Commun. Math. Phys. 229, 433–458 (2002)
Digital Object Identifier (DOI) 10.1007/s00220-002-0682-7

Communications in
**Mathematical
Physics**

# Fluctuations in the Composite Regime of a Disordered Growth Model

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Received: 6 November 2001 / Accepted: 8 April 2002
Published online: 6 August 2002 – 

**Abstract:** We continue to study a model of disordered interface growth in two dimensions. The interface is given by a height function on the sites of the one-dimensional integer lattice and grows in discrete time: (1) the height above the site $x$ adopts the height above the site to its left if the latter height is larger, (2) otherwise, the height above $x$ increases by 1 with probability $p_x$. We assume that $p_x$ are chosen independently at random with a common distribution $F$, and that the initial state is such that the origin is far above the other sites. Provided that the tails of the distribution $F$ at its right edge are sufficiently thin, there exists a nontrivial composite regime in which the fluctuations of this interface are governed by extremal statistics of $p_x$. In the quenched case, the said fluctuations are asymptotically normal, while in the annealed case they satisfy the appropriate extremal limit law.

## 1. Introduction

Disordered systems, which are, especially in the context of magnetic materials, often referred to as *spin glasses*, have been the subject of much research since the pioneering work in the 1970s. The vast majority of this work is nonrigorous, based on simulations and techniques for which a proper mathematical foundation is yet to be developed. (See [MPV] for early developments and [Tal] for a nice overview of the mean field approach.) As a result, there is a large number of new and intriguing phenomena observed in these models which await rigorous treatment. Among the most fundamental of issues are the existence and the nature of a phase transition into a *glassy* or *composite* phase: below a critical temperature, the dynamics of a strongly disordered system becomes extremely slow with strong correlations, aging and localization effects and possibly many local equilibria. We refer the reader to [NSv] and [BCKM] and other papers in the same volume for reviews and pointers to the voluminous literature and to [NSt1] and [NSt2] for some recent rigorous results. In view of the difficulties associated with a detailed understanding of realistic spinglass systems, other disordered models have been introduced, which are more amenable to existing probabilistic methods.

One of the most successful of such (deceptively) simple models is the one–dimensional random walk with random rates [FIN1]. In this model, the walker waits at a site $x \in \mathbf{Z}$ for an exponential time with mean $\tau_x$ before jumping to either of its two neighbors with equal probability. The disorder variables $\tau_x$ are i.i.d. and quenched, that is, chosen at the beginning. Provided that the distribution of $\tau_x$ has sufficiently fat tails, namely, if $P(\tau_x \geq t)$ decays for large $t$ as $t^{-\alpha}$ with $\alpha < 1$, the walk exhibits aging and localization effects ([FIN1, FIN2]). Various one–dimensional voter models and stochastic Ising models at zero temperature can be explicitly represented with random walks. This connection has been explored to demonstrate glassy phenomena such as aging and chaotic time dependence ([FIN1, FINS]). The positive temperature versions of such results remain open problems, even in one dimension.

In contrast with models which are exactly solvable in terms of random walks and are by now a classical subject in spatial processes ([Gri1, Lig]), techniques based on the RSK algorithm and random matrix theory have entered into the study of growth processes only recently ([BDJ, Joh1, Joh2, BR, PS, GTW1]). The purpose of this paper is to employ these new methods to prove the existence of a *pure phase* and a *composite phase* in a disordered growth model. It has been observed before in similar models [SK] that the role of temperature is for flat interfaces apparently played by their *slope*. In our case, the initial set is very far from flat and "temperature" is measured instead by the macroscopic direction (from the origin) of points on the boundary. We identify precisely the critical direction and demonstrate that the fluctuations asymptotics provide an order parameter that distinguishes the two phases. We emphasize that a hydrodynamic quantity, the asymptotic shape, has a discontinuity of the first derivative at the transition point, at which the shape changes from curved to flat. However, this does not signify the existence of a new phase as kinks are common in many random growth models [GG], thus a finer resolution is necessary.

The particular model we investigate is *Oriented Digital Boiling (ODB)* (Feb. 12, 1996, Recipe at [Gri2], [Gra, GTW1, GTW2]), arguably the simplest interacting model for a growing interface in the two–dimensional lattice $\mathbf{Z}^2$. The occupied set, which changes in discrete time $t = 0, 1, 2, \ldots$, is given by $\mathcal{A}_t = \{(x, y) : x \in \mathbf{Z}, y \leq h_t(x)\}$. The initial state is a long stalk at the origin:

$$h_0(x) = \begin{cases} 0, & \text{if } x = 0, \\ -\infty, & \text{otherwise,} \end{cases}$$

while the time evolution of the height function $h_t$ is determined thus:

$$h_{t+1}(x) = \max\{h_t(x-1), h_t(x) + \varepsilon_{x,t}\}.$$

Here $\varepsilon_{x,t}$ are independent Bernoulli random variables, with $P(\varepsilon_{x,t} = 1) = p_x$. Although this model is simplistic, note that it does involve the roughening noise (random increases) as well as the smoothing surface tension effect (neighbor interaction), the basic characteristics of many growth and deposition processes. (See Sects. 5.1, 5.2 and 5.4 of [Mea] for an overview of simple models of ODB type as well as some other disordered growth processes.)

We will assume, throughout this paper, that the disorder variables $p_x$ are initially chosen at random, independently with a common distribution $F(s) = P(p_x \leq s)$. We use $\langle \cdot \rangle$ to denote integration with respect to $dF$ and label by $p$ a generic random variable with distribution $F$.

It quickly turns out ([GTW1]), that fluctuation in ODB can be studied via equivalent increasing path problems. Start by constructing a random $m \times n$ matrix $A = A(F)$, with independent Bernoulli entries $\varepsilon_{i,j}$ and such that $P(\varepsilon_{i,j} = 1) = p_j$, where, again, $p_j \stackrel{d}{=} p$ are i.i.d. Label columns as usual, but start rows at the bottom. We call a sequence of 1's in $A$ whose positions have column index nondecreasing and row index strictly increasing an *increasing path* in $A$, and denote by $H = H(m, n)$ the length of the longest increasing path. Then, under a simple coupling, $h_t(x) = H(t - x, x + 1)$ ([GTW1]). Thus we will concentrate our attention on the random matrix $A$ rather than the associated growth model. From now on we will also replace $p_i$ with its *ordered* sample, so that $p_1 \geq p_2 \geq \cdots \geq p_n$ (see Sect. 2.2 of [GTW1]).

We initiated the study of ODB in a random environment in an earlier paper ([GTW2]), from which we now summarize the notation and the main results. Throughout, we denote by $b$ the right edge of the support of $dF$ and assume it is below 1, i.e.,

$$b = \min\{s : F(s) = 1\} < 1.$$

Moreover, we fix an $\alpha > 0$ and assume that $n = \alpha m$. (Actually, $n = \lfloor \alpha m \rfloor$, but we omit the obvious integer parts.) As mentioned above, we can expect different behaviors for different slopes on the boundary of the asymptotic shape, which translates to different $\alpha$'s. To be more precise, we define the following critical values:

$$\alpha_c = \left\langle \frac{p}{1-p} \right\rangle^{-1},$$

$$\alpha_c' = \left\langle \frac{p(1-p)}{(b-p)^2} \right\rangle^{-1}.$$

Note that the second critical value is nontrivial, i.e., $\alpha_c' > 0$, iff $\left\langle (b-p)^{-2} \right\rangle < \infty$. Next, define $c = c(\alpha, F)$ to be the time constant, $c = c(\alpha, F) = \lim_{m\to\infty} H/m$, which determines the limiting shape of $\mathcal{A}_t$, namely $\lim \mathcal{A}_t/t$, as $t \to \infty$. In Theorem 1 of [GTW2], it was found that $c$ exists a.s. and is given by

$$c(\alpha, F) = \begin{cases} b + \alpha(1-b)\,\langle p/(b-p) \rangle, & \text{if } \alpha \leq \alpha_c', \\ a + \alpha(1-a)\,\langle p/(a-p) \rangle, & \text{if } \alpha_c' \leq \alpha \leq \alpha_c, \\ 1, & \text{if } \alpha_c \leq \alpha. \end{cases}$$

Here $a = a(\alpha, F) \in [b, 1]$ is the unique solution to $\alpha \left\langle p(1-p)/(a-p)^2 \right\rangle = 1$.

In [GTW2], we also determined fluctuations in the *pure* regime $\alpha_c' < \alpha < \alpha_c$. (The *deterministic* regime $\alpha_c < \alpha$ has no fluctuations.) The *annealed* fluctuations ([GTW2], Theorem 2) about the deterministic shape $c$ grow as $\sqrt{m}$ and are asymptotically normal:

$$\frac{H - cm}{\tau_0 \sqrt{\alpha} \cdot m^{1/2}} \xrightarrow{d} N(0, 1)$$

as $m \to \infty$, where $\tau_0^2 = \mathrm{Var}((1-a)p/(a-p))$.

By contrast, *quenched* fluctuations conditioned on the state of the environment grow more slowly, as $m^{1/3}$, and satisfy the $F_2$–distribution known from random matrices ([TW1, TW2]). To formulate this result, we let $r_j = p_j/(1 - p_j)$, define $u_n$ to be the solution of

$$\frac{\alpha}{n}\sum_{j=1}^{n}\frac{r_j}{(1+r_ju)^2}=\frac{1}{(u-1)^2} \tag{1.1}$$

which lies in in the interval $(-r_1^{-1},\,0)$. This solution exists provided that $\alpha n^{-1}\sum_{j=1}^{n} r_j < 1$ which holds a.s. for large $n$ as soon as $\alpha < \alpha_c$. Next, set $c_n = c(u_n)$ where

$$c(u)=\frac{1}{1-u}-\frac{\alpha}{n}\sum_{j=1}^{n}\frac{r_ju}{1+r_ju}. \tag{1.2}$$

Then ([GTW2], Theorem 3) there exists a constant $g_0 \neq 0$ so that

$$P\left(\frac{H-c_nm}{g_0^{-1}m^{1/3}}\leq s \mid p_1,\ldots,p_n\right)\to F_2(s),$$

as $m\to\infty$, almost surely, for any fixed $s$.

For fluctuation results in this paper we need to impose some additional assumption on $F$, which are best expressed in terms of $G(x) = 1 - F((b-x)-)$, the distribution function for $b-p$. First we list our weaker conditions:

(a) If $x, y \to 0$ and $x \sim y$, then $G(x) \sim G(y)$.
(b) If $x, y \to 0$ and $x = O(y)$, then $G(x) = O(G(y))$.
(c) As $x \to 0$, $G(x) = o(x^2/\log x^{-1})$.

