

A LIMIT THEOREM FOR SHIFTED SCHUR MEASURES

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Abstract

To each partition $\lambda = (\lambda_1, \lambda_2, \dots)$ with distinct parts we assign the probability $Q_\lambda(x)P_\lambda(y)/Z$, where Q_λ and P_λ are the Schur Q -functions and Z is a normalization constant. This measure, which we call the shifted Schur measure, is analogous to the much-studied Schur measure. For the specialization of the first m coordinates of x and the first n coordinates of y equal to α ($0 < \alpha < 1$) and the rest equal to zero, we derive a limit law for λ_1 as $m, n \rightarrow \infty$ with $\tau = m/n$ fixed. For the Schur measure, the α -specialization limit law was derived by Johansson [J1]. Our main result implies that the two limit laws are identical.

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1. Introduction

The Schur measure (see [O2]) assigns to each partition $\lambda = (\lambda_1, \lambda_2, \dots)$ the weight

$$s_\lambda(x) s_\lambda(y),$$

where s_λ are the Schur functions (see, e.g., [M], [St]). Thus

$$\sum_{\substack{\lambda \in \mathcal{P} \\ \lambda_1 \leq h}} s_\lambda(x) s_\lambda(y) \tag{1.1}$$

is the (unnormalized) probability that λ_1 , the number of boxes in the first row of the associated Young diagram, is less than or equal to h . Here \mathcal{P} denotes the set of all partitions. The normalization constant, Z , is determined from the $h \rightarrow \infty$ limit

$$Z := \sum_{\lambda \in \mathcal{P}} s_\lambda(x) s_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}, \tag{1.2}$$

where the last equality is the Cauchy identity for Schur functions. A theorem of Gessel [Ge] expresses the partition sum (1.1) as an $h \times h$ Toeplitz determinant $D_h(\varphi)$. It follows from this that the normalization constant is also given by

$$Z = \lim_{h \rightarrow \infty} D_h(\varphi).$$

This limit can be explicitly computed by an application of the strong Szegő limit theorem (see, e.g., [BS]), and thereby the Cauchy identity reappears.

The Toeplitz determinant, or the Fredholm determinant coming from the Borodin-Okounkov identity (see [BO], [BW]), is the starting point in the analysis of limit laws for λ_1 . This analysis together with the Robinson-Schensted-Knuth (RSK) correspondence gives a new class of limit laws, first discovered in the context of random matrix theory (see [TW1], [TW2]), for a number of probability models. Indeed, the result of Baik, Deift, and Johansson [BDJ] for the limit law of the length, $\ell_N(\pi)$, of the longest increasing subsequence in a random permutation $\pi \in \mathcal{S}_N$ is the now-classic example. Exponential specialization of the Gessel identity together with the RSK correspondence* shows that

$$\sum_{N=0}^{\infty} \mathbb{P}(\ell_N \leq h) \frac{t^N}{N!}$$

is an $h \times h$ Toeplitz determinant with symbol $\varphi(z) = e^{\sqrt{t}(z+1/z)}$. An asymptotic

*RSK associates bijectively to each permutation π a pair of standard Young tableaux (P, Q) of the same shape λ such that $\ell_N(\pi) = \lambda_1$ (see, e.g., [S1]).

analysis of this Toeplitz determinant (using the steepest descent method for Riemann-Hilbert problems; see [DZ]) shows that

$$\lim_{n \rightarrow \infty} P\left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} < s\right) = F_2(s),$$

where F_2 is the limiting distribution of the largest eigenvalue (suitably centered and normalized) in the Gaussian unitary ensemble (see [TW1]). Similar results hold for longest increasing subsequences in symmetrized random permutations (see [BR3], [BR2]) and random words (see [AM], [J2], [ITW], [TW3]), for height fluctuations in various growth models (see [BR1], [GTW1], [GTW2], [J1]), and for tiling problems (see [J3]), as well as for extensions to the other rows of the Young diagram (see [BOO], [J2], [O1]).

In the theory of symmetric functions, there are many important generalizations of Schur functions (see [M]). These generalizations satisfy Cauchy identities, and it is natural to inquire about more general Gessel identities. However, one quickly sees that without determinantal formulas of the type that exist for Schur functions (the Jacobi-Trudi identity), Gessel identities seem unlikely. Nevertheless, the question of possible limit laws for sums of type (1.1) remains interesting.

This paper initiates work in this direction. Instead of Schur functions, we work with Schur Q -functions that have Pfaffian representations. These functions, introduced by Schur in 1911 in his analysis of the projective representation of the symmetric group, now have a combinatorial theory that parallels the combinatorial theory of Schur functions. This theory, due to Sagan [S] and Worley [Wo] (see also [Ste], [HH]), is based on a shifted version of the RSK algorithm. Whereas the usual RSK algorithm associates bijectively to each \mathbb{N} -matrix A a pair of semistandard Young tableaux, the shifted RSK algorithm associates bijectively to each \mathbb{P} -matrix* A a pair of shifted Young tableaux. There is a notion of increasing paths, and the length of the maximal path, $L(A)$, equals the number of boxes in the first row of the shifted tableau. Thus it is natural to assign to each partition λ into distinct parts, that is, a *strict partition*, the probability

$$P(\{\lambda\}) = \frac{1}{Z} Q_\lambda(x) P_\lambda(y), \tag{1.3}$$

where Q_λ and P_λ are the Schur Q -functions and Z is a normalization constant. We call this measure the *shifted Schur measure*.

