3 - \frac{n}{2}I\right)\sum_{\alpha=0}^{M} B_\alpha\sigma_3\right) + \Omega_2 \\
&= -\mathrm{diag}\sum_{\alpha=1}^{M} r_\alpha B_\alpha - \frac{1}{t}\mathrm{diag}\left(\sum_{\alpha,\gamma=0}^{M} B_\alpha B_\gamma\sigma_3 - \frac{n}{2}\sum_{\alpha=0}^{M}(\sigma_3 + I)B_\alpha\sigma_3\right) + \Omega_2 \\
&= -\mathrm{diag}\sum_{\alpha=1}^{M} r_\alpha B_\alpha - \frac{1}{t}\mathrm{diag}\left(\sum_{\alpha,\gamma=0}^{M} B_\alpha B_\gamma\sigma_3 - \frac{n}{2}\sum_{\alpha=0}^{M} B_\alpha(\sigma_3 + I)\right) + \Omega_2.
\end{aligned} \tag{3.67}$$

We also made use of the identities,

$$\mathrm{diag}(PL) = 0 \quad \text{if} \;\; \mathrm{diag}\,P = 0 \;\; \text{and} \;\; \mathrm{diag}\,L = L,$$

and

$$\mathrm{diag}\left(\sum_{\alpha=0}^{M} B_\alpha - \frac{n}{2}\sigma_3 - \frac{n}{2}I\right) = 0. \tag{3.68}$$

(The latter follows from (3.51).)

We are at last ready to evaluate $\partial \log D_n/\partial t$ in terms of B_α. To this end it is convenient to use

$$\mathrm{trace}\,\Gamma_1 = 0 \quad (\text{which follows from } \det Z \equiv 1)$$

to rewrite (3.11) in the form

$$\frac{\partial}{\partial t}\log D_n(t) = -\frac{1}{2}\mathrm{trace}(\Gamma_1\sigma_3), \tag{3.69}$$

and then use (3.67) and (3.68) to obtain

$$\frac{\partial}{\partial t}\log D_n(t) = \frac{1}{2}\sum_{\alpha=0}^{M} r_\alpha \mathrm{trace}(B_\alpha\sigma_3) + \frac{1}{2t}\sum_{\alpha,\gamma=0}^{M}\mathrm{trace}\,B_\alpha B_\gamma - \frac{n^2}{2t} - \frac{1}{2}\sum_{\alpha=1}^{M} r_\alpha k_\alpha. \tag{3.70}$$

For future comparison with the Jimbo–Miwa–Ueno τ-function it is convenient to use (3.68) one more time and rewrite (3.70) as

$$\frac{\partial}{\partial t}\log D_n(t) = \frac{1}{2}\sum_{\alpha=0}^{M} r_\alpha\,\mathrm{trace}(B_\alpha\sigma_3) + \frac{1}{2t}\sum_{j\neq i=1,2}\left(\sum_{\alpha=0}^{M} B_\alpha\right)_{ij}\left(\sum_{\gamma=0}^{M} B_\gamma\right)_{ji} - \frac{1}{2}\sum_{\alpha=1}^{M} r_\alpha k_\alpha. \tag{3.71}$$

Let us now perform the similar transformations with the right-hand side of Eq. (3.21). We first notice that its subscripts-free form can be written down as

$$\frac{\partial}{\partial r_\alpha}\log D_n(t) = \mathrm{trace}(Z^{-1}(r_\alpha)Z'(r_\alpha)E_\alpha), \tag{3.72}$$

where

$$E_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & -k_\alpha \end{pmatrix}$$

is the formal monodromy exponent at r_α (see (3.63)) and in transforming (3.21) into (3.72) we took into account that $\det Z(z) \equiv 1$. Secondly, by rewriting Eq. (3.33) as the equation

$$Z^{-1}(z)Z'(z) = Z^{-1}(z)B(z)Z(z) - \Omega(z),$$

we get the following representation of the product $Z^{-1}(r_\alpha)Z'(r_\alpha)E_\alpha$:

$$Z^{-1}(r_\alpha)Z'(r_\alpha)E_\alpha = \frac{t}{2}Z^{-1}(r_\alpha)\sigma_3 Z(r_\alpha)E_\alpha + \sum_{\substack{\gamma=0\\ \gamma\neq\alpha}}^{M}\frac{Z^{-1}(r_\alpha)B_\gamma Z(r_\alpha)E_\alpha}{r_\alpha - r_\gamma} - \frac{t}{2}\sigma_3 E_\alpha - \sum_{\substack{\gamma=0\\ \gamma\neq\alpha}}^{M}\frac{E_\gamma E_\alpha}{r_\alpha - r_\gamma}$$
$$+[Z^{-1}(r_\alpha)Z'(r_\alpha)E_\alpha, E_\alpha]. \tag{3.73}$$

(For notational convenience we set, as before, $r_0 := 0$ and $k_0 := -n-k$.)

Using Eq. (3.73) in the right-hand side of Eq. (3.72) and taking into account that

$$B_\alpha = Z(r_\alpha)E_\alpha Z^{-1}(r_\alpha),$$

we arrive to the following r_α-analog of (3.70),

$$\frac{\partial}{\partial r_\alpha}\log D_n(t) = \frac{t}{2}\mathrm{trace}(B_\alpha\sigma_3) + \sum_{\substack{\gamma=0\\ \gamma\neq\alpha}}^{M}\frac{\mathrm{trace}(B_\alpha B_\gamma)}{r_\alpha - r_\gamma} - \frac{k_\alpha t}{2} - \sum_{\substack{\gamma=0\\ \gamma\neq\alpha}}^{M}\frac{k_\alpha k_\gamma}{r_\alpha - r_\gamma}. \tag{3.74}$$

Combining Eqs. (3.71) and (3.74) we obtain the main result of this section which is the following equation for the total differential of the function $\log D_n(t, r_1, \ldots, r_M)$,

$$\mathrm{d}\log D_n = \frac{1}{2}\sum_{\substack{\alpha,\gamma=0\\ \alpha\neq\gamma}}^{M}\mathrm{trace}(B_\alpha B_\gamma)\frac{\mathrm{d}r_\alpha - \mathrm{d}r_\gamma}{r_\alpha - r_\gamma} + \frac{1}{2}\sum_{\alpha=0}^{M}\mathrm{trace}(B_\alpha\sigma_3)\mathrm{d}(r_\alpha t)$$
$$+\frac{1}{2}\sum_{\substack{1\leq i,j\leq 2\\ i\neq j}}\left(\sum_{\alpha=0}^{M} B_\alpha\right)_{ij}\left(\sum_{\gamma=0}^{M} B_\gamma\right)_{ji}\frac{\mathrm{d}t}{t} - \frac{1}{2}\sum_{\alpha=1}^{M}k_\alpha\mathrm{d}(r_\alpha t) - \frac{1}{2}\sum_{\substack{\alpha,\gamma=0\\ \alpha\neq\gamma}}^{M}k_\alpha k_\gamma\frac{\mathrm{d}r_\alpha - \mathrm{d}r_\gamma}{r_\alpha - r_\gamma}. \tag{3.75}$$

Eq. (3.75) describes the Toeplitz determinant $D_n(t)$ in terms of the solution of the Schlesinger system ((3.53)–(3.58)) up to a multiplicative constant (depending on n and k_α). Simultaneously, this equation shows, upon comparison with the expression (5.17) in [18] for the logarithmic derivative of the τ-function, that

$$D_n(t) = \mathrm{e}^{-(t/2)\sum_\alpha r_\alpha k_\alpha} \prod_{\substack{\alpha,\gamma=0\\ \alpha\neq\gamma}}^{M} |r_\alpha - r_\gamma|^{-(k_\alpha k_\gamma/2)} \tau_{\mathrm{JMU}}, \tag{3.76}$$

where we use the notation $\tau_{\mathrm{JMU}} \equiv \tau_{\mathrm{JMU}}(t, r_0, r_1, \dots, r_M)$ for the Jimbo–Miwa–Ueno τ-function corresponding to the linear system (3.59) and evaluated for the monodromy data given in (3.63) and (3.64).