Our stronger assumptions on $F$ require that there exists a $\gamma > 0$ so that:

(a$'$) The function $G(x)/x^\gamma$ is nonincreasing in a neighborhood of $x = 0$.
(b$'$) $G(x) = O(x^2/\log^\nu x^{-1})$ as $x \to 0$ for some $\nu > 2\gamma + 4$.

If $\alpha_c' > 0$, then automatically $G(x) = o(x^2)$ as $x \to 0$. The stronger assumptions thus do not require much more: for nicely behaved $G$ they amount to $G(x) = O(x^2/\log^\nu x^{-1})$ for some $\nu > 8$. The quenched and annealed fluctuations are now determined by the next two theorems.

**Theorem 1.** *Assume that* $0 < \alpha < \alpha_c'$, *let*

$$\tau^2 = b(1-b)\left(\frac{1}{\alpha}-\frac{1}{\alpha_c'}\right),$$

*and let* $\Phi$ *be the standard normal distribution function. If (a)–(c) hold, then for any fixed* $s$, *as* $m\to\infty$,

$$P\left(\frac{H-c_nm+2\tau\sqrt{n}}{\tau\sqrt{n}}\leq s \mid p_1,\ldots,p_n\right)\to\Phi(s).$$

*Here, the convergence is in probability if (a)–(c) hold, and almost sure if (a$'$) and (b$'$) hold.*

**Theorem 2.** *Assume that* $0 < \alpha < \alpha_c'$, *and that (a)–(c) hold. Then, for any fixed* $s$

$$P\left(H \leq cm-(1-\alpha/\alpha_c')\,m\,G^{-1}(s/n) \mid p_1,\ldots,p_n\right)\to e^{-s}$$

*in probability. In particular,*

$$P\left( H \le cm - (1-\alpha/\alpha_c')\, m\, G^{-1}(s/n) \right) \to e^{-s}.$$

Throughout, we follow the usual convention in defining $G^{-1}(x) = \sup\{y : G(y) < x\}$ to be the left continuous inverse of $G$, although any other inverse works as well.

Assume, for simplicity, that, as $x \to 0$, $G(x)$ behaves as $x^\eta$ for some $\eta > 2$. Then, in contrast with the pure regime, the annealed fluctuations in composite regime scale as $m^{1-1/\eta}$, while the quenched ones scale as $m^{1/2}$. In fact, this can be guessed from [GTW2]. Namely, as explained in Sect. 2 of that paper, the maximal increasing path has a nearly vertical segment of length asymptotic to $(1-\alpha/\alpha_c')m$ in (or near) the column of $A$ which uses the largest probability $p_1$. Therefore, this vertical part of the path dominates the fluctuations, as the rest presumably has $o(\sqrt{m})$ fluctuations. (These are most likely *not* of the order exactly $m^{1/3}$ as they correspond to the critical case $\alpha = \alpha_c'$. The precise nature of the critical fluctuations is an interesting open problem.) The variables in the $p_1$–column are Bernoulli with variances about $b(1-b)$, thus the contribution of the vertical part to the variance is about $(b(1-b)(1-\alpha/\alpha_c')m)^{1/2} = \tau\sqrt{n}$. The annealed case then simply picks up the variation in the extremal statistic $p_1$.

Simple as the above intuition may be, Theorems 1 and 2 are not so easy to prove and require considerable additional technical details. We also note the mysterious correction $2\tau\sqrt{n}$ in Theorem 1 for which we have no intuitive explanation.

The fluctuations results in [GTW2] and the present paper thus sharply distinguish between two different phases of *one* particular growth model. Nevertheless, it seems natural to speculate that this phenomenon is universal in the sense that it occurs in other one–dimensional finite range dynamics of ODB type, started from a variety of initial states. Indeed, such universality has been established in other random matrix contexts [Sos]. Fluctuations of higher–dimensional versions seem much more elusive; it appears that a glassy transition should take place, but the fluctuation scalings could be completely different.

To elucidate, we present some simulation results. In all of them, we start from the flat substrate $h_0 \equiv 0$ and use $F(s) = 1-(1-2s)^\eta$, so that $b = 1/2$. It is expected that, as $\eta$ increases, the quenched fluctuation experiences a sudden jump from $1/3$ to $1/2$. We simulate two dynamics, the ODB and the two–sided digital boiling (abbreviated simply as DB), given by

$$h_{t+1}(x) = \max\{h_t(x-1), h_t(x+1), h_t(x) + \varepsilon_{x,t}\}.$$

The top of Fig. 1 illustrates the ODB on 600 sites (with periodic boundary), run until time 600. The occupied sites are periodically colored so that the sites which become occupied at the same time are given the same color. On the left, $\eta = 1$ (i.e., $p$ is uniform on $[0, 1/2]$ and $\alpha_c' = 0$), while $\eta = 3$ (and $\alpha_c' > 0$) on the right. The darkly colored sites thus give the height of the surface at different times and provide a glimpse of its evolution. In the pure regime ($\eta = 1$), the boundary of the growing set reaches a local equilibrium ([SK]), while in the composite regime ($\eta = 3$) the boundary apparently divides into domains, which are populated by different equilibria and grow sublinearly. This is the mechanism that causes increasing fluctuations. The bottom of Fig. 1 confirms this observation; it features a log–log plot of quenched standard deviation (estimated over 1000 independent trials) of $h_t(0)$ vs. $t$ up to $t = 10\,000$. The $\eta = 1$ case is drawn with $+$'s and the $\eta = 3$ case with $\times$'s; the two least squares approximations lines (with slopes 0.339 and 0.517, respectively) are also drawn. We note that the asymptotic speed

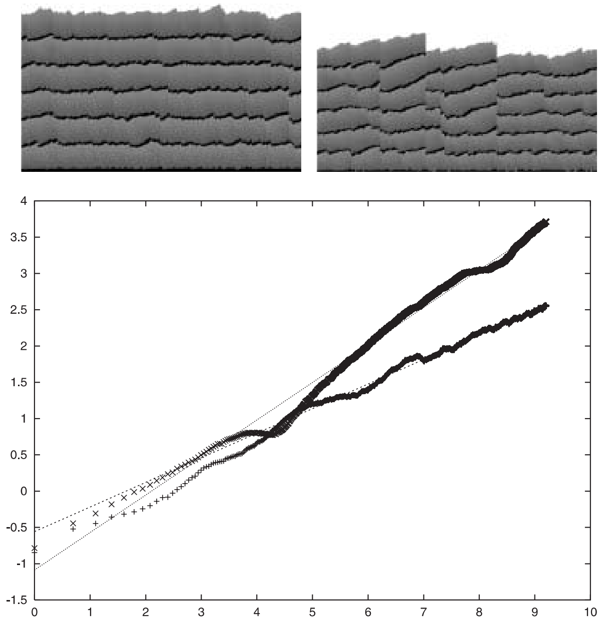


**Fig. 1.** Evolution and quenched deviation in the two phases of disordered ODB

of this flat interface is known: $\lim_{t\to\infty} h_t(0)/t = \sup_{\alpha>0}(\alpha+1)c(\alpha)$. Here is the reason: if ODB dynamics $h_t^i$, $h_t$ start from initial states $h_0^i$, $h_0 = \sup_i h_0^i$, respectively, and are coupled by using the same coin flips $\varepsilon_{x,t}$, then $h_t = \sup_i h_t^i$ for every $t$.

Perhaps surprisingly, it appears that the phase transition in the DB does *not* occur at $\eta = 2$, and in general the delineation is much murkier. At this point, we cannot even eliminate the possibility of continuous dependence of fluctuation exponent on $\eta$. In Fig. 2, we present the results of simulations for $\eta = 0.2$ (left) and $\eta = 1$ (right). The top figures only show evolution near time $t = 5000$, as no difference is readily apparent at earlier times. The plot of quenched deviations is analogous to the one in Fig. 1, with the least squares slopes 0.395 ($\eta = 0.2$) and 0.49 ($\eta = 1$).

The organization of the rest of the paper is as follows. Section 2 reviews the set-up from [GTW1, GTW2], in Sect. 3 we prove the relevant asymptotic properties of the order statistics and of the solutions of (1.1) and (1.2), and demonstrate how Theorem 2 follows from Theorem 1. Section 4 is a detailed analysis of the asymptotic behavior of steepest descent curves. The proof of convergence in probability in Theorem 1 is then concluded in Sect. 4. Finally, Sect. 5 strengthens the results of Sect. 3 (under the stronger conditions) so that almost sure convergence is implied.

## 2. The Basic Set-up

We recall how we approached these problems in [GTW1, GTW2]. The starting point is the identity

$$P(H \le h) = \det(I - K_h),$$

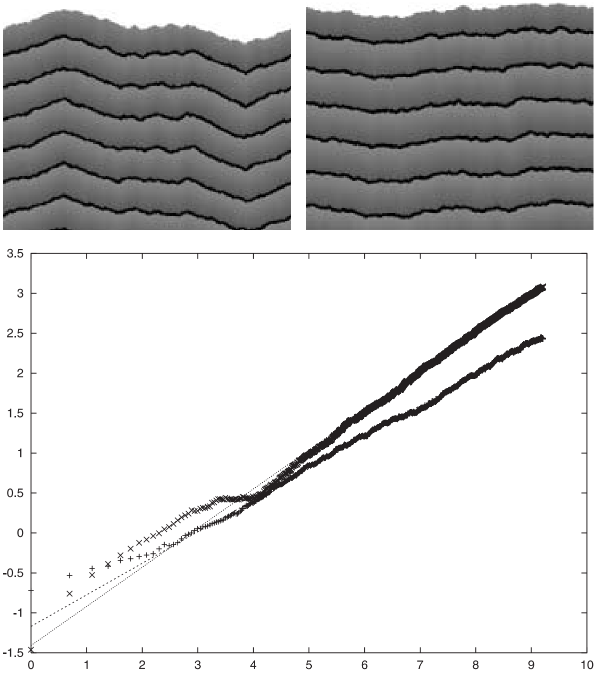


**Fig. 2.** Evolution and quenched deviation in disordered DB

where $K_h$ is the infinite matrix acting on $\ell^2(\mathbf{Z}^+)$ with $(j, k)$–entry

$$K_h(j,k) = \sum_{\ell=0}^{\infty} (\varphi_-/\varphi_+)_{h+j+\ell+1} \; (\varphi_+/\varphi_-)_{-h-k-\ell-1}.$$

The subscripts denote Fourier coefficients and the functions $\varphi_\pm$ are given by

$$\varphi_+(z) = \prod_{j=1}^{n} (1 + r_j z), \quad \varphi_-(z) = (1 - z^{-1})^{-m}.$$

The matrix $K_h$ is the product of two matrices, with $(j, k)$–entries given by

$$(\varphi_+/\varphi_-)_{-h-j-k-1} = \frac{1}{2\pi i} \int \prod_{j=1}^{n} (1 + r_j z) \; (z-1)^m \, z^{-m+h+j+k} \, dz,$$

$$(\varphi_-/\varphi_+)_{h+j+k+1} = \frac{1}{2\pi i} \int \prod_{j=1}^{n} (1 + r_j z)^{-1} \; (z-1)^{-m} \, z^{m-h-j-k-2} \, dz.$$

The contours for both integrals go around the origin once counterclockwise; in the second integral 1 is on the inside and all the $-r_j^{-1}$ are on the outside.

