

ON THE LIMIT OF SOME TOEPLITZ-LIKE DETERMINANTS*

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Abstract. In this article we derive, using standard methods of Toeplitz theory, an asymptotic formula for certain large minors of Toeplitz matrices. Bump and Diaconis obtained the same asymptotics using representation theory, with an answer having a different form.

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Our Toeplitz-like matrices are of the form

$$M = (c_{p_i - q_j}), \quad i, j = 0, 1, \dots,$$

where $\{p_i\}$ and $\{q_i\}$ are sequences of integers satisfying $p_i = q_i = i$ for i sufficiently large, say for $i \geq m$. These are a particular class of finite-rank perturbations of Toeplitz matrices. If the p_i and the q_i are all different, then after rearranging the first m rows and columns, these become minors of the Toeplitz matrix (c_{i-j}) obtained by removing finitely many rows and columns.

Recently Bump and Diaconis [1], using the representation theory of the symmetric group, obtained an asymptotic formula for the determinants of large sections of these minors. Here we use “Toeplitz” methods to obtain the asymptotics in quite a different form, although the answers must be the same.

We assume that $\{c_i\}$ is the sequence of Fourier coefficients of a bounded function φ , so that $(c_{i-j}) = T(\varphi)$ in the usual notation. We assume also that $T(\varphi)$ is invertible on the space $\ell^2(\mathbf{Z}^+)$, and that is almost all. (We shall explain this below.)

For convenience in notation we consider the $(m+n) \times (m+n)$ sections of M ,

$$M_{m+n} = (c_{p_i - q_j}), \quad i, j = 0, 1, \dots, m+n-1,$$

and denote by $T_n(\varphi)$ the $n \times n$ Toeplitz matrix $(c_{i-j})_{i,j=0,1,\dots,n-1}$.

If $T(\varphi)$ is invertible, then φ has a factorization $\varphi = \varphi^- \varphi^+$ where the functions $(\varphi^+)^{\pm 1}$ and $(\varphi^-)^{\pm 1}$ belong to H^2 and $\overline{H^2}$, respectively.¹ This is the “Wiener–Hopf factorization” of φ . In terms of these factors (and with subscripts denoting Fourier coefficients) our limit formula is

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\det M_{m+n}}{\det T_n(\varphi)} = \det \left(\sum_{k=1}^{\infty} (\varphi^-)_{p_i+k-m} (\varphi^+)_{-q_j-k+m} \right)_{i,j=0,\dots,m-1}.$$

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¹Recall that H^2 consists of the L^2 functions whose Fourier coefficients with negative index all vanish. The sequences of Fourier coefficients with nonnegative indices of φ^+ and $\overline{\varphi^-}$ are, up to constant factors, $T(\varphi)^{-1}\delta$, respectively, $T(\overline{\varphi})^{-1}\delta$, where δ is the sequence $\{1, 0, 0, \dots\}$.

(The sum in the determinant on the right side of (1) has only finitely many nonzero terms since the Fourier coefficient $(\varphi^-)_k$ vanishes for $k > 0$ and $(\varphi^+)_k$ vanishes for $k < 0$.)

Under some additional hypotheses the strong Szegő limit theorem gives the asymptotics of the Toeplitz determinant:

$$\det T_n(\varphi) \sim G(\varphi)^n E(\varphi),$$

where

$$G(\varphi) = \exp \left\{ \frac{1}{2\pi} \int \log \varphi(\theta) d\theta \right\}, \quad E(\varphi) = \exp \left\{ \sum_{k=1}^{\infty} k(\log \varphi)_k (\log \varphi)_{-k} \right\}.$$

Our result holds without these extra hypotheses. What we do need, which is a little stronger than the invertibility of $T(\varphi)$, is the uniform invertibility of the $T_n(\varphi)$. That is, we require that the $T_n(\varphi)$ be invertible for sufficiently large n and that the norms of the inverse matrices $T_n(\varphi)^{-1}$ be bounded as $n \rightarrow \infty$. This holds in all “normal” cases where $T(\varphi)$ is invertible. The known counterexamples are not simple. (For a discussion of these points, see [2], especially Chapter 2.) If φ has a continuous logarithm, for example, then the $T_n(\varphi)$ are uniformly invertible.²