At first our analysis is for general parameters x and y appearing in the shifted Schur measure, and we find that there is indeed a Gessel identity. (It follows from the Ishikawa-Wakayama Pfaffian summation formula [IW].) Then we specialize the

*Informally, a \mathbb{P} -matrix is an \mathbb{N} -matrix where we allow the nonzero entries to be either marked or unmarked. Precise definitions are given below.

measure by choosing the first m coordinates of x and the first n coordinates of y equal to α ($0 < \alpha < 1$) and the rest equal to zero. We call this α -specialization and denote the resulting specialized shifted Schur measure by P_σ , where $\sigma = (m, n, \alpha)$ denotes the parameters of the measure. Now, however, the matrix on the right side of the Gessel identity is not Toeplitz, and so the earlier analytical methods are not immediately available to us. Nevertheless, we do find that the distribution function for $L(A) = \lambda_1$ can be expressed in terms of the Fredholm determinant of an operator which is a perturbation of a direct sum of products of Hankel operators. In the end we can show that the trace norms of the perturbations tend to zero and are able to determine the asymptotics.

We assume* that $m/n = \tau$ is a constant satisfying $\alpha^2 < \tau < \alpha^{-2}$. Our main result is the following.

MAIN THEOREM

Let P_σ denote the α -specialized shifted Schur measure with τ satisfying the stated restriction. Then there exist constants $c_1 = c_1(\alpha, \tau)$ and $c_2 = c_2(\alpha, \tau)$ such that

$$\lim_{n \rightarrow \infty} P_\sigma \left(\frac{\lambda_1 - c_1 n}{c_2 n^{1/3}} < s \right) = F_2(s).$$

For $\tau = 1$, the constants have a particularly simple form, namely,

$$c_1(\alpha, 1) = \frac{4\alpha}{1 - \alpha^2} \quad \text{and} \quad c_2(\alpha, 1) = \frac{(2\alpha(1 + 6\alpha^2 + \alpha^4))^{1/3}}{1 - \alpha^2}.$$

Expressions for c_1 and c_2 in general are given in §6. For the Schur measure, the corresponding α -specialization limit law was derived by Johansson [J1]; as the main theorem shows, the two limit laws are identical.† The table of contents provides a description of the layout of this paper.

2. Schur Q -functions

This section and §3 summarize the properties of the Schur Q -functions and the shifted RSK algorithm which we need in this paper. The material is not new on our part. It is presented to establish the notation used in subsequent sections and as a convenience to the reader. A complete presentation can be found in the books by Macdonald [M] and Hoffman and Humphreys [HH], in the papers by Sagan [S] and Stembridge [Ste], and in the dissertation of Worley [Wo].

*The stated restriction on τ is very likely unnecessary for the validity of the final result. Some details of the proof would be different in the other cases, but we did not carry them out.

†We note that our α is related to Johansson's q by $q = \alpha^2$. For the α -specialized Schur measure, $c_1(\alpha, 1) = 2\alpha/(1 - \alpha)$ and $c_2(\alpha, 1) = \alpha^{1/3}(1 + \alpha)^{1/3}/(1 - \alpha)$.

If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of n , we denote this by $\lambda \vdash n$. The length of λ is denoted by $\ell(\lambda)$. Let \mathcal{P}_n denote the set of all partitions of n , and let $\mathcal{P} := \bigcup_{n=0}^\infty \mathcal{P}_n$. (\mathcal{P}_0 is the empty partition.) Introduce $\mathcal{D}_n \subset \mathcal{P}_n$, the set of partitions of n into *distinct parts*. For example,

$$\mathcal{D}_6 = \{\{6\}, \{5, 1\}, \{4, 2\}, \{3, 2, 1\}\}.$$

Let $\mathcal{D} := \bigcup_{n=0}^\infty \mathcal{D}_n$, the set of all partitions into distinct parts. If $\lambda \vdash n$ is a partition with distinct parts, we denote this by $\lambda \models n$ and call λ a *strict* partition of n .

Associated to a strict partition λ is a *shifted shape* $S(\lambda)$. One starts with the usual Young diagram $Y(\lambda)$ and for $i = 1, 2, \dots, \ell(\lambda)$ simply indents the i th row to the right by $i - 1$ boxes. The result is $S(\lambda)$. We usually use $\lambda \in \mathcal{D}$ to denote both a strict partition and the shifted shape $S(\lambda)$.

We let \mathbb{N} denote the set of positive integers, and

$$\mathbb{P} = \{1', 1, 2', 2, 3', 3, \dots\}$$

with the ordering

$$1' < 1 < 2' < 2 < 3' < 3 < \dots$$

We call the elements either *marked* or *unmarked*, depending on whether the element is primed or not. When we do not wish to distinguish a marked element m' from the unmarked element m , we write m^* . A (*generalized*) *shifted Young tableau*, T , is an assignment of elements of \mathbb{P} to a shifted shape λ having the following properties.

- (T1) T is weakly increasing across rows and down columns.
- (T2) For each integer m^* , there is at most one m' in each row and at most one m in each column of T . (Thus the marked elements are strictly increasing across rows of T and the unmarked elements are strictly increasing down columns of T .)