If $h = c_n\, m + h'$ we have

$$(\varphi_+/\varphi_-)_{-h-j-k-1} = \frac{1}{2\pi i}\int \psi(z)\, z^{h'+j+k}\, dz, \tag{2.1}$$

$$(\varphi_-/\varphi_+)_{h+j+k+1} = \frac{1}{2\pi i}\int \psi(z)^{-1}\, z^{-h'-j-k-2}\, dz, \tag{2.2}$$

where

$$\psi(z) = \prod_{j=1}^{n}(1 + r_j z)\ (z-1)^m\, z^{-(1-c_n)\,m}.$$

The idea is to apply steepest descent to the above integrals. If $\sigma(z) = m^{-1}\log\psi(z)$, then

$$\sigma'(z) = \frac{\alpha}{n}\sum_{j=1}^{n}\frac{r_j}{1+r_j z} + \frac{1}{z-1} + \frac{c_n - 1}{z} \tag{2.3}$$

and, with $u_n$ and $c_n$ as defined above, $\sigma'(u_n) = \sigma''(u_n) = 0$. The steepest descent curves both pass through $u_n$. As $n \to \infty$ the zeros/poles $-r_j^{-1}$ accumulate on the half-line $(-\infty,\ \xi]$, where $\xi = 1 - b^{-1}$. In the pure regime the points $u_n$ and the curves are bounded away from this half-line, behave regularly and have nice limits. However in the composite regime the points and curves come very close to $\xi$, their behavior is not so simple, and we apply steepest descent not quite as described.

## 3. Preliminary Lemmas I: Properties of $p_n$, $u_n$, and $c_n$

Until Sect. 5, we assume that all limits are in probability, unless otherwise indicated. To prove the first part of Theorem 1 and Theorem 2, we thus assume that (a)–(c) hold.

We let $q_j = b - p_j$, so that $q_1, \cdots, q_n$ are chosen independently according to the distribution function $G$, then ordered so that $q_1 \le q_2 \le \cdots \le q_n$.

Let $t_1 < t_2 < \ldots < t_n$ be an ordered sample of i.i.d. uniform (0, 1) random variables. Then we may construct the $G$–sample by setting $q_j = G^{-1}(t_j)$. We will also use the well-known fact that, given $t_j$, the conditional distribution of $t_1, \ldots t_{j-1}$ is that of an ordered sample of $j-1$ uniforms on $[0, t_j]$.

**Lemma 3.1.** *There exist a positive constant $c_1$ so that $x \le G(G^{-1}(x)) \le x/c_1$ for $x \in (0,1)$. Moreover, $G(G^{-1}(x)) \sim x$ as $x \to 0$.*

*Proof.* Write the complement of the range of $G$ as $\cup_i I_i$, where $I_i$ are disjoint and either of the form $[a_i, b_i)$ or $(a_i, b_i)$. If $x \in (0,1)$ is in the range of $G$, then $G(G^{-1}(x)) = x$, otherwise, if $x \in I_i$, $G(G^{-1}(x)) = b_i$. By (a), $b_i \sim a_i$ if $a_i \to 0$. The last sentence in the statement is then proved, and the first follows. □

**Lemma 3.2.** *With $c_1$ as in Lemma 3.1, for $\eta < 1$ and $j \ge 2$,*

$$P\left(G(q_1) > \eta G(q_j)\right) \le (1 - c_1\eta)^{j-1}.$$

*Proof.* By Lemma 3.1 and remarks preceding it,

$$P\left(G(q_1) > \eta G(q_j)\right) \le P\left(t_1 > c_1\eta t_j\right) = (1 - c_1\eta)^{j-1}.\quad \square$$

**Lemma 3.3.** $\lim_{n\to\infty} P\left(q_1 \le G^{-1}(s/n)\right) = 1 - e^{-s}$.

*Proof.* Fix an $\varepsilon > 0$. First, by monotonicity of $G^{-1}$, $t_1 \le s/n$ implies $q_1 \le G^{-1}(s/n)$. Second, by Lemma 3.1 and the monotonicity of $G$ we have that, for large enough $n$, $q_1 \le G^{-1}(s/n)$ implies $t_1 \le G(G^{-1}(t_1)) = G(q_1) \le G(G^{-1}(s/n)) \le (1+\varepsilon)s/n$. These give the inequalities $P(q_1 \le G^{-1}(s/n)) \ge 1-(1-s/n)^n$, and $P(q_1 \le G^{-1}(s/n)) \le 1-(1-(1+\varepsilon)s/n)^n$. The statement of the lemma now follows upon first letting $n \to \infty$ and then $\varepsilon \to 0$. □

*Remark.* It follows from Lemma 3.3, and the fact that $G(x) = o(x^2)$ near $x = 0$, that $n^{1/2}q_1 \to \infty$ as $n \to \infty$.

**Lemma 3.4.** *With high probability $q_1/q_2$ is bounded away from 1 as $n \to \infty$. More precisely, for every $\eta > 0$ there is a $\delta > 0$ such that $P(q_1 \le (1-\delta)\, q_2) \ge 1-\eta$ for large enough $n$.*

*Proof.* It follows from Lemma 3.1 that for every $\eta > 0$ there exists a $\delta_1 > 0$ so that the following implication holds for $t_2 < \delta_1$: if $G(q_1) > (1-\delta_1)G(q_2)$ then $t_1 > (1-\eta)t_2$. Furthermore, by the assumption (a), there exists a $\delta \in (0, \delta_1)$ so that, for $t_2 < \delta$, $q_1 > (1-\delta)q_2$ implies $G(q_1) > (1-\delta_1)G(q_2)$. Therefore,

$$P(q_1 > (1-\delta)q_2) \le P(t_1 > (1-\eta)t_2) + P(t_2 > \delta) = \eta + P(t_2 > \delta),$$

and the proof is concluded since $t_2 \to 0$ a.s. □

**Lemma 3.5.** $n^{-1}\sum_1^n q_1/q_j^3 \to 0$ *as* $n \to \infty$.

*Proof.* For any fixed $k$ we have $n^{-1}\sum_{j=1}^k q_1/q_j^3 \le k/nq_1^2 \to 0$. Also, $n^{-1}\sum_{j=k+1}^n q_j^{-2} < \left\langle q^{-2}\right\rangle + 1$ a.s. for large $n$.

Let $\delta > 0$ be given. By the above paragraph, it suffices to show that

$$\limsup_{n\to\infty} P\left(\frac{q_1}{q_{k+1}} > \delta\right)$$

will be arbitrarily small for sufficiently large $k$. Now, from the assumption (b), it follows that for some $\eta > 0$ we have $G(q_1) > \eta G(q_{k+1})$ whenever $q_1 > \delta q_{k+1}$ and $q_1 < \eta$. With this $\eta$ (which we may assume is less than 1) we have, from Lemma 3.2,

$$P\left(\frac{q_1}{q_{k+1}} > \delta\right) \le (1-c_1\eta)^k + P(q_1 \ge \eta),$$

which is clearly enough. □

From now on $\{\varphi_n\}$ will denote a sequence of random variables satisfying $\varphi_n = o(q_1)$. Since $q_1 \gg n^{-1/2}$ we shall assume when convenient that also $\varphi_n \gg n^{-1/2}$. In the statement of the next lemma, the expression $O(\varphi_n)$ could have been replaced by the less awkward $o(q_1)$. The reasons for the present statement are that the substitute for this lemma (Lemma 6.2) when we consider almost sure convergence will have this form, and that the same sequence $\{\varphi_n\}$ will appear in later lemmas.

**Lemma 3.6.** *Let* $\{v_n\}$ *be a sequence of points in a disc with diameter the real interval* $[-r_1^{-1} - O(\varphi_n),\ \xi]$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=2}^{n} \frac{r_j}{(1+r_j v_n)^2} = \left\langle \frac{r}{(1+r\xi)^2} \right\rangle.$$

*Proof.* Write $v_n = (b_n - 1)/b_n$. Then if we recall that $\xi = (b-1)/b$ and $p_j = b - q_j$ we see that $b - b_n$ lies in a disc with diameter $[0,\ q_1 + O(\varphi_n)]$ and that

$$\frac{1}{n} \sum_{j=2}^{n} \frac{r_j}{(1+r_j v_n)^2} = \frac{1}{n} \sum_{j=2}^{n} \frac{b_n^2 (b-q_j)(1-b+q_j)}{(b_n - b + q_j)^2}.$$

If we subtract from this the same expression with $b_n$ replaced by $b$, that is,

$$\frac{1}{n} \sum_{j=2}^{n} \frac{b^2 (b-q_j)(1-b+q_j)}{q_j^2}, \tag{3.1}$$

we obtain

$$\frac{1}{n} \sum_{j=2}^{n} (b-q_j)(1-b+q_j) \left[ \frac{b_n^2}{(b_n - b + q_j)^2} - \frac{b^2}{q_j^2} \right]. \tag{3.2}$$

We shall show that this is $o(1)$. Assuming this for the moment, we can finish the proof by first noting that we may, with error $o(1)$, start the sum in (3.1) at $n = 1$ since $q_i \gg n^{-1/2}$, and then (3.1) has the a.s. limit

$$\left\langle \frac{b^2 (b-q)(1-b+q)}{q^2} \right\rangle = \left\langle \frac{r}{(1+r\xi)^2} \right\rangle.$$

It remains to show that (3.2) is $o(1)$. If we replace the numerator $b^2$ on the right by $b_n^2$, the error is $o(1)$, since $n^{-1} \sum q_j^{-2}$ is a.s. bounded. If we make this replacement then what we obtain is bounded by a constant times

$$\frac{b}{n} \sum_{j=2}^{n} \left| \frac{(b_n - b)^2 - 2(b_n - b) q_j}{q_j^2 (b_n - b + q_j)^2} \right|.$$

