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A SYSTEM OF DIFFERENTIAL EQUATIONS FOR THE AIRY PROCESS

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Abstract
The Airy process $\tau \to A_\tau$ is characterized by its finite-dimensional distribution functions

$$\Pr\left(A_{\tau_1} < \xi_1, \ldots, A_{\tau_m} < \xi_m\right).$$

For $m = 1$ it is known that $\Pr\left(A_\tau < \xi\right)$ is expressible in terms of a solution to Painlevé II. We show that each finite-dimensional distribution function is expressible in terms of a solution to a system of differential equations.

I. Introduction

The Airy process $\tau \to A_\tau$, introduced by Prähofer and Spohn [6], is the limiting stationary process for a certain $1+1$-dimensional local random growth model called the polynuclear growth model (PNG). It is conjectured that the Airy process is, in fact, the limiting process for a wide class of random growth models. (This class is called the $1+1$-dimensional KPZ universality class in the physics literature [5].) The PNG model is closely related to the length of the longest increasing subsequence in a random permutation [2]. This fact together with the result of Baik, Deift and Johansson [3] on the limiting distribution of the length of the longest increasing subsequence in a random permutation shows that the distribution function $\Pr\left(A_\tau < \xi\right)$ equals the limiting distribution function, $F_2(\xi)$, of the largest eigenvalue in the Gaussian Unitary Ensemble [7]. F_2 is expressible either as a Fredholm determinant of a certain trace-class operator (the Airy kernel) or in terms of a solution to a nonlinear differential equation (Painlevé II). The finite-dimensional distribution functions

$$\Pr\left(A_{\tau_1} < \xi_1, \ldots, A_{\tau_m} < \xi_m\right)$$

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are expressible as a Fredholm determinant of a trace-class operator (the extended Airy kernel) [4, 6]. It is natural to conjecture [4, 6] that these distribution functions are also expressible in terms of a solution to a system of differential equations. It is this last conjecture which we prove.

II. Statement

The Airy process is characterized by the probabilities

$$\Pr\left(A_{\tau_1} < \xi_1, \, \dots \, , A_{\tau_m} < \xi_m\right) = \det\,(I-K),$$

where K is the operator with $m \times m$ matrix kernel having entries

$$K_{ij}(x,y) = L_{ij}(x,y)\, \chi_{(\xi_j,\infty)}(y)$$

and

$$L_{ij}(x,y) = \begin{cases} \displaystyle\int_0^\infty e^{-z\,(\tau_i-\tau_j)}\, \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz & \text{if } i \ge j, \\ \displaystyle -\int_{-\infty}^0 e^{-z\,(\tau_i-\tau_j)}\, \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz & \text{if } i < j. \end{cases}$$

We assume throughout that $\tau_1 < \cdots < \tau_m$, and think of K as acting on the m-fold direct sum of $L^2(\alpha, \infty)$ where $\alpha < \min \xi_j$.

To state the result we let $R = K\,(I-K)^{-1}$ and let $A(x)$ denote the $m \times m$ diagonal matrix $\mathrm{diag}\,(\mathrm{Ai}(x))$ and $\chi(x)$ the diagonal matrix $\mathrm{diag}\,(\chi_j(x))$, where $\chi_j = \chi_{(\xi_j,\infty)}$. Then we define the matrix functions $Q(x)$ and $\tilde{Q}(x)$ by

$$Q = (I-K)^{-1} A, \quad \tilde{Q} = A\,\chi\,(I-K)^{-1}$$

(where for $\tilde{Q}$ the operators act on the right). These and $R(x,y)$ are functions of the ξ_j as well as x and y. We define the matrix functions q, $\tilde{q}$ and r of the ξ_j only by

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j), \quad r_{ij} = R_{ij}(\xi_i,\, \xi_j).$$ [3]

Finally we let τ denote the diagonal matrix $\mathrm{diag}\,(\tau_j)$.

Our differential operator is $\mathcal{D} = \sum_j \partial_j$, where $\partial_j = \partial/\partial\xi_j$, and the system of equations is

$$\begin{aligned} \mathcal{D}^2\, q &= \xi\, q + 2\, q\, \tilde{q}\, q - 2\, [\tau,\, r]\, q, & (1) \\ \mathcal{D}^2\, \tilde{q} &= \tilde{q}\, \xi + 2\, \tilde{q}\, q\, \tilde{q} - 2\, \tilde{q}\, [\tau,\, r], & (2) \\ \mathcal{D}\, r &= -q\, \tilde{q} + [\tau,\, r]. & (3) \end{aligned}$$

Here the brackets denote commutator and ξ denotes the diagonal matrix $\mathrm{diag}\,(\xi_j)$.