An example of a shifted tableau of shape $(7, 5, 3, 2, 1)$ is

$$\begin{array}{cccccc} 1' & 1 & 2' & 2 & 2 & 5' & 6 \\ & & 2' & 2 & 3' & 4 & 5 \\ & & & & 3' & 4 & 5 \\ & & & & & 6 & 7' \\ & & & & & & 7'. \end{array}$$

To each shifted tableau T we associate a monomial

$$x^T = x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} \dots,$$

where a_m is the number of times m^* appears in T . Thus as far as the monomial is concerned, we do not distinguish between marked and unmarked elements. In the above example,

$$x^T = x_1^2 x_2^5 x_3^2 x_4^2 x_5^3 x_6^2 x_7^2.$$

Let λ be a strict partition of n . The Schur Q -function, the generating function of shifted tableaux of shape λ , is

$$Q_\lambda(x) := \sum_T x^T, \tag{2.1}$$

where the sum runs over all shifted tableaux of shape $\lambda \models n$. The Schur Q -function is the analogue of the Schur function s_λ when one replaces semistandard Young tableaux of shape λ by shifted tableaux of shape λ . (Of course, here λ must be a strict partition.) It is convenient to introduce the Schur P -function

$$P_\lambda(x) = \frac{1}{2^{\ell(\lambda)}} Q_\lambda(x).$$

We remark that a shifted tableau T of shifted shape $\lambda \models n$ is called *standard* if it has no marked elements and uses each unmarked letter $1, 2, \dots, n$ exactly once. Schur showed that the number of standard shifted tableaux of shape $\lambda \models n$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, is

$$f_s^\lambda = \frac{n!}{\lambda_1! \lambda_2! \dots \lambda_\ell!} \prod_{1 \leq i < j \leq \ell} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}. \tag{2.2}$$

This should be compared with the number of standard Young tableaux

$$f^\lambda = \frac{n!}{\ell_1! \ell_2! \dots \ell_k!} \prod_{1 \leq i < j \leq k} (\ell_i - \ell_j),$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\ell_j = \lambda_j + k - j$. The number of semistandard Young tableaux of shape λ which can be formed using the integers $1, 2, \dots, n$ is

$$d^\lambda(n) = s_\lambda(\overbrace{1, \dots, 1}^n, 0, 0, \dots).$$

Similarly, the number of shifted tableaux of shape λ which can be formed using the integers $1^*, 2^*, \dots, n^*$ is

$$d_s^\lambda(n) = Q_\lambda(\overbrace{1, \dots, 1}^n, 0, 0, \dots).$$

This specialization of Q_λ is important below.

The Schur Q -functions satisfy a Cauchy identity

$$\sum_{\lambda \in \mathcal{D}} Q_\lambda(x) P_\lambda(y) = \prod_{i,j=1}^\infty \frac{1 + x_i y_j}{1 - x_i y_j} = Z. \tag{2.3}$$

The right-hand side enumerates all matrices A whose entries are chosen from $\mathbb{P} \cup \{0\}$: the denominator counts matrices with entries in $\mathbb{N} \cup \{0\}$, while the numerator accounts for the primes. We call these matrices \mathbb{P} -matrices. The above product is frequently specialized to $x = (x_1, x_2, \dots, x_m, 0, \dots)$ and $y = (y_1, y_2, \dots, y_n, 0, \dots)$. We use the same symbol Z to denote this specialization. It is clear from the context how to interpret Z .

Define symmetric functions q_k by

$$Q(t) := \prod_{i=1}^{\infty} \frac{1 + tx_i}{1 - tx_i} = \sum_{k=0}^{\infty} q_k(x)t^k. \tag{2.4}$$

(When necessary to indicate the dependence upon x , we write $Q(t, x)$.) It follows from $Q(t)Q(-t) = 1$ that

$$q_{2m} = \sum_{r=1}^{m-1} (-1)^{r-1} q_r q_{2m-r} + \frac{1}{2} (-1)^m q_m^2,$$

which shows that $q_{2m} \in \mathbb{Q}[q_1, q_2, \dots, q_{2m-1}]$ and hence by induction on m ,

$$q_{2m} \in \mathbb{Q}[q_1, q_3, q_5, \dots, q_{2m-1}].$$

Denote by Γ the subring of Λ generated by the q_r ,

$$\Gamma = \mathbb{Z}[q_1, q_3, \dots].$$

If $\lambda = (\lambda_1, \lambda_2, \dots)$, we let

$$q_\lambda := q_{\lambda_1} q_{\lambda_2} \dots$$

It is known that the q_λ with λ strict form a \mathbb{Z} -basis of Γ .

We now give the classical definition of the Schur Q -function. (Of course, in this presentation it is a theorem.) If λ is a strict partition of length at most n , then Q_λ equals the coefficient of $t^\lambda := t_1^{\lambda_1} t_2^{\lambda_2} \dots$ in

$$Q(t_1, t_2, \dots, t_n) = \prod_{i=1}^n Q(t_i) \prod_{i < j} F(t_i^{-1} t_j),$$

where

$$F(y) = \frac{1 - y}{1 + y} = 1 + 2 \sum_{r \geq 1} (-1)^r y^r$$

and Q is defined by (2.4); in particular, for $r > s$,

$$Q_{(r,s)} = \left(\frac{1}{2\pi i}\right)^2 \iint t_1^{-r-1} t_2^{-s-1} F(t_2/t_1) Q(t_1) Q(t_2) dt_1 dt_2, \tag{2.5}$$

where the contours could be chosen to be circles with $|t_2| < |t_1|$.*

Here are some additional properties of Schur Q -functions.

- (1) The Q_λ, λ strict, form a \mathbb{Z} -basis of Γ .
- (2) Using (2.5) and the expansion for F , we have for $r > s \geq 0$,

$$Q_{(r,s)} = q_r q_s + 2 \sum_{i=1}^s (-1)^i q_{r+i} q_{s-i}. \tag{2.6}$$

For $r \leq s$, we define $Q_{(r,s)} = -Q_{(s,r)}$. Now let λ be a strict partition that we write in the form $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2n})$, where $\lambda_1 > \lambda_2 > \dots > \lambda_{2n} \geq 0$. Define the $2n \times 2n$ antisymmetric matrix

$$M_\lambda = (Q_{(\lambda_i, \lambda_j)});$$

then we have

$$Q_\lambda = \text{pf}(M_\lambda), \tag{2.7}$$

where pf denotes the Pfaffian.