Since $|b - b_n| \le q_1 + O(\varphi_n) = q_1 + o(q_1)$ it follows from Lemma 3.4 that $|b_n - b + q_j|$ is at least a constant times $q_j$ for large $n$ and so the above is at most a constant times

$$\frac{1}{n} \sum_{j=2}^{n} \frac{|b_n - b|}{q_j^3} \le \frac{1}{n} \sum_{j=2}^{n} \frac{q_1}{q_j^3},$$

and by Lemma 3.5 this is $o(1)$. □

We denote

$$\theta = 1 - \alpha/\alpha_c', \qquad \beta = \left( \frac{(1-b)\,\alpha}{b^3\,\theta} \right)^{1/2}. \tag{3.3}$$

**Lemma 3.7.** *We have* $u_n = -r_1^{-1} + \beta n^{-1/2} + o(n^{-1/2})$ *as* $n \to \infty$.

*Proof.* We show first that $u_n \geq \xi$ cannot occur for arbitrarily large $n$. If it did, then we would have, using Eq. (1.1) for $u_n$,

$$b^2 = \frac{1}{(\xi-1)^2} \leq \frac{1}{(u_n-1)^2} \leq \frac{\alpha}{n}\frac{r_1}{(1+r_1\xi)^2} + \frac{\alpha}{n}\sum_{j=2}^{n}\frac{r_j}{(1+r_j\xi)^2}.$$

It follows from the remark following Lemma 3.3 that the first term on the right is $o(1)$ and from Lemma 3.6 that the second term on the right has limit

$$\alpha\left\langle\frac{r}{(1+r\xi)^2}\right\rangle = \alpha b^2\left\langle\frac{p(1-p)}{(b-p)^2}\right\rangle < b^2$$

since we are in the composite regime. This contradiction shows that $u_n \leq \xi$ for sufficiently large $n$, and so $u_n \in [-r_1^{-1},\ \xi]$. By Lemma 3.6 again,

$$\frac{\alpha}{n}\sum_{j=2}^{n}\frac{r_j}{(1+r_ju)^2} = \frac{1}{(u-1)^2} \to \alpha\left\langle\frac{r}{(1+r\xi)^2}\right\rangle = b^2\alpha/\alpha_c'.$$

Therefore Eq. (1.1) for $u_n$ becomes

$$\frac{\alpha}{n}\frac{r_1}{(1+r_1u_n)^2} = \frac{1}{(\xi-1)^2} - \alpha\left\langle\frac{r}{(1+r\xi)^2}\right\rangle + o(1) = b^2\theta + o(1).$$

Since $r_1 = b/(1-b) + o(1)$ we find that the solution is as stated. □

Next, we see how $c_n$ behaves.

**Lemma 3.8.** *We have $c_n = c(\alpha, F) - \theta\, q_1 + o(q_1)$ as $n \to \infty$, where $\theta$ is given in (3.3).*

*Proof.* Write

$$c_n = \frac{1}{1-u_n} - \frac{\alpha}{n}\sum_{j=2}^{n}\frac{r_ju_n}{1+r_ju_n} - \frac{\alpha}{n}\frac{r_1u_n}{1+r_1u_n}. \tag{3.4}$$

By Lemma 3.7, the last term above is $O(n^{-1/2})$. Equation (1.1) tells us that

$$\frac{d}{du}\left(\frac{1}{1-u} - \frac{\alpha}{n}\sum_{j=1}^{n}\frac{r_ju}{1+r_ju}\right)\Big|_{u=u_n} = 0,$$

and so

$$\frac{d}{du}\left(\frac{1}{1-u} - \frac{\alpha}{n}\sum_{j=2}^{n}\frac{r_ju}{1+r_ju}\right)\Big|_{u=u_n} = \frac{\alpha}{n}\frac{r_1}{(1+r_1u_n)^2}$$
$$= \frac{\alpha}{r_1\beta^2} + o(1) = \frac{\alpha(1-b)}{b\beta^2} + o(1).$$

By Lemma 3.6 and its proof, with an error $o(1)$ the derivative of the expression in the parentheses above equals in $[u_n, \xi]$ what it equals at $u = \xi$, so the above holds with $u_n$ replaced by any point in this interval. From this and (3.4) we get

$$c_n = c(u_n) = c(\xi) - \frac{\alpha(1-b)}{b\beta^2}(\xi - u_n) + o(\xi - u_n).$$

We have

$$\xi - u_n = 1 - b^{-1} - r_1^{-1} + O(n^{-1/2}) = p_1^{-1} - b^{-1} + O(n^{-1/2}) = \frac{q_1}{b^2} + o(q_1),$$

where we have used the fact that $q_1 \gg n^{-1/2}$. Thus

$$c_n = c(\xi) - \frac{\alpha(1-b)}{b^3\beta^2} q_1 + o(q_1).$$

Finally, as $\left\langle (b-p)^2 \right\rangle < \infty$, we can use the central limit theorem to conclude that $c(\xi) = c(\alpha, F) + O(n^{-1/2})$, which completes the proof. □

*Remark.* Lemmas 3.3 and 3.8 show that Theorem 2 follows from the part of Theorem 1 on convergence in probability.

## 4. Preliminary Lemmas II: Steepest Descent Curves

Now we go to our integrals (2.1) and (2.2). We are not going to apply steepest descent with $\psi$ as the main integrand, but rather with the function $\psi_1$ which is $\psi$ with the factor $1 + r_1 z$ removed. It is convenient to introduce the notation

$$\psi_1(z, c) = \prod_{j=2}^{n} (1 + r_j z)\ (z-1)^m\, z^{-(1-c)m},$$

where $c > 0$. (This parameter is not to be confused with the time constant $c = c(\alpha, F)$ defined earlier.) Thus $\psi_1(z) = \psi_1(z, c_n)$ in this notation. We also define the integrals

$$I^+(c) = \frac{1}{2\pi i} \int (1+r_1 z)\, \psi_1(z, c)\, dz, \quad I^-(c) = \frac{1}{2\pi i} \int (1+r_1 z)^{-1}\, \psi_1(z, c)^{-1} z^{-2}\, dz.$$

(Since $I^+(c) = 0$ when $c \geq 1$ we always assume that $c < 1$.) Notice that these are exactly the integrals (2.1) and (2.2) when we set

$$c = c_n + (h' + j + k)/m.$$

Since $j, k \geq 0$ and we will eventually set $h' = sn^{1/2}$, we may also assume that

$$c \geq c_n - O(n^{-1/2}). \tag{4.1}$$

To apply steepest descent to $I^\pm(c)$ we must locate the critical points and determine the critical values of $\psi_1(z, c)$. Thus we define

$$\sigma_1(z, c) = \frac{1}{m} \log\, \psi_1(z, c),$$

so that

$$\sigma_1'(z, c) = \frac{\alpha}{n} \sum_{j=2}^{n} \frac{r_j}{1 + r_j z} + \frac{1}{z-1} + \frac{c-1}{z}.$$

As before, if the parameter $c$ does not appear we take it to be $c_n$, e.g., $\sigma_1(z) = \sigma_1(z, c_n)$. So

$$\sigma_1'(z) = \frac{1}{m} \log\, \psi_1(z) = \sigma'(z) - \frac{\alpha}{n} \frac{r_1}{1 + r_1 z}.$$

Using $\sigma'(u_n) = \sigma''(u_n) = 0$ we get from the above and Lemma 3.7 that

$$\sigma_1'(u_n) = -\frac{\alpha}{\beta\sqrt{n}}(1 + o(1)), \quad \sigma_1''(u_n) = \frac{\alpha}{\beta^2}(1 + o(1)). \tag{4.2}$$

To determine the critical values of $\sigma_1(z, c)$ let us first find the value of $c$ for which its derivative has a double zero. (This is the analogue of the quantity $c_n$ for $\sigma(z)$.) For this we use the analogue of (1.1) and (1.2) but where the terms corresponding to $j = 1$ are dropped from the sums. If we call the solution of (1.1) $\bar{u}$ and set $\bar{c} = c(\bar{u})$ then $\sigma_1'(z, \bar{c})$ has a double zero at $\bar{u}$. In analogy with $u_n$, we know that $\bar{u}$ is to the right of and within $O(n^{-1/2})$ of $-r_2^{-1}$. As for $\bar{c}$, we use Lemma 3.8, its analogue where the sums in (1.1) and (1.2) start with $j = 2$, as well as Lemma 3.4, to see that to a first approximation

$$\bar{c} = c_n - \theta(q_2 - q_1)$$

and that $q_2 - q_1 \gg n^{-1/2}$. From this and (4.1) we see that $c > \bar{c}$.

Using subscripts for derivatives now, we have

$$\sigma_{1z}(\bar{u}, \bar{c}) = \sigma_{1zz}(\bar{u}, \bar{c}) = 0,$$

and we want to see how the critical points $u_c^{\pm}$ of $\sigma_1(z, c)$ move away from $\bar{u}$ as $c$ increases from $\bar{c}$. (Here we take $u_c^- < u_c^+$.) The function $\sigma_{1z}(z, \bar{c})$ vanishes at $\bar{u}$ and is otherwise positive in $(-r_2^{-1}, 0)$. It follows that for $c$ close to but larger than $\bar{c}$ we have $u_c^- < \bar{u} < u_c^+$. Differentiating $\sigma_{1z}(u_c^{\pm}, c) = 0$ with respect to $c$ gives

$$0 = \sigma_{1zz}(u_c^{\pm}, c)\, \frac{du_c^{\pm}}{dc} + \sigma_{1zc}(u_c^{\pm}, c) = \sigma_{1zz}(u_c^{\pm}, c)\, \frac{du_c^{\pm}}{dc} + \frac{1}{u_c^{\pm}}. \tag{4.3}$$

Since $u_c^{\pm} < 0$ it follows that $du_c^{+}/dc \neq 0$, and so each of $u_c^{\pm}$ is either a decreasing or increasing function of $c$ for $c > \bar{c}$. From their behavior that we already know for $c$ close to $\bar{c}$ we deduce that $u_c^+$ increases and $u_c^-$ decreases as $c$ increases. In particular, $u_c^-$ is even closer to $-r_2^{-1}$ than $\bar{u}$.

We remark that from (4.3) and the signs of $du_c^{+}/dc$ we deduce

$$\sigma_{1zz}(u_c^+, c) > 0, \quad \sigma_{1zz}(u_c^-, c) < 0. \tag{4.4}$$

Next we shall determine the asymptotics of the critical values $\sigma(u_c^{\pm}, c)$. The sequence $\{\varphi_n\}$ is as described before Lemma 3.6.

**Lemma 4.1.** *For $c - c_n = O(\varphi_n)$,*

$$\sigma_1(u_c^+, c) = \sigma_1(-r_1^{-1}, c) - \frac{r_1\beta^2}{2\alpha}\left(c - c_n + \frac{2\alpha}{r_1\beta}(1 + o(1))n^{-1/2}\right)^2, \tag{4.5}$$

*and for all $c \geq c_n$,*

$$\sigma_1(u_c^+, c) < \sigma_1(-r_1^{-1}, c) - \eta n^{-1/2}\,(c - c_n) + O(n^{-1}). \tag{4.6}$$

*for some $\eta > 0$. Moreover for all $c$*

$$\sigma_1(u_c^-, c) > \sigma_1(-r_1^{-1}, c) + \varphi_n^2$$

*when $n$ is sufficiently large.*

*Remark.* In these and analogous inequalities below we think of $\sigma_1$ as actually meaning $\Re\sigma_1$.