This can be interpreted as a system of ordinary differential equations if we replace the variables $\xi_1, \dots, \xi_m$ by $\xi_1 + \xi, \dots, \xi_m + \xi$, where $\xi_1, \dots, \xi_m$ are fixed and ξ variable. Then $\mathcal{D} = d/d\xi$, and the ξ_j are regarded as parameters.

To get a representation for $\det\,(I-K)$ observe that

$$\partial_j\, K = -L\, \delta_j, \tag{4}$$

[3] We always interpret $R_{ij}(x,\, \xi_j)$ as the limit $R_{ij}(x,\, \xi_j+)$. These quantities are independent of our choice of α.

where the last factor denotes multiplication by the diagonal matrix with all entries zero except for the $j^{\rm th}$, which equals $\delta(x-\xi_j)$. We deduce that

$$\partial_j \log\, \det(I-K) = -{\rm Tr}\,(I-K)^{-1}\,\partial_j K = R_{jj}(\xi_j,\,\xi_j).$$

Hence $\mathcal{D}\,\log\,\det(I-K) = {\rm Tr}\; r$, and so it follows from (3) that

$$\mathcal{D}^2\,\log\,\det(I-K) = -{\rm Tr}\; q\,\tilde q$$

since the trace of $[\tau,\,r]$ equals zero. This gives the representation

$$\det(I-K) = \exp\left\{-\int_0^\infty \eta\,{\rm Tr}\; q(\xi+\eta)\,\tilde q(\xi+\eta)\;d\eta\right\}.$$

Here the determinant is evaluated at $(\xi_1,\,\ldots,\,\xi_m)$ and in the integral $\xi+\eta$ is shorthand for $(\xi_1+\eta,\,\ldots,\,\xi_m+\eta)$.
If $m=1$ the commutators drop out, $q=\tilde q$, equations (1) and (2) are Painlevé II and these are the previously known results.
Note Added in Proof: After the submission of this manuscript, Adler and van Moerbeke [1] found a PDE involving different quantitites than ours for the case $m=2$.

III. Proof

The proof will follow along the lines of the derivation in [7] for the case $m=1$. There the kernel was "integrable" in the sense that its commutator with M, the operator of multiplication by x, was of finite rank. The same was then true of the resolvent kernel, which was useful. But now our kernel is not integrable, so there will necessarily be some differences.
With $D=d/dx$ we compute that

$$[D,\,K]_{ij} = -{\rm Ai}(x)\,{\rm Ai}(y)\,\chi_j(y) + L_{ij}(x,\xi_j)\,\delta(y-\xi_j) + (\tau_i-\tau_j)\,K_{ij}(x,\,y).$$

Equivalently,

$$[D,\,K] = -A(x)\;A(y)\,\chi(y) + L\,\delta + [\tau,\,K],$$

where $\delta=\sum_j \delta_j$, multiplication by the matrix ${\rm diag}\,(\delta(x-\xi_j))$, and L is the operator with kernel $L_{ij}(x,y)$. (For clarity we sometimes write the kernel of an operator in place of the operator itself.) To obtain $[D,\,R]$ we replace K by $K-I$ in the commutators and left- and right-multiply by $\rho=(I-K)^{-1}$. The result is

$$[D,\,R] = -Q(x)\,\tilde Q(y) + R\,\delta\,\rho + [\tau,\,\rho].^4 \qquad (5)$$

We have already defined the matrix functions Q and $\tilde Q$ and we define

$$P=(I-K)^{-1}A', \quad u=(\tilde Q,\,{\rm Ai}) = \int \tilde Q(x)\,{\rm Ai}(x)\,dx.$$

It follows from (5) and the fact that τ and A commute that

$$Q' = P - Q\,u + R\,\delta\,Q + [\tau,\,Q].^5 \qquad (6)$$

[4]Because of the fact $\rho\,L\,\chi=R$ and our interpretation of $R_{ij}(x,\,\xi_j)$ as $R_{ij}(x,\,\xi_j+)$ we are able to write $R\,\delta\,\rho$ in place of $\rho\,L\,\delta\,\rho$.