3. Shifted RSK algorithm

For later convenience, we use a nonstandard labeling of matrix A : rows are numbered starting at the lower left-hand corner of A , and columns have the usual left-to-right labeling. To each \mathbb{P} -matrix A we (bijectively) associate a biword w_A as follows. For a fixed column index we scan the matrix for increasing values of the row index. If the (i, j) -entry is unmarked with value a_{ij} , we repeat the pair $\binom{j}{i} a_{ij}$ times in w_A . If the (i, j) -element is marked, the i of the first pair $\binom{j}{i}$ appearing in w_A is marked. For example, if

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1' & 0 & 2' \\ 3' & 0 & 1 \end{pmatrix},$$

then

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ 1' & 1 & 1 & 2' & 3 & 3 & 3 & 1 & 2' & 2 \end{pmatrix}.$$

A description of the shifted RSK algorithm is more involved than that of the usual RSK algorithm, though the general features remain the same. Namely, there is a row bumping (and column bumping) procedure that, when iterated on a sequence

*This requires that $x \in \ell^1$ and that the poles x_i^{-1} lie outside the contours. Notice that if the t_1 -contour were deformed to one inside the t_2 -contour, then since $Q(t)Q(-t) = 1$, the residue at the pole $t_1 = -t_2$ crossed would be $2t_2^{-r-s-1}$. The integral of this equals zero as long as r and s are not both zero. This shows that the contours can also be chosen so that $|t_2| > |t_1|$. Equivalently, the integral representation holds for $r < s$ as well.

α whose elements are in \mathbb{P} , gives a shifted tableau S , the insertion tableau. (This is applied to the sequence in the bottom half of the biword w_A .) The top half of w_A gives a recording tableau T . We now state the final result, referring the interested reader to either [HH] or [S].

THEOREM (Sagan [S], Worley [Wo])

There is a bijective correspondence between \mathbb{P} -matrices $A = (a_{ij})$ and ordered pairs (S, T) of shifted tableaux of the same shape, such that T has no marked letters on its main diagonal. The correspondence has the property that $\sum_i a_{ij}^$ is the number of entries t of T for which $t^* = j$, whereas $\sum_j a_{ij}^*$ is the number of entries s of S for which $s^* = i$. (Recall that the $*$ means we do not distinguish between a marked or unmarked form of an integer.)*

We designate the matrix S , respectively, T , to be of type s , respectively, t .

An important property of the RSK algorithm is its relationship to increasing subsequences of maximal length in the biword w_A (equivalently, increasing paths in the matrix A of maximal weight). The shifted RSK algorithm of Sagan and Worley shares a similar property once the notion of an increasing subsequence is properly formulated. Let $\psi \in \mathbb{P}^m$, $\phi \in \mathbb{P}^n$, and denote by $\psi \sqcup \phi$ the concatenation $(\psi_1, \dots, \psi_m, \phi_1, \dots, \phi_n)$. Denote by $\text{rev}(\phi)$ the reverse of ϕ , $\text{rev}(\phi) = (\phi_n, \phi_{n-1}, \dots, \phi_1)$. Given a sequence α from \mathbb{P} , an *ascent pair* (ψ, ϕ) for α is a pair of subsequences ψ of $\text{rev}(\alpha)$ and ϕ of α such that if $\psi \in \mathbb{P}^m$ and $\phi \in \mathbb{P}^n$, then

- (1) $\psi \sqcup \phi$ is weakly increasing with respect to the ordering of \mathbb{P} ;
- (2) for all $k \in \mathbb{N}$, at most one (unmarked) k appears in ψ ;
- (3) for all $k \in \mathbb{N}$, at most one (marked) k' appears in ϕ .

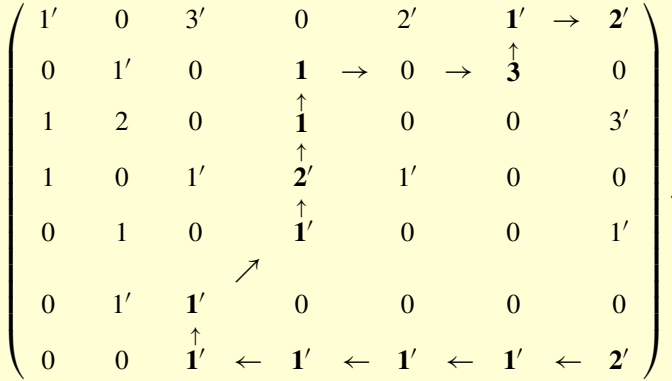
Thus the unmarked symbols are strictly increasing in ψ and the marked symbols are strictly increasing in ϕ . The length of $\psi \sqcup \phi$ is defined to be $m + n - 1$. (Note that the length here is one less than the length defined in either [HH] or [S].) Let $\alpha \in \mathbb{P}^n$, and let $L(\alpha)$ denote the length of the longest ascent pair (ψ, ϕ) of α . Then we have the following.

THEOREM (Sagan [S], Worley [Wo])

If α is a sequence from \mathbb{P} and T is the shifted tableau of shape $(\lambda_1, \lambda_2, \dots, \lambda_k)$ resulting from the insertion of α (following the rules of the shifted RSK algorithm), then $L(\alpha) = \lambda_1$.

Here is an example of an increasing path displayed in the \mathbb{P} -matrix A . In this example,

$L(A) = 16$.*



A quick way to compute this length is to apply a modified patience sorting algorithm (see [AD]) to the lower row of the associated biword. (We leave this as an exercise for the interested reader.) Of course, another way is to apply the shifted RSK algorithm and count the number of boxes in the first row.