*Proof.* Consider first the case $c = c_n$. We have

$$\sigma_1(u_n+\zeta) = \sigma_1(u_n) + \sigma_1'(u_n)\,\zeta + \zeta^2 \int_0^1 (1-t)\,\sigma_1''(u_n+t\zeta)\,dt.$$

If $\zeta = O(\varphi_n)$ then it follows from Lemma 3.6 that $\sigma_1''(u_n+t\zeta) = \sigma_1''(u_n) + o(1)$. Hence, by (4.2), we have for such $\zeta$

$$\sigma_1(u_n+\zeta) = \sigma_1(u_n) - \frac{\alpha}{\beta\sqrt{n}}\zeta + \left(\frac{\alpha}{2\beta^2} + o(1)\right)\zeta^2. \tag{4.7}$$

This has zero derivative for

$$\zeta = \frac{\beta}{\sqrt{n}}(1+o(1))$$

and it follows that

$$u_{c_n}^+ = u_n + \frac{\beta}{\sqrt{n}}(1+o(1)) = -r_1^{-1} + \frac{2\beta}{\sqrt{n}}(1+o(1)). \tag{4.8}$$

(This critical value must be $u_{c_n}^+$ rather than $u_{c_n}^-$ since the latter is within $O(n^{-1/2})$ of $-r_2^{-1}$.) From this and (4.7), taking $\zeta = -r_1^{-1} - u_n = -(\beta + o(1))n^{-1/2}$ and $\zeta = u_{c_n}^+ - u_n = (\beta + o(1))n^{-1/2}$ and subtracting, it follows that

$$\sigma_1(u_{c_n}^+) = \sigma_1(-r_1^{-1}) - 2(\alpha + o(1))n^{-1}. \tag{4.9}$$

To determine the behavior of $u_c^+$ and $\sigma_1(u_c^+, c)$ for more general $c$ we assume first that

$$c = c_n + o(1), \quad u_c^+ = u_n + O(\varphi_n) = -r_1^{-1} + O(\varphi_n).$$

Then

$$\sigma_{1zz}(u_c^+, c) = \sigma_1''(u_n) - \frac{c - c_n}{u_c^{+2}} = \frac{\alpha}{\beta^2} + o(1)$$

by (4.2). Therefore (4.3) gives

$$\frac{du_c^+}{dc} = -(\beta^2/\alpha + o(1))/u_c = r_1\frac{\beta^2}{\alpha}(1+o(1)),$$

whence

$$\begin{aligned} u_c^+ &= u_{c_n}^+ + r_1\frac{\beta^2}{\alpha}(c-c_n)(1+o(1)) \\ &= -r_1^{-1} + \frac{2\beta}{\sqrt{n}}(1+o(1)) + r_1\frac{\beta^2}{\alpha}(c-c_n)(1+o(1)), \end{aligned} \tag{4.10}$$

by (4.8). This holds if $c - c_n = O(\varphi_n)$ since this assures that $u_c^+ = u_n + O(\varphi_n)$. The above gives

$$\log(-u_c^+) = \log(-r_1^{-1}) - 2r_1\beta(1+o(1))n^{-1/2} - r_1^2\frac{\beta^2}{\alpha}(c-c_n)(1+o(1)). \tag{4.11}$$

(Again, real parts are tacitly meant.)

To determine, $\sigma_1(u_c^+, c)$ we use $\sigma_{1z}(u_c^+, c) = 0$ to deduce

$$\frac{d}{dc}\sigma_1(u_c^+, c) = \log u_c^+. \tag{4.12}$$

We continue to assume that $c - c_n = O(\varphi_n)$ so our estimates hold. Integrating (4.12) using the first part of (4.10) gives (since $u_{c_n}^+ \to -r_1^{-1}$)

$$\begin{aligned}\sigma_1(u_c^+, c) = &\ \sigma_1(u_{c_n}^+) + (c - c_n)\ \log u_{c_n}^+ - \frac{1}{2}r_1^2\frac{\beta^2}{\alpha}\,(c - c_n)^2\,(1 + o(1))\\ = &\ \sigma_1(-r_1^{-1}) - 2(\alpha + o(1))\,n^{-1} + \log(-r_1^{-1})(c - c_n)\\ &-2r_1\beta(c - c)\,n^{-1/2}(1 + o(1)) - \frac{1}{2}r_1^2\frac{\beta^2}{\alpha}\,(c - c_n)^2\,(1 + o(1)),\end{aligned}$$

by (4.9) and (4.11). This gives (4.5).

For all $c \geq c_n$ we use the fact that $\log(-u_c^+)$ is a decreasing function of $c$, since $u_c^+$ increases, and integrate (4.12) with respect to $c$ from $c_n$ to $c$, which gives

$$\sigma_1(u_c^+, c) \leq \sigma_1(u_{c_n}^+) + \log(-u_{c_n}^+)(c - c_n).$$

Using (4.9) and (4.8) give (4.6).

For the lower bound for $\sigma_1(u_c^-, c)$, we assume first that $c \leq c_n$. By (4.1) this implies in particular that $c - c_n = O(n^{-1/2})$. Now $\sigma_1(z)$ is decreasing on the interval $(u_c^-,\ u_c^+)$ and $u_c^+ - u_c^- \gg \varphi_n$. To see the last inequality, note that, from Lemma 3.6, $\sigma_{1zz}(u_n + \zeta, c) \neq 0$ for $\zeta = O(\varphi_n)$ and $c - c_n = o(1)$. Therefore $\sigma_{1z}(u_n + \zeta, c)$ can vanish for at most one such $\zeta$ and, since $u_c^+ - u_n = O(\varphi_n)$, we must have $u_n - u_c^- \gg \varphi_n$.

Take any sequence $\varphi_n = o(q_1)$ and write

$$\sigma_1(u_c^-, c) \geq \sigma_1(u_c^+ - \varphi_n, c) = \sigma_1(u_c^+ - \varphi_n) + (c - c_n)\log(\varphi_n - u_c^+).$$

(As usual, we imagine real parts having been taken.) If we apply (4.7) with $\zeta = u_c^+ - u_n$ and with $\zeta = u_c^+ - \varphi_n - u_n$ and subtract, we obtain

$$\sigma(u_c^+ - \varphi_n) - \sigma(u_c^+) = \frac{\alpha}{\beta}n^{-1/2}\varphi_n(1 + o(1)) + \frac{\alpha}{2\beta^2}\left(-2\varphi_n(u_c^+ - u_n) + \varphi_n^2\right)(1 + o(1)).$$

By subtracting the first parts of (4.10) and (4.8) we see that this equals

$$o(n^{-1/2}\varphi_n) + \frac{\alpha}{2\beta^2}\varphi_n^2.$$

Since $\varphi_n \gg n^{-1/2}$, as we may assume, we obtain

$$\sigma_1(u_c^+ - \varphi_n) > \sigma_1(u_c^+) + \eta\varphi_n^2$$

for some $\eta > 0$. Also, since $c - c_n > -\eta n^{-1/2}$ for some $\eta$ and $\log(1 - \varphi_n/u_c^+)$ is positive and $O(\varphi_n)$ we have

$$(c - c_n)\log(\varphi_n - u_c^+) \geq (c - c_n)\log(-u_c^+) - \eta n^{-1/2}\varphi_n.$$

Putting these together gives

$$\sigma_1(u_c^-, c) > \sigma_1(u_c^+, c) + \eta\varphi_n^2$$

for some $\eta > 0$.

This was for $c \le c_n$. For $c > c_n$ we use what we get from (4.12) by replacing $^+$ with $^-$, subtracting the two, and integrating. Together with using the already proved inequality for $c = c_n$ this gives

$$\sigma_1(u_c^-, c) - \sigma_1(u_c^+, c) > \eta\varphi_n^2 + \int_{c_n}^{c} \log(u_c^-/u_c^+)\, dc.$$

The logarithm is nonnegative. Hence $\sigma_1(u_c^-, c) - \sigma_1(u_c^+, c) > \eta\varphi_n^2$ for all $c$.

If $c - c_n = O(\varphi_n)$ then using this and (4.5) give

$$\sigma_1(u_c^-, c) > \sigma_1(-r_1^{-1}) + \log(r_1^{-1})(c - c_n) + \eta\varphi_n^2$$

with a different $\eta$. If $c \ge c_n$ we use

$$\sigma_1(u_c^-, c) - \sigma_1(u_{c_n}^-) = \int_{c_n}^{c} \log(-u_c^-)\, dc.$$

Since $u_c^-$ is decreasing and is less than $-r_1^{-1}$ when $c = c_n$ this gives

$$\begin{aligned}\sigma_1(u_c^-, c) &\ge \sigma_1(u_{c_n}^-) + \log(r_1^{-1})(c - c_n)\\ &\ge \sigma_1(u_{c_n}^+) + \log(r_1^{-1})(c - c_n) + \varphi_n^2.\end{aligned}$$

Combining this with (4.5) for $c = c_n$ shows that

$$\sigma_1(u_c^-, c) \ge \sigma_1(-r_1^{-1}) + \log(r_1^{-1})(c - c_n) + \eta\varphi_n^2$$

holds for these $c$ as well. Since $\{\varphi_n\}$ was an arbitrary sequence satisfying $\varphi_n = o(q_1)$ the last statement of the lemma follows. □

Next we consider the steepest descent curves, which we denote by $C^\pm(c)$ corresponding to the integrals $I^\pm(c)$. It follows from (4.4) that $C^+(c)$ passes through $u_c^+$ because on the curve $|\psi_1(z, c)|$ has a maximum at that point; similarly, $C^-(c)$ passes through $u_c^-$. We have enough information to evaluate the portions of these integrals taken over the immediate neighborhoods of these points, but we also have to show that the integrals over the rest of the curves are negligible. This requires not only that the integrands are much smaller there, which they are, but also that the curves themselves are not too badly behaved.