[5]The meaning of δ here and later is this: If U and V are matrix functions then $U\,\delta\,V$ is the matrix with i,j entry $\sum_k U_{ik}(\xi_k)\,V_{kj}(\xi_k)$. Thus $R\,\delta\,Q$ is the matrix function with i,j entry $\sum_k R_{ik}(x,\,\xi_k)\,Q_{kj}(\xi_k)$. This makes it compatible with our use of δ also as a multiplication operator so that, for example, $(R\,\delta\,\rho)\,(A) = R\,\delta\,(\rho\,A)$.

Next, it follows from (4) that

$$\partial_j R = -R\,\delta_j\,\rho, \tag{7}$$

and it follows from this that $\partial_j Q = -R\,\delta_j\,Q$. Summing over j, adding to (6) and evaluating at ξ_k give

$$\mathcal{D}\,Q(\xi_k) = P(\xi_k) - Q(\xi_k)\,u + [\tau,\,Q(\xi_k)].$$

If we define $p_{ij} = P_{ij}(\xi_i)$ then we obtain

$$\mathcal{D}\,q = p - q\,u + [\tau,\,q]. \tag{8}$$

Next we use the facts that $D^2 - M$ commutes with L and that M commutes with χ. It follows that

$$[D^2 - M,\,K] = [D^2 - M,\,L\,\chi] = L\,[D^2 - M,\,\chi] = L\,[D^2,\,\chi] = L\,(\delta\,D + D\,\delta).$$

It follows from this that

$$[D^2 - M,\,\rho] = \rho\,L\,\delta\,D\,\rho + \rho\,L\,D\,\delta\,\rho.$$

Applying both sides to A and using the fact that $(D^2 - M)A = 0$ we obtain

$$Q''(x) - x\,Q(x) = \rho\,L\,\delta\,Q' + \rho\,L\,D\,\delta\,Q. \tag{9}$$

The first term on the right equals $R\,\delta\,Q'$. For the second term observe that

$$\rho\,L\,D\,\chi = \rho\,L\,\chi\,D + \rho\,L\,[D,\,\chi] = R\,D + \rho\,L\,\delta,$$

so we can interpret that term as $-R_y\,\delta\,Q$ (the subscript denotes partial derivative) where $-R_y(x,y)$ is interpreted as not containing the delta-function summand which arises from the jumps of R. With this interpretation of R_y we can write the second term on the right as $-R_y\,\delta\,Q$. Thus,

$$Q''(x) - x\,Q(x) = R\,\delta\,Q' - R_y\,\delta\,Q.$$

Using this we obtain from (6)

$$P' = x\,Q(x) + R\,\delta\,Q' - R_y\,\delta\,Q + Q'\,u - R_x\,\delta\,Q - [\tau,\,Q'],$$

and then from (6) once more

$$P' = x\,Q(x) + R\,\delta\,(P - Q\,u + R\,\delta\,Q + [\tau,\,Q]) - R_y\,\delta\,Q$$

$$+(P - Q\,u + R\,\delta\,Q + [\tau,\,Q])\,u - R_x\,\delta\,Q - [\tau,\,P - Q\,u + R\,\delta\,Q + [\tau,\,Q]\,].$$

It follows from (5) that

$$R_x + R_y = -Q(x)\,\tilde{Q}(y) + R\,\delta\,R + [\tau,\,\rho].$$

(We replaced $R\,\delta\,\rho$ by $R\,\delta\,R$ since, recall, R_y does not contain delta-function summands.) We use this and also the identity $R\delta[\tau,Q] - [\tau,R\delta Q] = -[\tau,R\delta]Q$, and the fact that δ and τ commute. The result is that

$$P' = x\,Q(x) + R\,\delta\,P + Q(x)\,\tilde{Q}\delta Q + (P - Q\,u + [\tau,\,Q])\,u$$
$$-2[\tau,\,R]\,\delta\,Q - [\tau,\;P - Q\,u + [\tau,\,Q]\,].$$