4. A Gessel identity

The Gessel identity [Ge] (see also [TW3]) states that the sum

$$\sum_{\substack{\lambda \in \mathcal{D} \\ \lambda_1 \leq h}} s_\lambda(x) s_\lambda(y)$$

equals a certain $h \times h$ Toeplitz determinant. The proof begins by expressing the Schur functions s_λ as determinants (the Jacobi-Trudi identity) and proceeds by recognizing this sum of products of determinants as the expansion of a single determinant of the product of two (nonsquare) matrices. (This expansion is called the Cauchy-Binet expansion.)

We are interested in sums of the form

$$\sum_{\substack{\lambda \in \mathcal{D} \\ \lambda_1 \leq h}} Q_\lambda(x) P_\lambda(y), \tag{4.1}$$

where now, as we have seen, the Q_λ and P_λ are given by Pfaffians. What is needed is a Pfaffian version of the Cauchy-Binet formula. Fortunately, Ishikawa and Wakayama [IW] have such a formula (see also Stembridge [Ste]).

Introduce

$$I_r^h = \{I = (i_1, \dots, i_r) : 1 \leq i_1 < \dots < i_r \leq h\},$$

*Note, for example, that the path segment $1' \rightarrow 2'$ in the upper right-hand corner contributes a weight of two, not three, since the marked elements are strictly increasing.

and denote by A_J the submatrix formed from A by taking those rows and columns indexed by $J \in I_r^h$. Then the Pfaffian summation formula is the following.

THEOREM (Ishikawa and Wakayama [IW])

Let $A = (a_{ij})_{0 \leq i, j \leq h}$ and $B = (b_{ij})_{0 \leq i, j \leq h}$ be $(h + 1) \times (h + 1)$ skew-symmetric matrices, $h \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{\substack{0 \leq r \leq h \\ r \text{ even}}} \sum_{I \in I_r^h} \gamma^{|I|} \text{pf}(A_I) \text{pf}(B_I) + \sum_{\substack{0 \leq r \leq h \\ r \text{ odd}}} \sum_{I \in I_r^h} \gamma^{|I|} \text{pf}(A_{0I}) \text{pf}(B_{0I}) \\ = (-1)^{h(h-1)/2} \text{pf} \begin{pmatrix} -A & I_{h+1} \\ -I_{h+1} & C \end{pmatrix}, \end{aligned}$$

where $C = (C_{ij})_{0 \leq i, j \leq h}$ is the $(h + 1) \times (h + 1)$ antisymmetric matrix

$$C_{ij} = \begin{cases} \gamma^j b_{0j} & \text{if } i = 0, j \geq 1, \\ \gamma^i b_{i0} & \text{if } i \geq 1, j = 0, \\ \gamma^{i+j} b_{ij} & \text{if } i \geq 1, j \geq 1. \end{cases}$$

Here $|I| = \sum i_k$, and A_{0I} , where $I = \{i_1, i_2, \dots\}$, stands for A_J , where $J = \{0, i_1, i_2, \dots\}$.

Observe that I_r^h is the set of partitions with exactly r distinct parts such that the largest part is at most h . For any such partition, $r \leq h$. In (4.1) we break the sum into two sums—the first sum over distinct partitions with an even number of parts and the second sum over distinct partitions with an odd number of parts. Recalling the Pfaffian representation (2.7) of Q_λ , we note that if λ has an odd number of parts, then we extend the partition by appending zero, giving us a vector of even length. Thus the sum appearing in the Pfaffian summation formula is (up to a reversal of labels) the sum over distinct partitions satisfying $\lambda_1 \leq h$. From (2.7), we see that $A_h(x)$ is the $(h + 1) \times (h + 1)$ antisymmetric matrix

$$\begin{pmatrix} 0 & -q^t \\ q & \widehat{Q}_h(x) \end{pmatrix}.$$

Here q is the $(h \times 1)$ -matrix with elements $q_r(x)$ ($r = 1, 2, \dots, h$), $\widehat{Q}_h(x)$ is the $h \times h$ antisymmetric matrix with elements $Q_{(r,s)}(x)$, and q^t denotes the transpose of q . (Recall that $Q_{(r,0)} = q_r$.) The Pfaffian representation for P_λ is obtained from the

Q_λ Pfaffian representation by inserting the factor $2^{-\ell(\lambda)}$. The matrix B_h is

$$\begin{aligned} B_h(y) &= \begin{pmatrix} 0 & -\frac{1}{2} q^t \\ \frac{1}{2} q & \frac{1}{4} \widehat{Q}_h(y) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} A_h(y) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

Applying the summation formula then gives for $h \in \mathbb{N}$,

$$\begin{aligned} \sum_{\substack{\lambda \in \mathcal{D} \\ \lambda_1 \leq h}} Q_\lambda(x) P_\lambda(y) &= \text{pf} \begin{pmatrix} -A_h(x) & I \\ -I & B_h(y) \end{pmatrix} \\ &= (\det(I - A_h(x)B_h(y)))^{1/2}. \end{aligned}$$

The \pm -factors are accounted for by the reversal of labels or, more simply, because we are computing probabilities.