Remark. It follows from (3.76) that τ_{JMU} vanishes as $r_\alpha \to r_\gamma$ for some pair (α, γ). This fact, of course, can be established directly from the definition of the τ-function.

4. Summary of the results

Recall that $D_n(\phi)$ denotes the Toeplitz determinant associated with the symbol

$$\phi(z)=\mathrm{e}^{tz}\prod_{\alpha=1}^{M}\left(\frac{z-r_\alpha}{z}\right)^{k_\alpha}, \quad -1 < r_\alpha < 0, \quad \alpha = 1, \dots, M, \quad r_\alpha \neq r_\beta, \quad \alpha \neq \beta, \quad k_\alpha \in \mathbf{N}, \quad \sum k_\alpha = k, \quad t \in \mathbf{R},$$

and that the generating function $G_I(n; \{p_i\}, t)$ is given by the formula

$$G_I(n; \{p_i\}, t) = D_n(\phi), \qquad r_\alpha = -p_{i_\alpha}.$$

We also denote by $D_{n+1}(\phi|z)$ the Toeplitz determinant $D_{n+1}(\phi)$ whose last row is replaced by the row $(1, z, z^2, \dots, z^n)$, and we shall assume that $D_n(\phi) \neq 0$.

The following theorem identifies $D_n(\phi)$ as an object of the theory of integrable systems; more specifically, as an object of the theory of generalized Schlesinger equations developed in [18,19].

Theorem 1. *Let Z denote the 2×2 matrix function defined by*

$$Z(z) = \begin{pmatrix} \dfrac{D_{n+1}(\phi|z)}{D_n(\phi)} & -\dfrac{\mathrm{i}}{2\pi}\displaystyle\int_C \dfrac{D_{n+1}(\phi|z')}{D_n(\phi)}(z')^{-n}\phi(z')\dfrac{\mathrm{d}z'}{z-z'} \\ -\dfrac{\bar{D}_n(\bar{\phi}|1/\bar{z})}{\bar{D}_n(\bar{\phi})}z^{n-1} & \dfrac{\mathrm{i}}{2\pi}\displaystyle\int_C \dfrac{\bar{D}_n(\bar{\phi}|1/\bar{z}')}{\bar{D}_n(\bar{\phi})}(z')^{-1}\phi(z')\dfrac{\mathrm{d}z'}{z-z'} \end{pmatrix}, \tag{4.1}$$

where C is the unit circle $|z| = 1$ oriented counterclockwise. Introduce the 2×2 matrices $B_\alpha := B_\alpha(t) := B_\alpha(\{r_\alpha\}, t)$, $\alpha = 0, 1, \dots, M$, by the equations,

$$B_\alpha = -Z(r_\alpha)\begin{pmatrix} 0 & 0 \\ 0 & k_\alpha \end{pmatrix} Z^{-1}(r_\alpha), \qquad \alpha = 0, \dots, M, \tag{4.2}$$

where

$$r_0 := 0, \qquad \text{and} \qquad k_0 := -n - k.$$

(*The invertibility of Z follows from statement 4 below*.) *Then the following statements hold:*

1. $$\mathrm{d}\log D_n(t,r_1,\dots,r_M)=\frac{1}{2}\sum_{\substack{\alpha,\gamma=0\\ \alpha\neq\gamma}}^{M}\mathrm{trace}(B_\alpha B_\gamma)\frac{\mathrm{d}r_\alpha-\mathrm{d}r_\gamma}{r_\alpha-r_\gamma}+\frac{1}{2}\sum_{\alpha=0}^{M}\mathrm{trace}(B_\alpha\sigma_3)\,\mathrm{d}(r_\alpha t)$$
$$+\frac{1}{2}\sum_{\substack{1\le i,j\le 2\\ i\neq j}}\left(\sum_{\alpha=0}^{M}B_\alpha\right)_{ij}\left(\sum_{\gamma=0}^{M}B_\gamma\right)_{ji}\frac{\mathrm{d}t}{t}-\frac{1}{2}\sum_{\alpha=1}^{M}k_\alpha\,\mathrm{d}(r_\alpha t)$$
$$-\frac{1}{2}\sum_{\substack{\alpha,\gamma=0\\ \alpha\neq\gamma}}^{M}k_\alpha k_\gamma\frac{\mathrm{d}r_\alpha-\mathrm{d}r_\gamma}{r_\alpha-r_\gamma}. \tag{4.3}$$