It follows from (7) that $\partial_j P = -R\,\delta_j\,P$. Summing over j, adding to the above and evaluating at ξ_k give

$$\mathcal{D}\,P(\xi_k) = \xi_k\,Q(\xi_k) + Q(\xi_k)\,\tilde{Q}\delta Q + (P(\xi_k) - Q(\xi_k)\,u + [\tau,\,Q(\xi_k)])\,u$$
$$-2\,[\tau,\,R(\xi_k,\,\cdot\,)]\,\delta\,Q - [\tau,\;P(\xi_k) - Q(\xi_k)\,u + [\tau,\,Q(\xi_k)]\,].$$

Hence $\mathcal{D}\,p$ is equal to

$$\xi\,q + q\,\tilde{q}\,q + (p - q\,u + [\tau,\,q])\,u - 2\,[\tau,\,r]\,q - [\tau,\,p - q\,u + [\tau,\,q]\,].$$

Equivalently, in view of (8),

$$\mathcal{D}\,p = \xi\,q + q\,\tilde{q}\,q + \mathcal{D}\,q\cdot u - 2\,[\tau,\,r]\,q - [\tau,\,\mathcal{D}\,q]. \tag{10}$$

Let us compute $\mathcal{D}\,u$. We have

$$u_{ij} = \int\!\!\int \mathrm{Ai}(x)\,\chi_i(x)\,\rho_{ij}(x,y)\,\mathrm{Ai}(y)\,dx\,dy,$$

and so

$$\partial_k\,u_{ij} = -\delta_{ik}\int \mathrm{Ai}(\xi_k)\,\rho_{kj}(\xi_k,y)\,\mathrm{Ai}(y)\,dy$$
$$-\int\!\!\int \mathrm{Ai}(x)\,\chi_i(x)\,[R_{ik}(x,\xi_k)\,\rho_{kj}(\xi_k,y)]\,\mathrm{Ai}(y)\,dx\,dy,$$

where we use (7) again. This is equal to

$$-\delta_{ik}\,\mathrm{Ai}(\xi_k)\,Q_{kj}(\xi_k) - \Big(\tilde{Q}_{ik}(\xi_k) - \delta_{ik}\,\mathrm{Ai}(\xi_k)\Big)\,Q_{kj}(\xi_k),$$

and so

$$\partial_k\,u_{ij} = -\tilde{Q}_{ik}(\xi_k)\,Q_{kj}(\xi_k). \tag{11}$$

This gives

$$\mathcal{D}\,u = -\tilde{q}\,q. \tag{12}$$

Next, we find from (7) and (5) that

$$\mathcal{D}\,R(\xi_j,\,\xi_k) = -Q(\xi_j)\,\tilde{Q}(\xi_k) + [\tau,\,R(\xi_j,\,\xi_k)].$$

This gives $\mathcal{D}\,r = -q\,\tilde{q} + [\tau,\,r]$, which is equation (3).
To get equation (1) we apply $\mathcal{D}$ to (8) and use (10) and (12). We find that

$$\mathcal{D}^2\,q = \xi\,q + q\,\tilde{q}\,q + \mathcal{D}\,q\cdot u - 2\,[\tau,\,r]\,q - [\tau,\,\mathcal{D}\,q] - \mathcal{D}\,q\cdot u + q\,\tilde{q}\,q + [\tau,\,\mathcal{D}\,q]$$
$$= \xi\,q + 2\,q\,\tilde{q}\,q - 2\,[\tau,\,r]\,q,$$

which is (1).
Finally, to get equation (2) we use the fact that $\chi_j(y)\,\rho_{jk}(y,x)$ is equal to $\chi_k(x)$ times $\rho'_{kj}(x,y)$, where ρ' is the resolvent kernel for the matrix kernel with i,j entry $L_{ji}(x,y)\,\chi_j(y)$. Hence $\tilde{Q}_{jk}(x)$ is equal to $\chi_k(x)$ times the $Q_{kj}(x)$ associated with L_{ji}. Consequently for all the differentiation formulas we have for the $Q_{kj}(\xi_k)$, etc., there are analogous formulas for the $\tilde{Q}_{jk}(\xi_k)$, etc.. The difference is that we have to reverse subscripts and replace r by r^t and τ by $-\tau$. The upshot is that, by computations analogous to those used to derive (1), we derive another equation which can be obtained from (1) by making the replacements $q \to \tilde{q}^t$, $\tilde{q} \to q^t$, $r \to r^t$, $\tau \to -\tau$ and then taking transposes. The result is equation (2).

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