Introducing the antisymmetric matrix

$$K_h(x) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} A_h(x) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

or, more explicitly,

$$K_h(x)_{rs} = \begin{cases} -\frac{1}{\sqrt{2}} q_s(x), & r = 0, s \geq 1, \\ \frac{1}{\sqrt{2}} q_r(x), & r \geq 1, s = 0, \\ \frac{1}{2} Q_{(r,s)}(x), & r \geq 1, s \geq 1, \end{cases} \tag{4.2}$$

we obtain our Gessel identity,

$$\sum_{\substack{\lambda \in \mathcal{D} \\ \lambda_1 \leq h}} Q_\lambda(x) P_\lambda(y) = (\det(I - K_h(x)K_h(y)))^{1/2}. \tag{4.3}$$

Observe that it follows from (4.3) and the Cauchy identity (2.3) that

$$\lim_{h \rightarrow \infty} \det(I - K_h(x)K_h(y)) = Z^2. \tag{4.4}$$

5. Shifted Schur measure

Let $\mathcal{P}_{m,n}$ denote the set of \mathbb{P} -matrices of size $m \times n$. For $A \in \mathcal{P}_{m,n}$, we recall that $L(A)$ denotes the length of the longest increasing path in A . Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ with $0 \leq x_i < 1$ and $0 \leq y_i < 1$. We assume that the matrix elements a_{ij} are distributed independently with a geometric distribution with parameter $x_i y_j$. Specifically, for $k \geq 1$,

$$P(a_{ij} = k) = P(a_{ij} = k') = \left(\frac{1 - x_i y_j}{1 + x_i y_j} \right) (x_i y_j)^k$$

and

$$P(a_{ij} = 0) = \frac{1 - x_i y_j}{1 + x_i y_j}.$$

We have, of course,

$$\sum_{k^*=0}^{\infty} P(a_{ij} = k^*) = \frac{1 - x_i y_j}{1 + x_i y_j} + 2 \sum_{k \geq 1} \left(\frac{1 - x_i y_j}{1 + x_i y_j} \right) (x_i y_j)^k = 1.$$

Let $\mathcal{P}_{m,n,s,t}$ ($t \in \mathbb{N}^m, s \in \mathbb{N}^n$) denote the set of $A \in \mathbb{P}_{m,n}$ satisfying, for $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\sum_{1 \leq j \leq n} a_{ij}^* = s_i \quad \text{and} \quad \sum_{1 \leq i \leq m} a_{ij}^* = t_j.$$

Then for $A \in \mathcal{P}_{m,n,s,t}$ we have

$$P(\{A\}) = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left(\frac{1 - x_i y_j}{1 + x_i y_j} \right) x^s y^t = \frac{1}{Z} x^s y^t.$$

Since the right-hand side does not depend upon the A chosen in $\mathcal{P}_{m,n,s,t}$, the conditional probability

$$P\left(L \leq h \mid \sum_j a_{ij}^* = s_i, \sum_i a_{ij}^* = t_j\right)$$

is uniform. Note that this uses both the independence and the geometric distribution of the random variables a_{ij} .

By the shifted RSK correspondence, to each $A \in \mathcal{P}_{m,n,s,t}$ we associate bijectively a pair (S, T) of shifted tableaux of the same shape $\lambda \models N$ ($N := \sum_{i,j} a_{ij}^*$) of types s and t , respectively. The condition $L(A) \leq h$ becomes $\lambda_1 \leq h$. Hence

$$\begin{aligned} P_{m,n}(L \leq h) &= \sum_{A \in \mathcal{P}_{m,n} / \mathcal{P}_{m,n,s,t}} P(\ell(A) \leq h \mid A \in \mathcal{P}_{m,n,s,t}) P(A \in \mathcal{P}_{m,n,s,t}) \\ &= \sum_{A \in \mathcal{P}_{m,n} / \mathcal{P}_{m,n,s,t}} \frac{1}{|\mathcal{P}_{m,n,s,t}|} \frac{1}{Z} x^s y^t |_{\mathcal{P}_{m,n,s,t}} \\ &= \frac{1}{Z} \sum_{N \geq 0} \sum_{\substack{\lambda \models N \\ \lambda_1 \leq h}} Q_\lambda(x) P_\lambda(y). \end{aligned}$$

(Here $P_{m,n}$ denotes probability before α -specialization.) Thus, by (4.3),

$$P_{m,n}(L \leq h) = \frac{1}{Z} (\det(I - K_h(x)K_h(y)))^{1/2}. \tag{5.1}$$

The above uses the combinatorial definition (2.1) of the Schur Q -function. The reason for the occurrence of $P_\lambda(y)$ (instead of $Q_\lambda(y)$) is that the recording tableau T has no

marked elements on the diagonal, which accounts for the factor $2^{-\ell(\lambda)}$. (There are exactly $2^{\ell(\lambda)}$ entries on the main diagonal in a marked shifted tableau of shape λ , and so there are $2^{\ell(\lambda)}$ ways to mark and unmark the diagonal elements.)

Observe the consequence that the distribution function $P_{m,n}(L \leq h)$ is a symmetric function of $x = (x_1, \dots, x_m)$ and of $y = (y_1, \dots, y_n)$.

6. Proof of the main theorem

6.1. An operator formulation

We begin by deriving an alternative representation for $\det(I - K_h(x)K_h(y))$ in terms of Toeplitz and Hankel operators on the Hilbert space $\ell^2(\mathbb{Z}_+)$ ($\mathbb{Z}_+ := \mathbb{N} \cup 0$). These may well be of independent interest in the theory of Schur Q -functions. We assume at first only that $x, y \in \ell^1$ together with uniform estimates $x_j, y_j \leq c < 1$

To set notation, we let $\{e_j\}_{j \geq 0}$ denote the canonical basis of $\ell^2(\mathbb{Z}_+)$. Since the vector e_0 occurs frequently, we denote e_0 by e , and it is convenient to set $e_{-1} = 0$. The backward shift operator Λ is characterized by

$$\Lambda e_j = e_{j-1},$$

and its adjoint Λ^* is the forward shift operator. The two satisfy

$$\Lambda \Lambda^* = I \quad \text{and} \quad \Lambda^* \Lambda = I - e \otimes e, \tag{6.1}$$

where for vectors u and v we denote by $u \otimes v$ the operator sending a vector f to $u(v, f)$.