To see what is needed, let $\Gamma^\pm$ be arcs of steepest descent curves for a function $\rho$, curves on which $\Im\rho$ is constant. In analogy with our $C^\pm(c)$ we assume $\Re\rho$ is increasing on $\Gamma^-$ as we move away from the critical point and decreasing on $\Gamma^+$. If $s$ measures arc length on $\Gamma^\pm$ we have for $z \in \Gamma^\pm$,

$$\frac{dz}{ds} = \mp\frac{|\rho'(z)|}{\rho'(z)}. \tag{4.13}$$

If the arc goes from $a$ to $b$ then

$$\int_{\Gamma^\pm} |\rho'(z)|\, ds = \mp\int_{\Gamma} \rho'(z)\, dz = \mp(\rho(b) - \rho(a)).$$

Hence the length of $\Gamma^\pm$ is at most

$$\frac{|\rho(b) - \rho(a)|}{\min_{z\in\Gamma^\pm} |\rho'(z)|}. \tag{4.14}$$

This is to be modified if $\rho'$ has a simple zero at $z = a$, for example. In this case we replace $\rho'(z)$ by $\rho'(z)/(z-a)$. (This is seen by making the variable change $z = a+\sqrt{\xi}$.)

Our goal is Lemma 4.5 below. In order to use the length estimate (4.14) to deduce the bounds of the lemma, we must first locate regions in which our curves are located, and then find lower bounds for $\sigma_1'(z, c)$ in these regions. (Upper bounds for $|\sigma_1(z, c)|$ will be easy.) These will be established in the next lemmas.

For $r > 0$ define $n(r) = \#\{j : r_j \geq r\}$.

**Lemma 4.2.** *The curves $C^{\pm}(c)$ lie in the regions*

$$\left\{z : |\arg(r^{-1} + z)| \leq \pi \frac{cn}{\alpha n(r) + cn}\right\}$$

*for all $r$ and in $|z + r_2^{-1}| \geq \delta n^{-1}$ if $\delta$ is small enough.*

*Proof.* For a point $z$ on either of the curves, say in the upper half-plane, we have

$$\begin{aligned} c\pi &= \frac{\alpha}{n}\sum_{j=2}^{n} \arg(r_j^{-1} + z) + \arg(z-1) + (c-1)\,\arg z \\ &\geq \frac{\alpha n(r)}{n} \arg(r^{-1} + z) + c\,\arg(r^{-1} + z), \end{aligned}$$

which gives the first statement of the lemma. For the second, observe that if $\zeta = O(\varphi_n)$ then $\sigma_1'(r_2^{-1} + \zeta, c) = \alpha/n\zeta + O(1)$. This shows, first, that $u_c^-$ lies to the right of the circle $|\zeta| = \delta\, n^{-1}$ if $\delta$ is small enough and, second, that $1/\sigma_1'(z, c)$, thought of as a vector, points outward from this circle if $\delta$ is small enough. Since a point of $C^-(c)$ moves in the direction of $1/\sigma_1'(z, c)$ as it moves away from $u_c^-$ (see (3.7) of [GTW2]), the curve can never pass inside the circle. Therefore the entire disc $|\zeta| \leq \delta\, n^{-1}$ lies to the left of $C^-(c)$. This gives the second statement for $C^-(c)$ and it follows also for $C^+(c)$ since this is to the right of $C^-(c)$. $\square$

The next lemma, together with (4.13) and the length estimate (4.14), will imply that for $z$ large the curves will move in the direction of $z$ and are well-behaved. If we take any $\bar{r} < b/(1-b)$ then a positive proportion of the $r_j$ are greater than $\bar{r}$ and so by Lemma 4.2 the curves lie in a region

$$\left\{z : |\arg(\bar{r}^{-1} + z)| \leq \pi(1-\delta)\right\} \tag{4.15}$$

for some $\delta > 0$.

**Lemma 4.3.** *We have $z\,\sigma_1'(z, c) \to c+\alpha$ as $n \to \infty$ and $z \to \infty$ through region (4.15).*

*Proof.* We have

$$z\,\sigma_1'(z, c) = c + \alpha + O(n^{-1}) + O(z^{-1}) + \frac{\alpha}{n}\sum_{j=2}^{n} \frac{1}{1 + r_j z},$$

and it suffices to show that the last term tends to 0 as $n \to \infty$ and $z \to \infty$ through region (4.15). If $z$ is in this region and $r < \bar{r}/2$ then $|1+rz| \geq \delta(1+r|z|)$ for another $\delta$. The same bound will hold for all $r \leq b/(1-b)$ if $z$ is large enough. Choose $M$ large and break the sum on the right, with its factor $n^{-1}$, into two parts, the terms where $r_j|z| < M$ and the terms where $r_j|z| \geq M$. We find that its absolute value is at most

$$n^{-1}(n - n(M/|z|)) + \frac{1}{\delta M}.$$

The first term tends to 0 as $z \to \infty$ while the second could have been arbitrarily small to begin with. □

*Remark.* If $P(p = 0)$ is positive then the above has to be modified. We replace $c + \alpha$ by $c + \alpha\, P(p > 0)$.

Because of the above lemma we need only consider $z$ in a bounded set. We use the fact that by Lemma 4.2 with $r = r_2$ our curves lie a region

$$\left\{ z : |\arg(r_2^{-1} + z)| \leq \pi(1 - \delta n^{-1}),\ \ |r_2^{-1} + z| \geq \delta n^{-1} \right\}. \tag{4.16}$$

**Lemma 4.4.** *For all $z$ in any bounded subset of the region (4.16) we have*

$$|\sigma_1'(z, c)| \geq \delta\, n^{-6} \left| \frac{(z - u_c^-)\,(z - u_c^+)}{z(z-1)} \right|$$

*for some $\delta > 0$ independent of $c$.*

*Proof.* To obtain the lower bound we write

$$\phi(s; z) = \phi(s_2,\ s_3, \cdots, s_n;\ z) = \frac{\alpha}{n} \sum_{j=2}^{n} \frac{1}{s_j + z} + \frac{1}{z-1} + \frac{c-1}{z}.$$

Of course $\sigma_1'(z, c) = \phi(r_2^{-1}, r_3^{-1}, \cdots, r_n^{-1})$. Think of $s_2 = r_2^{-1}$ and $z$ as fixed, and consider the problem of finding $\inf\ |\phi(s; z)|$, where $s_3, \cdots, s_n$ are subject to the conditions

$$s_j \geq s_2,\ \ \phi(s; u_c^{\pm}) = 0.$$

If we take sequences so that the inf is approached in the limit, then some $s_j$ may tend to infinity, others may tend to $s_2$, and the rest, if any, tend to values strictly greater than $s_2$. Thus our inf is equal to the minimum of $|\phi(s; z)|$, where $\phi$ now has the form

$$\phi(s_2,\ s_3, \cdots, s_{n'};\ z) = \frac{\alpha}{n} \sum_{j=2}^{n'} \frac{n_j}{s_j + z} + \frac{1}{z-1} + \frac{c-1}{z}$$

with $n' \leq n$, $\sum n_j = n - 1$, and the $s_j$ with $j > 2$ satisfying $s_j > s_2$ and the constraints $\phi(s; u_c^{\pm}) = 0$.

Notice that the minimum cannot be zero since $\phi(s;\ z)$, thought of for the moment as a function of $z$, has $n'$ finite zeros. It has zeros at $u_c^{\pm}$ and one between each pair of consecutive $-s_j$ since all the coefficients of $1/(s_j + z)$ are positive. This accounts for all $n'$ zeros, so our $z$ cannot be one of them.

We apply Lagrange multipliers to find the minimum of $|\phi(s; z)|^2$ over $s_3, \cdots, s_{n'}$, achieved at interior points. There are two constraints, hence two multipliers $\lambda$ and $\mu$.

If $p + iq$ is the value $\phi(s; z)$, where its absolute value achieves its minimum, then the equations we get are

$$\Re\,(p - iq)\,\frac{1}{(s_j + z)^2} = \frac{\lambda}{(s_j + u_c^-)^2} + \frac{\mu}{(s_j + u_c^+)^2},$$

where we have divided by the factor $n_j$ appearing in all terms. This is the same sixth degree polynomial equation for all the $s_j$. It follows that there are at most six different $s_j$. Assuming there are exactly six (if there are fewer the argument is the same and the final estimate is better) we change notation again and write these as $s_3, \cdots, s_8$ so that the minimum is achieved for

$$\phi(s_2,\ s_3, \cdots, s_8;\ z) = \frac{\alpha}{n}\sum_{j=2}^{8}\frac{n_j}{s_j + z} + \frac{1}{z-1} + \frac{c-1}{z}$$

with other $n_j$.

This has eight zeros. Two of them are $u_c^{\pm}$ and the other six, lying between consecutive $-s_j$, we denote by $u_1, \cdots, u_6$. We have the factorization

$$\phi(s;\ z) = \frac{1-c}{u_c^-\ u_c^+}\,\frac{(z - u_c^-)\,(z - u_c^+)}{z(z-1)}\,\frac{\prod_{i=1}^{6}(1 - z/u_i)}{\prod_{j=2}^{8}(1 - z/s_j)},$$

and it remains to find a lower bound for this. Near $z = 0$ we have $\sigma_1'(z, c) = (1 - c)z^{-1} - 1 + \alpha\,\langle r\rangle + o(1)$, so if $c$ is close to 1 then $(1 - c)/u_c^+ = 1 - \alpha\,\langle r\rangle + o(1)$. In particular this is bounded away from zero. Thus the first factor above is bounded away from zero. As for the factors in the products, observe first that each factor $1 - z/s_j$ is bounded since $z$ and all factors $1/s_j$ are. For the others, we use again the fact that the curves lie in a region (4.16). In any bounded subset of this region each $|1 - z/u_i| \geq \eta n^{-1}$ for some $\eta > 0$. (If $z$ is in a neighborhood of 0 this is clear since each $u_i < 0$. Otherwise write $1 - z/u_i = z(z^{-1} - u_i^{-1})$.) Therefore the product of these is bounded below by a constant times $n^{-6}$. This completes the proof. □

Now we can show that the curves $C^{\pm}(c)$ are not too badly behaved.