THEOREM. *If the $T_n(\varphi)$ are uniformly invertible, then (1) holds.*

Proof. We use the fact from linear algebra that if we have a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A and D square and if D is invertible, then

$$\det M = \det D \det(A - BD^{-1}C).$$

In our case $M = M_{m+n}$, $A = (c_{p_i - q_j})_{i,j=0,\dots,m-1}$ and D is the Toeplitz matrix $T_n(\varphi)$. If the indices for B and C start at 0, then their entries are given by

$$(2) \quad B_{i,k} = c_{p_i - m - k}, \quad C_{k,j} = c_{m+k - q_j} \quad (0 \leq i, j \leq m - 1, \quad 0 \leq k \leq n - 1),$$

and we are interested in the limit as $n \rightarrow \infty$ of the i, j entry of the $m \times m$ matrix $BT_n(\varphi)^{-1}C$.

Now we use the fact that if the $T_n(\varphi)$ are uniformly invertible, then $T_n(\varphi)^{-1}$ converges strongly to the infinite Toeplitz matrix $T(\varphi)^{-1}$. (See [2, Proposition 2.2].) It follows that each entry of $BT_n(\varphi)^{-1}C$ converges to the corresponding entry of $BT(\varphi)^{-1}C$, where now B and C are the $m \times \infty$ and $\infty \times m$ matrices, respectively, with entries given by (2) but now with $k \in \mathbf{Z}^+$. What remains is to show that the i, j entry of $A - BT(\varphi)^{-1}C$ is given by the summand on the right side of (1).

It is convenient to extend the range of the row or column indices of our Toeplitz matrices so that one or the other can run over \mathbf{Z} rather than \mathbf{Z}^+ . So we introduce the notations $T^{(r)}(\varphi)$ and $T^{(c)}(\varphi)$ for the matrices in which the row, respectively, the column, index runs over \mathbf{Z} . With this notation, we see that

$$B_{i,k} = T^{(r)}(\varphi)_{p_i - m, k}, \quad C_{k,j} = T^{(c)}(\varphi)_{k, q_j - m}.$$

²For such a φ the Wiener–Hopf factors φ^- and φ^+ are, up to constant factors, the exponentials of the portions of the Fourier series of $\log \varphi$ corresponding to negative, respectively, positive, indices.

Thus

$$(BT(\varphi)^{-1}C)_{i,j} = (T^{(r)}(\varphi)T(\varphi)^{-1}T^{(c)})_{p_i-m, q_j-m}.$$

Now it is well known and easy to check (see [2, Proposition 1.13]) that

$$T(\varphi) = T(\varphi^-)T(\varphi^+), \quad T(\varphi)^{-1} = T(\varphi^+)^{-1}T(\varphi^-)^{-1}$$

and just as easy to check that

$$T^{(r)}(\varphi) = T^{(r)}(\varphi^-)T(\varphi^+), \quad T^{(c)}(\varphi) = T(\varphi^-)T^{(c)}(\varphi^+).$$

It follows that

$$T^{(r)}(\varphi)T(\varphi)^{-1}T^{(c)}(\varphi) = T^{(r)}(\varphi^-)T^{(c)}(\varphi^+).$$

Hence

$$\begin{aligned} (BT(\varphi)^{-1}C)_{i,j} &= \sum_{k=0}^{\infty} (\varphi^-)_{p_i-m-k} (\varphi^+)_{m+k-q_j} \\ &= \sum_{k=-\infty}^{\infty} (\varphi^-)_{p_i-m-k} (\varphi^+)_{k-q_j+m} - \sum_{k=-\infty}^{-1} (\varphi^-)_{p_i-m-k} (\varphi^+)_{k-q_j+m} \\ &= \varphi_{p_i-q_j} - \sum_{k=1}^{\infty} (\varphi^-)_{p_i-m+k} (\varphi^+)_{-k-q_j+m}. \end{aligned}$$

The first term on the right is $A_{i,j}$, and so the theorem is established.

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