Suppressing temporarily the parameters x and y , we define L to be the matrix with entries

$$L_{jk} = \left(\frac{1}{2\pi i}\right)^2 \iint \frac{Q(z) Q(\zeta)}{z^{j+1} \zeta^{k+1}} \frac{dz d\zeta}{z + \zeta}, \tag{6.2}$$

where Q is defined as in (2.4). Here the contours can be taken to be concentric circles of different radii near the unit circle. Since $Q(z) Q(-z) = 1$, the residue at $\zeta = -z$ in the integral defining L is zero, so we may freely choose whether the z -contour lies inside or outside the ζ -contour without affecting the value of the integral. (Equivalently, L is symmetric.)

From (2.5), we see that

$$Q_{(j,k)} = L_{j-1,k} - L_{j,k-1},$$

where we set $L_{-1,k} = 0$. Note that $L_{j-1,0} = q_j$. The matrix elements K_{jk} given in

(4.2) are then

$$K_{jk} = \begin{cases} -\frac{1}{\sqrt{2}} L_{0,k-1}, & j = 0, k \neq 0, \\ \frac{1}{\sqrt{2}} L_{j-1,0}, & j \neq 0, k = 0, \\ \frac{1}{2} (L_{j-1,k} - L_{j,k-1}), & j, k > 0. \end{cases}$$

By introducing the vector $q = q(x) = (q_0(x), q_1(x), \dots)$, the operator K can be written

$$K = \frac{1}{2} (\Lambda^* L - L \Lambda) + \frac{1}{2} \omega (e \otimes q - q \otimes e), \tag{6.3}$$

where $\omega = 1 - \sqrt{2}$.

The operator L is expressible in terms of Toeplitz and Hankel matrices acting on $\ell^2(\mathbb{Z}_+)$. Recall that $T(\psi)$, the Toeplitz matrix with symbol ψ , has (j, k) -entry ψ_{j-k} (subscripts denote Fourier coefficients here), while the Hankel matrix $H(\psi)$ has (j, k) -entry ψ_{j+k+1} . If we assume that the contours in (6.2) are chosen so that $|\zeta| < |z|$ and expand $(z + \zeta)^{-1}$ in powers of ζ/z , we obtain

$$L_{ij} = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2\pi i}\right)^2 \iint z^{-i-k-2} \zeta^{k-j-1} Q(z) Q(\zeta) dz d\zeta.$$

Now make the substitution $\zeta \rightarrow -\zeta^{-1}$ to obtain

$$(-1)^j \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i}\right)^2 \iint z^{-i-k-2} \zeta^{-k+j-1} \frac{Q(z)}{Q(\zeta)} dz d\zeta,$$

where $\tilde{Q}(\zeta) = Q(\zeta^{-1})$. The z -integral gives Q_{i+k+1} , while the ζ -integral gives $(\tilde{Q}^{-1})_{k-j}$. It follows that

$$L = H(Q) T(\tilde{Q}^{-1}) J, \tag{6.4}$$

where J is the diagonal matrix with diagonal entries $(-1)^j$. If in the last integrals we make the substitutions $z \rightarrow -z$, $\zeta \rightarrow -\zeta$, we find that also

$$L = -J H(Q^{-1}) T(\tilde{Q}). \tag{6.5}$$

If we reintroduce our parameters x and y , which we now write for notational convenience as subscripts, and use the two representations of L , we see that

$$\begin{aligned} \det(I + L_x L_y) &= \det(I - H(Q_x) T(\tilde{Q}_x^{-1}) H(Q_y^{-1}) T(\tilde{Q}_y)) \\ &= \det(I - T(\tilde{Q}_y) H(Q_x) T(\tilde{Q}_x^{-1}) H(Q_y^{-1})). \end{aligned}$$

Here we have used the general identity $\det(I - AB) = \det(I - BA)$, valid if one of the operators is trace class and the other bounded, and the fact that the Hankel

operators are Hilbert-Schmidt under our assumptions on x and y . Another general fact is

$$T(\psi_1) H(\psi_2) + H(\psi_1) T(\tilde{\psi}_2) = H(\psi_1 \psi_2). \tag{6.6}$$

In particular, if ψ_1 is a minus function (Fourier coefficients with positive index all vanish), then $T(\psi_1) H(\psi_2) = H(\psi_1 \psi_2)$. From this we find that

$$T(\tilde{Q}_y) H(Q_x) = H(Q_x \tilde{Q}_y) \quad \text{and} \quad T(\tilde{Q}_x^{-1}) H(Q_y^{-1}) = H(\tilde{Q}_x^{-1} Q_y^{-1}),$$

so the product of these equals $H(\phi) H(\tilde{\phi}^{-1})$, where

$$\phi(z) := Q_x(z) \tilde{Q}_y(z). \tag{6.7}$$

Since yet another general identity is

$$T(\psi_1) T(\psi_2) = T(\psi_1 \psi_2) - H(\psi_1) H(\tilde{\psi}_2), \tag{6.8}$$

we have $I - H(\phi) H(\tilde{\phi}^{-1}) = T(\phi) T(\phi^{-1})$, and we have shown that

$$\det(I + L_x L_y) = \det T(\phi) T(\phi^{-1}). \tag{6.9}$$

If a symbol ϕ has geometric mean 1 and is sufficiently well behaved, then the strong Szegő limit theorem says that

$$\lim_{h \rightarrow \infty} \det T_h(\phi) = E(\phi) := \exp \left(\sum_{n=1}^{\infty} n (\log \phi)_n (\log \phi)_{-n} \right),$$

where $T_h(\phi) = (\phi_{j-k})_{j,k=0,\dots,h-1}$. In the case of our symbol given by (6.7), we find that

$$E(\phi) = \left(\prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j} \right)^2 = Z^2,$$

where Z is as in the right side of (2.3). But there is another formula for $E(\phi)$, namely (see [W]),

$$E(\phi) = \det T(\phi) T(\phi^{-1}),$$

and so from (6.9) we have the identity

$$\det(I + L_x L_y) = Z^2.$$

In the case of Schur functions, the right-hand side of the Gessel identity is a Toeplitz determinant, and the Cauchy identity for Schur functions emerges as a consequence of the Szegő limit theorem. In view of the last identity and the connection between the operators L and K on $\ell^2(\mathbb{Z}_+)$, it is tempting to try to find, using these, an independent derivation of (2.3). It follows from (6.3) that $I - K(x) K(y)$ and

$I + L_x L_y$ differ by a finite-rank operator. This operator cannot contribute to the determinant, but we do not see, a priori, why this is so. So such an independent derivation eludes us.