**Lemma 4.5.** *For some constant $A > 0$ the length of $C^+(c)$ is $O(n^A)$ and*

$$\int_{C^-(c)} |z|^{-2}\,|dz| = O(n^A).$$

*Proof.* It follows from Lemma 4.3 that $C^+(c)$ lies in a bounded set. For, this lemma implies that the vectors $1/\sigma_1'(z, c)$ point outward from a large circle $|z| = R$, and since by (4.13) $C^+(c)$ goes in the direction opposite to $1/\sigma_1'(z, c)$, a point of the curve starting at $u_c^+$ can never pass outside the circle. Also, some disc $|z| \leq \delta(1 - c)$ is disjoint from $C^+(c)$ because $1/\sigma_1'(z, c)$ points outward from a small enough circle $|z| = \delta(1 - c)$ and so $C^+(c)$ cannot cross into it. It follows that $\sigma_1'(z, c)$, and so also $\sigma_1(z, c)$, is bounded on any portion of $C^+(c)$ close to $z = 0$. A similar argument shows that some disc $|z - 1| \leq \delta$ lies entirely inside $C^+(c)$. Finally, we know that $u_c^-$ is within $O(n^{-1/2})$ of $-r_2^{-1}$ and if $\zeta = o(q_1)$ then $\sigma_1'(r_2^{-1} + \zeta, c) = \alpha/n\zeta + O(1)$. In particular $u_c^-$ lies in a region $|\zeta| \geq \delta n^{-1}$ for some $\delta > 0$. Since also $\sigma_1'' = -\alpha/n\zeta^2 + O(1)$, by Lemma 3.6, we deduce that $\sigma_1''(z, c) = O(n)$ when $|z - u_c^-| \leq \delta n^{-1}/2$, thus for such

$z$ we have $\sigma_1(z,c) = \sigma_1(u_c^-,c) + O(n|z-u_c^-|^2)$. But it follows from Lemma 4.1 that $\sigma_1(u_c^-,c) - \sigma_1(u_c^+,c) > \varphi_n^2$, and then, since $n^{-1} = o(\varphi_n^2)$, $\sigma_1(u_c^+,c) < \sigma_1(z,c)$ for $|z-u_c^-| \le \delta n^{-1}/2$. As the maximum of $\sigma_1(z,c)$ on $C^+(c)$ occurs at $u_c^+$, this shows that the distance from $C^+(c)$ to $u_c^-$is at least $\delta n^{-1}/2$. With these facts established we use the lower bound of Lemma 4.4, the length estimate (4.14) (extended as in the remark following it), and the obvious upper bound for $|\sigma_1(z,c)|$ in the region (4.16) to deduce that the length of $C^+(c)$ is $O(n^A)$ for some constant $A$.

As for the integral over $C^-(c)$, we observe that, since $c < 1$ and $cm$ is an integer, $1-c$ is at least a constant times $n^{-1}$. Since $C^-(c)$ lies outside a disc $|z| \le \delta(1-c)$, we have $z^{-1} = O(n)$ on $C^-(c)$. A lower bound for the distance from $C^-(c)$ to $u_c^+$ is obtained using the fact that $\sigma_1(u_c^-,c) - \sigma_1(u_c^+,c) > \varphi_n^2$. Since $\sigma_1'$ is bounded in a neighborhood of $u_c^+$, we have $\sigma_1(u_c^-,c) > \sigma_1(z,c)$ for $|z-u_c^+|$ less than $\varphi_n^2$ times a sufficiently small constant. This shows that $C^-(c)$ is at least this far from $u_c^+$. We apply the other bounds as before; we think of the integral over the portion of $C^-(c)$ outside a large circle as the sum of integrals over the arcs from $a_k$ to $a_{k+1}$, where $a_k$ is the point of $C^-(c)$ where $|z| = k$. Lemma 4.3 and (4.14) are used again here. □

## 5. Asymptotic Evaluation of the Integrals

We evaluate $I^+(c)$ first when $c - c_n = O(\varphi_n)$. Then $\sigma_{1zz}(u_c^+,c) = \alpha/\beta^2 + o(1)$ and so if we set $z = u_c^+ + \zeta$ we have

$$\sigma_1(z,c) = \sigma_1(u_c^+,c) + \frac{\alpha}{2\beta^2}(1+o(1))\zeta^2$$

as long as $\zeta = O(\varphi_n)$. If $|\zeta| = \varphi_n$ then the real part of the second term above is less than a negative constant times $\varphi_n^2$ and, since this real part decreases as we go out $C^+(c)$, it is at least this negative whenever $|\zeta| \ge \varphi_n$. If we recall that this gets multiplied by $m$ in the exponent and the fact that $C^+(c)$ has the length at most a power of $n$ (by Lemma 4.5), we see that the contribution of this part of the integral is $O\left(e^{m\sigma(u_c^+,c) - n\varphi_n^2 + O(\log n)}\right)$. It follows from Lemma 3.3 and assumption (c) that with high probability $q_1 \gg \log n/n^{1/2}$, and we could have chosen $\varphi_n$ to satisfy this also. Thus, with error $o(e^{m\sigma(u_c^+,c)})$ the integral $I^+(c)$ is equal to

$$\frac{1}{2\pi i}\int_{|\zeta|<\varphi_n}(1+r_1(u_c^+ + \zeta))\, e^{(n/2\beta^2)(1+o(1))\zeta^2}\, dz\, e^{m\sigma_1(u_c^+,c)}$$

(since $\alpha m = n$). Since $\varphi_n \gg n^{-1/2}$, in the limit after making the variable change $\zeta \to n^{-1/2}\zeta$ the integration can be taken over $(-i\infty, i\infty)$ (downward really, but we can reverse the directions of integrations), the linear factor $\zeta$ contributes zero, and by (4.10),

$$1 + r_1 u_c^+ = r_1\left(2\beta n^{-1/2} + \frac{r_1\beta^2}{\alpha}(c-c_n) + o(n^{-1/2} + |c-c_n|)\right).$$

Thus the integral is asymptotically equal to $\beta\sqrt{2\pi}\, i n^{-1/2}$ times the above and, by (4.5),

$$I^+(c) = \frac{r_1\beta^2}{\sqrt{2\pi}}n^{-1}\left(2 + \frac{r_1\beta}{\alpha}n^{1/2}(c-c_n) + o(1+n^{1/2}|c-c_n|\right)$$
$$\times\psi_1(-r_1,c)^{-1}\, e^{-\frac{r_1\beta^2}{2\alpha}m\left(c-c_n+\frac{2\alpha}{r_1\beta}(1+o(1))n^{-1/2}\right)^2}.$$

This assumes that $c - c_n = O(\varphi_n)$. For all $c \geq c_n$ we use the second part of Lemma 4.1 and again the fact that $C^+(c)$ has the length at most a power of $n$. We deduce

$$I^+(c) = O\left(\psi_1(-r_1^{-1}, c)\, e^{-\eta n^{1/2}\,(c-c_n)+O(\log n)}\right)$$

for $c \geq c_n$.

For the integral over $C^-(c)$ we use the last part of Lemma 4.1 and the second part of Lemma 4.5. These imply that the integral over $C^-$ is

$$O\left(\psi_1(-r_1^{-1}, c)^{-1}\, e^{-n\varphi_n^2+O(\log n)}\right) = o(\psi_1(-r_1^{-1}, c)).$$

But our integral for $I^-(c)$ is *not* taken over $C^-(c)$. Recall that the original contour must have all the $-r_j^{-1}$ on the outside whereas $-r_1^{-1}$ is inside (more precisely, on the other side of) $C^-(c)$. Therefore if we deform the contour to $C^-(c)$ we pass through the pole at $-r_1^{-1}$. Thus

$$I^-(c) = r_1\, \psi_1(-r_1^{-1}, c)^{-1} + o(\psi_1(-r_1^{-1}, c)).$$

Now recall that in $I^+(c)$ we set $c - c_n = h' + j + \ell$, in $I^-(c)$ we set $c - c_n = h' + \ell + k$ and then we sum over $\ell$ to get the matrix product. Recall also that $\psi_1(-r_1^{-1}, c) = \psi_1(-r_1^{-1})\,(-r_1)^{-m(c-c_n)}$. The factors $(-r_1)^{-m(c-c_n)}$ in $I^+(c)$ and $(-r_1)^{m(c-c_n)}$ in $I^-(c)$ will combine to give $(-r_1)^{m(k-j)}$ which can be eliminated without affecting the determinant. It follows that we can modify the expressions for $I^{\pm}(c)$ by removing these factors. We can also remove the factors $\psi_1(-r_1^{-1})^{\pm 1}$ since they cancel upon multiplying. Thus our replacements are

$$I^+(c) \to \frac{r_1\beta^2}{\sqrt{2\pi}} n^{-1}\left(2 + \frac{r_1\beta}{\alpha} n^{1/2}(c - c_n)\right) e^{-\frac{r_1\beta^2}{2\alpha} m\left(c - c_n + \frac{2\alpha}{r_1\beta}(1+o(1))n^{-1/2}\right)^2},$$

if $c - c_n = O(\varphi_n)$, and

$$I^+(c) \to O\left(e^{-\eta n^{1/2}(c-c_n)+O(\log n)}\right),$$

if $c > c_n$. Furthermore, $I^-(c) \to r_1 + o(1)$.

Recall next that we set $h' = sn^{1/2}$ and in $I^+(c)$, $c = c_n + sn^{1/2} + \lfloor xn^{1/2} \rfloor + \lfloor zn^{1/2} \rfloor$, so that

$$c - c_n = (s + x + z + o(1))n^{1/2}/m = \alpha(s + x + z + o(1))n^{-1/2},$$

and eventually we multiply by $n$ because of the scaling. Take first the case $c - c_n = O(\varphi_n)$, that is, $x + z = O(n^{1/2}\varphi_n)$. Since $m = n/\alpha$ and $r_1\,\beta = \tau^{-1}\,(1 + o(1))$ the modified $I^+(c)$ equals

$$\frac{r_1^2\beta^3}{\sqrt{2\pi}} n^{-1}(2\tau + s + x + z + o(1 + x + y))e^{-(2\tau+s+x+z+o(1))^2/2\tau^2}.$$

On the other hand, $I^-(c)$ is equal to $r_1$ with error $o(1)$. The result of multiplying these together, multiplying by $n$, and integrating with respect to $z$ over $(0,\ \infty)$, is asymptotically equal to

$$\frac{1}{\sqrt{2\pi}\,\tau} e^{-(2\tau+s+x)^2/2\tau^2}. \tag{5.1}$$

This holds for $c - c_n = O(\varphi_n)$. If $c - c_n \geq \varphi_n$ we have, for our modified $I^+(c)$, the estimate

$$O\left(e^{-\eta n^{1/2}(c-c_n)+O(\log n)}\right) = O(n^{-1}).$$

Integrating the square of this over a region $x + z = O(n^{1/2})$ will give $o(1)$.

It follows that the matrix product scales to the operator on $(0, \infty)$ with kernel (5.1). This is a rank one kernel so its Fredholm determinant equals one minus its trace, which equals

$$\frac{1}{\sqrt{2\pi}\,\tau}\int_{-\infty}^{2\tau+s} e^{-x^2/2\tau^2}.$$

This establishes the convergence in probability statement of Theorem 1.

*Remark.* One could rightly object that to scale a product to a trace class operator we should know that each factor scales in Hilbert-Schmidt norm. In our case the second limiting kernel is a constant and the product is not even Hilbert-Schmidt. But we could have multiplied the kernel of the first operator by $(1 + x)\,(1 + z)$ and the kernel of the second operator by $(1+z)^{-1}\,(1+y)^{-1}$. This would not have affected the determinant of the product, both operators would have scaled in Hilbert-Schmidt norm and the product would have scaled in trace norm to the rank one kernel

$$\frac{1}{\sqrt{2\pi}\,\tau}\,e^{-(2\tau+s+x)^2/2\tau^2}\,\frac{1+x}{1+y}$$

which has the same Fredholm determinant.