2. *The matrices B_α satisfy the system of nonlinear PDEs* (*generalized Schlesinger equations*),

$$\frac{\partial B_\alpha}{\partial t}=\frac{n-tr_\alpha}{2t}[B_\alpha,\sigma_3]+\sum_{\gamma=0}^{M}\frac{[B_\gamma,B_\alpha]}{t},\quad \alpha=0,1,\dots,M. \tag{4.4}$$

$$\frac{\partial B_\alpha}{\partial r_\gamma}=\frac{[B_\alpha,B_\gamma]}{r_\alpha-r_\gamma},\quad \alpha\neq\gamma=1,\dots,M, \tag{4.5}$$

$$\frac{\partial B_0}{\partial r_\alpha}=\frac{[B_0,B_\alpha]}{r_0-r_\alpha}, \tag{4.6}$$

$$\frac{\partial B_\alpha}{\partial r_\alpha}=\sum_{\gamma\neq\alpha}\frac{[B_\gamma,B_\alpha]}{r_\gamma-r_\alpha},\quad \alpha=1,\dots,M. \tag{4.7}$$

3. *Eqs*. (4.4)–(4.7) *are the compatibility conditions for the system of linear equations*,

$$\frac{\partial\Phi(z)}{\partial z}=\left(\frac{t}{2}\sigma_3+\sum_{\alpha=0}^{M}\frac{B_\alpha}{z-r_\alpha}\right)\Phi(z), \tag{4.8}$$

$$\frac{\partial\Phi(z)}{\partial t}=\left(\frac{z}{2}\sigma_3-\frac{n}{2t}\sigma_3-\frac{n}{2t}I+\frac{1}{t}\sum_{\alpha=0}^{M}B_\alpha\right)\Phi(z), \tag{4.9}$$

$$\frac{\partial\Phi(z)}{\partial r_\alpha}=-\frac{B_\alpha}{z-r_\alpha}\Phi(z),\quad \alpha=1,\dots,M, \tag{4.10}$$

which in turn implies that the system (4.4)–(4.7) *describes the isomonodromy deformations of the z—Eq.* (4.8).

4. *The function Z is alternatively defined as an unique solution of the matrix Riemann–Hilbert problem*,
   - *Z is holomorphic for all $z\notin C$*,
   - $Z(z)z^{-n\sigma_3}\to I,\quad z\to\infty,$
   - $Z_+(z)=Z_-(z)\begin{pmatrix}1 & z^{-n}\phi(z)\\ 0 & 1\end{pmatrix},\quad z\in C.$

(*In particular, we have that* $\det Z\equiv 1$.) *Eq*. (4.3) *can be rewritten in terms of Z as*

$$\mathrm{d}\log D_n=-(\mathrm{res}_{z=\infty}(z^nZ_{22}(z)))\,\mathrm{d}t-\sum_{\alpha=1}^{M}k_\alpha(Z_{11}(r_\alpha)Z'_{22}(r_\alpha)-Z_{21}(r_\alpha)Z'_{12}(r_\alpha))\,\mathrm{d}r_\alpha. \tag{4.11}$$

Also,

$$\frac{D_{n+1}}{D_n} = Z_{12}(0). \tag{4.12}$$

5. *The function*

$$\Phi(z) := Z(z)\,\mathrm{e}^{(tz/2)\sigma_3}\begin{pmatrix} 1 & 0 \\ 0 & z^n\psi^{-1}(z)\end{pmatrix}, \quad \psi(z) = \prod_{\alpha=1}^{M}\left(\frac{z-r_\alpha}{z}\right)^{k_\alpha},$$

satisfies the linear system (4.8)–(4.10) *with the matrices* B_α *given by* (4.2).