## 6. Almost Sure Convergence

What is needed, and all that is needed, is an "almost sure" substitute for Lemma 3.6 under assumptions (a′) and (b′). We begin with a lemma on extreme order statistics of uniform random variables, part or all of which may well be in the literature.

**Lemma 6.1.** *Let $a > 1$ be arbitrary. Then, almost surely,*

$$t_1 \geq \frac{\eta}{n\,\log^a n}, \qquad \frac{t_1}{t_2} \leq 1 - \frac{1}{\log^a n},$$

*for sufficiently large $n$. Here, $\eta$ is a positive constant depending on $a$.*

*Proof.* We use the notation $t_{n,j}$ for our $t_j$ to display their dependence on $n$. We have

$$P(t_{n,1} \leq \delta) = 1 - (1-\delta)^n \sim n\delta \ \text{ if } n\delta = o(1).$$

In particular

$$P\left(t_{2^k,1} \leq \frac{2^{-k}}{k^a}\right) \sim \frac{1}{k^a}.$$

It follows that, a.s. for sufficiently large $k$ we have

$$t_{2^k,1} > \frac{2^{-k}}{k^a}.$$

Take any $n$ and let $k$ be such that $2^{k-1} < n \le 2^k$. From the above we have, a.s. for sufficiently large $n$

$$t_{n,1} \ge t_{2^k,1} > \frac{2^{-k}}{k^a} \ge \frac{\eta}{n \log^a n},$$

for some $\eta$.

For the ratio we use the fact that

$$P\left(\frac{t_{n,j}}{t_{n,j+1}} > 1-\delta\right) = 1-(1-\delta)^j \sim j\delta \text{ if } j\delta = o(1). \tag{6.1}$$

Now suppose that

$$\frac{t_{n,1}}{t_{n,2}} > 1 - \frac{1}{\log^a n}, \tag{6.2}$$

and let $k$ be such that $2^{k-1} < n \le 2^k$. Take any $J$ (which will eventually be of order $\log k$). Then there are two possibilities:

(1) $t_{2^k,j} \le t_{n,1}$ for all $j \le J$;
(2) $t_{2^k,j} > t_{n,1}$ for some $j \le J$.

Consider possibility (1) first. Let $G_n$ be the event that $t_{n,1} \le a \log\log n/n$. By Ex. 4.3.2 of [Gal], $P(G_n \text{ eventually}) = 1$. Moreover,

$$\begin{aligned} P(\{t_{2^k,j} \le t_{n,1} \text{ for all } j \le J\} \cap G_n) &\le P(t_{2^k,j} \le 2\log\log n/n \text{ for all } j \le J) \\ &\le \binom{2^k}{J!}\left(2\,\frac{\log\log n}{n}\right)^J \le e^{J\log\log k - J\log J + AJ}, \end{aligned}$$

for some constant $A$. If $J = B\log k$ then the bound above equals $e^{-B(\log B - A)\log k}$, so if we choose $B$ large enough the sum over $k$ of these probabilities will be finite. With this $J$, (1) can therefore a.s. occur for only finitely many $k$.

Next consider possibility (2) and let $j$ be the smallest integer $\le J$ such that $t_{2^k,j} > t_{n,1}$. Then $t_{2^k,j} \le t_{n,2}$ and $t_{n,1} = t_{2^k,\ell}$ for some $\ell < j$. It follows that $t_{2^k,j-1}/t_{2^k,j} > t_{n,1}/t_{n,2}$ and by (6.2) this is at least $1 - C/k^a$, for some constant $C$ (which will change from appearance to appearance). Therefore, by (6.1),

$$\begin{aligned} &P((6.2) \text{ and } (2) \text{ both happen}) \\ &\quad \le P(t_{2^k,j-1}/t_{2^k,j} > 1 - C/k^a \text{ for some } j \le J) \le CJ^2/k^a \le C\log^2 k/k^a. \end{aligned}$$

It follows that (2) and (6.2) can happen together only for finitely many $n$. The upshot is that a.s. the inequality (6.2) can occur for only finitely many $n$, which completes the proof. □

We are now ready to prove our substitute for Lemma 3.6. Recall that we can set $q_j = G^{-1}(t_j)$. The assumption (a′) implies that $G$ is continuous near 0, so that $G(G^{-1}(x)) = x$ for small $x$.

**Lemma 6.2.** *Suppose (a′) and (b′) are satisfied. Then there exists a sequence $\varphi_n \gg \log n/n^{1/2}$ such that a.s. for any sequence $\{v_n\}$ lying in the disc with diameter the real interval $[-r_1^{-1} - O(\varphi_n),\ \xi]$ we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=2}^{n} \frac{r_j}{(1+r_j v_n)^2} = \left\langle \frac{r}{(1+r\xi)^2} \right\rangle.$$

*Proof.* From the proof of Lemma 3.6 we see that we want to show that, for some sequence $\varphi_n$ as described, we have a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=2}^{n} \frac{q_1}{q_j\,(q_j - (q_1 + O(\varphi_n)))^2} = 0.$$

Assumption (a′) implies that

$$\frac{x}{y} \geq \left( \frac{G^{-1}(x)}{G^{-1}(y)} \right)^{\gamma},$$

when $x \leq y$ are small enough. Therefore, it follows from the second part of Lemma 6.1, that a.s. for large $n$,

$$\frac{q_1}{q_2} \leq 1 - \frac{\eta}{\log^a n} \tag{6.3}$$

for another constant $\eta > 0$. Set

$$\psi_n = \frac{1}{2} \frac{\eta}{\log^a n} q_2.$$

Let us show that $\psi_n \gg \log n / n^{1/2}$. Assumption (a′) implies that $G^{-1}(x)$ is at most a constant times $x^{1/\gamma}$, thus the fact that $t_1 = O(\log\log n/n)$ shows that $q_1$ is at most a constant times $(\log\log n/n)^{1/\gamma}$. Furthermore, assumption (b′) gives, with a slightly smaller $\nu$, $x^2 \gg G(x) \log^\nu x^{-1}$. Applying this with $x = q_1 = G^{-1}(t_1)$ and using the first part of Lemma 6.1 gives

$$q_1^2 \gg \frac{1}{n \log^a n} \log^\nu q_1^{-1}.$$

We therefore deduce that

$$q_1^2 \gg \frac{1}{n} \log^{\nu - a} n \tag{6.4}$$

for a slightly smaller $\nu$ than in (b′). By (6.3), the same holds for $q_2$ and so

$$\psi_n^2 \gg \frac{1}{n} \log^{\nu - 3a} n$$

and $\psi_n \gg \log n / n^{1/2}$ as long as $\nu - 3a > 2$. Since $a > 1$ is arbitrary the requirement becomes $\nu > 5$. But from (a′) and (b′) we see that necessarily $\gamma > 2$, so that $\nu > 8$.

If $j \geq 2$, then (6.3) and the inequality $q_2 \leq q_j$ imply that

$$q_j - (q_1 + \psi_n) \geq \frac{1}{2} \frac{\eta}{\log^a n} q_j.$$

We take for $\{\varphi_n\}$ any sequence satisfying

$$\frac{\log n}{n^{1/2}} \ll \varphi_n \ll \psi_n.$$

At this point we follow the proof of Lemma 3.6 to see that the expression

$$\frac{\log^{2a} n}{n} \sum_{j=2}^{n} \frac{q_1}{q_j^3} \tag{6.5}$$

needs to go to 0 a.s. to conclude the proof of this lemma. This is what we will demonstrate.

For any $k_n$, if we separate the sum in (6.5) over $j \le k_n$ from the sum over $j > k_n$, we see that (6.5) is at most

$$\frac{\log^{2a} n}{n\, q_1^2}\, k_n + \log^{2a} n\, \frac{q_1}{q_{k_n+1}}\, \frac{1}{n} \sum_{j=1}^{n} \frac{1}{q_j^2}. \tag{6.6}$$

We first determine $k_n$ so the second term in (6.6) goes a.s. to 0. By strong law, $n^{-1} \sum q_j \to \langle q^{-2} \rangle$ a.s., so $\log^{2a} n\, q_1/q_{k_n+1}$ needs to go to 0. We have, for each $\delta > 0$,

$$\begin{aligned} &P\left(\log^{2a} n \frac{q_1}{q_{k_n+1}} \ge \delta\right) \\ &= P\left(\frac{G^{-1}(t_1)}{G^{-1}(t_{k_n+1})} \ge \frac{\delta}{\log^{2a} n}\right) \le P\left(\frac{t_1}{t_{k_n+1}} \ge \left(\frac{\delta}{\log^{2a} n}\right)^{\gamma}\right) \\ &= \left(1 - \left(\frac{\delta}{\log^{2a} n}\right)^{\gamma}\right)^{k_n} \le e^{-\left(\frac{\delta}{\log^{2a} n}\right)^{\gamma} k_n}. \end{aligned}$$

This is summable over $n$ if we choose

$$k_n = \lfloor \log^a n \left(\log^{2a} n\right)^{\gamma} \rfloor + 1.$$

With this choice, the second summand in (6.6) therefore goes to 0 a.s.

On the other hand, the first term in (6.6) is with the same choice of $k_n$ at most a constant times

$$\frac{\log^{(2\gamma+3)a} n}{n\, q_1^2},$$

and from (6.4) this is $o(1)$ times $\log^{(2\gamma+4)a-\nu} n$. Since $a > 1$ was arbitrary and $\nu > 2\gamma + 4$, we can make $(2\gamma + 4)a - \nu < 0$ and then the first summand in (6.6) goes to 0 a.s. This completes the proof. □

With this lemma in place of Lemma 3.6 the reader will find that all subsequent limits and estimates in Sects. 4 and 5 will hold almost surely, thus giving the second statement of the theorem. The reason our sequence had to satisfy $\varphi_n \gg \log n/n^{1/2}$ is that errors of the form $O\left(e^{-n\varphi_n^2 + O(\log n)}\right)$ appeared in the evaluation of $I^{\pm}(c)$ and these had to be $o(1)$.

*Acknowledgement.* This work was partially supported by National Science Foundation grants DMS-9703923, DMS–9802122, and DMS–9732687, as well as the Republic of Slovenia's Ministry of Science Program Group 503. Special thanks go to Harry Kesten, who supplied the main idea for the proof of Lemma 6.1. The authors are also thankful to the referee for the careful reading of the manuscript and suggestions for its improvement.

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Communicated by H. Spohn