6. *The matrices* B_α *are alternatively defined as the solution of the inverse monodromy problem for the linear equation* (4.8) *characterized by the following monodromy data:*
   - *the formal monodromy exponents at the singular points* r_α, ∞ *are given by the equations*

$$E_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & -k_\alpha \end{pmatrix}, \qquad E_\infty = \begin{pmatrix} -n & 0 \\ 0 & 0 \end{pmatrix},$$

   - the corresponding connection matrices are:

$$C_\alpha = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad C_\infty = I.$$

   - the Stokes matrices at the irregular singular point, $z = \infty$, are trivial.

7. $$D_n(\phi) = \mathrm{e}^{-(t/2)\sum_\alpha r_\alpha k_\alpha} \prod_{\substack{\alpha,\gamma=0 \\ \alpha\neq\gamma}}^{M} |r_\alpha - r_\gamma|^{-(k_\alpha k_\gamma/2)}\tau_{\mathrm{JMU}}, \tag{4.13}$$

where $\tau_{\mathrm{JMU}} \equiv \tau_{\mathrm{JMU}}(t, r_0, r_1, \dots, r_M)$ *denotes the Jimbo–Miwa–Ueno* τ*-function corresponding to the linear system* (4.8) *and evaluated for the monodromy data indicated. Eq.* (4.13) *in turn implies the following representation for the generating function* $G_I(n; \{p_i\}, t)$,

$$G_I(n; \{p_i\}, t) = \mathrm{e}^{t/2} \prod_{\substack{\alpha,\gamma=0 \\ \alpha\neq\gamma}}^{M} |p_{i_\alpha} - p_{i_\gamma}|^{-(k_\alpha k_\gamma/2)}\tau_{\mathrm{JMU}}(t, 0, -p_{i_1}, \dots, -p_{i_M}), \quad p_{i_0} := 0. \tag{4.14}$$

Remark 1. In the uniform case, i.e. when $M = 1$ and $k_1 = k$, the linear system (4.8) reduces to the 2×2 system of linear ODEs which has two regular singular points and one irregular point of Poincaré index 1. In this case, as it is shown in [18,19], the isomonodromy Eqs. (4.4)–(4.7) reduce to the special case of the fifth Painlevé equation. Consequently this suggests that the uniform generating function $G_I(n; t)$ can be expressed in terms of a solution of the fifth Painlevé equation. That this is so was obtained earlier in [30] via a direct analysis of the Toeplitz determinant $D_n(t)$.

Remark 2. The methods developed in this paper can be easily generalized to any symbol $\phi(z) := \phi(z, t)$ such that

$$\partial_z \log\phi, \qquad \partial_t \log\phi$$

are rational in z.

Remark 3. In virtue of the Fredholm determinant formula (2.2) for the Toeplitz determinant $D_n(\phi)$, Eq. (4.13) can be interpreted as an example of the general relation [26] between the Jimbo–Miwa–Ueno isomonodromy

τ-function and the Sato–Segal–Wilson τ-function defined via an appropriate determinant bundle (see also [14] for another example of this relation).

Remark 4. The generalized Schlesinger system (4.4)–(4.7) appeared earlier in [17] in connection with the sine kernel Fredholm determinant considered on a union of intervals. The corresponding monodromy data, and hence the solution, are different from the ones related to the Toeplitz determinant $D_n(t)$. For instance, the sine kernel monodromy matrices are not trivial (see [17]; see also [14] for higher matrix dimensional generalizations); in fact, each of them equals the identity matrix plus a one-dimensional projection.

Remark 5. From the point of view of the asymptotic analysis of the Toeplitz determinant, the most important statement of Theorem 1 is Statement 4. It allows one to apply the Riemann–Hilbert asymptotic methods of [3,6,11].

Remark 6. This paper has been primarily concerned with the isomonodromy/Riemann–Hilbert aspect of our integrable system. Presumably an analysis of the additional compatibility conditions, which arise if one extends (4.8)–(4.10) by a relevant n-difference equation, would lead to a Toda like system, see [1,25,33].

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