To continue now, we let P_h be the projection operator onto the subspace of $\ell^2(\mathbb{Z}_+)$ spanned by $\{e_0, e_1, \dots, e_h\}$. Thus if K is the operator on $\ell^2(\mathbb{Z}_+)$, then $K_h = P_h K P_h$. Instead of working directly with the product $K(x) K(y)$, it is convenient to write (2×2) -matrices with operator entries. Thus $\det(I - K_h(x) K_h(y))$ is the determinant of

$$\begin{pmatrix} P_h & 0 \\ 0 & P_h \end{pmatrix} \begin{pmatrix} I & K(x) \\ K(y) & I \end{pmatrix} \begin{pmatrix} P_h & 0 \\ 0 & P_h \end{pmatrix},$$

thought of as acting on $P_h \ell^2(\mathbb{Z}_+) \oplus P_h \ell^2(\mathbb{Z}_+)$. To simplify notation, we use P_h to denote also

$$\begin{pmatrix} P_h & 0 \\ 0 & P_h \end{pmatrix},$$

and we set

$$\mathcal{K} = \begin{pmatrix} 0 & K(x) \\ K(y) & 0 \end{pmatrix}.$$

Thus

$$\det(I - K_h(x) K_h(y)) = \det P_h(I + \mathcal{K})P_h.$$

It follows from (4.4) and the infinite-dimensional version of Jacobi's theorem on the principal $(n \times n)$ -minor of the inverse of a (finite) matrix* that this may be written

$$\det(I - K_h(x) K_h(y)) = Z^2 \det((I - P_h)(I + \mathcal{K})^{-1}(I - P_h)). \tag{6.10}$$

Thus our first goal is to compute $(I + \mathcal{K})^{-1}$.

Using the easily verified fact

$$\Lambda^* L + L \Lambda = q \otimes q - e \otimes e \tag{6.11}$$

and (6.3), we see that $K = \Lambda^* L + R^- = -L \Lambda + R^+$, where

$$R^\pm = \frac{1}{2}(\pm q \otimes q \mp e \otimes e + \omega(e \otimes q - q \otimes e)). \tag{6.12}$$

Thus

$$I + \mathcal{K} = \begin{pmatrix} I & \Lambda^* L_x + R_x^- \\ \Lambda^* L_y + R_y^- & I, \end{pmatrix},$$

where subscripts have the usual meaning.

*The infinite-dimensional result follows by replacing the operator \mathcal{K} by $P_N \mathcal{K} P_N$, applying the finite-dimensional result, and taking the $(N \rightarrow \infty)$ -limit. We use the fact that \mathcal{K} is trace class, which holds since the Hankel operators are trace class. Of course, all this requires that the infinite-dimensional operator be invertible. This follows from the limit results we establish, as we see at the end of §6.3.

6.2. Calculation of $(I + \mathcal{K})^{-1}$

The fundamental objects that appear here are $I + L_x L_y$ and $I + L_y L_x$, and we begin by showing that they are invertible and computing their inverses. Define

$$H_1 = H(Q_x \tilde{Q}_y^{-1}) \quad \text{and} \quad H_2 = H(Q_y \tilde{Q}_x^{-1}).$$

We prove the basic identities

$$(I + L_x L_y)^{-1} = I - H_1 H_2 \quad \text{and} \quad (I + L_y L_x)^{-1} = I - H_2 H_1. \quad (6.13)$$

We do this with the help of (6.6) and (6.8). Using these and (6.4), we find that

$$\begin{aligned} H_1 H_2 L_x &= H(Q_x \tilde{Q}_y^{-1}) H(Q_y \tilde{Q}_x^{-1}) H(Q_x) T(\tilde{Q}_x^{-1}) J \\ &= H(Q_x \tilde{Q}_y^{-1}) [T(Q_y) - T(Q_y \tilde{Q}_x^{-1}) T(\tilde{Q}_x)] T(\tilde{Q}_x^{-1}) J \\ &= [H(Q_x) T(\tilde{Q}_x^{-1}) - H(Q_x \tilde{Q}_y^{-1}) T(Q_y \tilde{Q}_x^{-1})] J \\ &= L_x - H_1 T(Q_y \tilde{Q}_x^{-1}) J. \end{aligned}$$

Thus

$$(I - H_1 H_2) L_x = H_1 T(Q_y \tilde{Q}_x^{-1}) J. \quad (6.14)$$

From (6.14) and (6.5) we obtain similarly

$$\begin{aligned} (I - H_1 H_2) L_x L_y &= -H_1 T(Q_y \tilde{Q}_x^{-1}) H(Q_y^{-1}) T(\tilde{Q}_y) \\ &= H_1 H(Q_y \tilde{Q}_x^{-1}) T(\tilde{Q}_y^{-1}) T(\tilde{Q}_y) = H_1 H_2. \end{aligned}$$

This establishes the first identity of (6.13), and the second is obtained by interchanging x and y .

Beginning the calculation of $(I + \mathcal{K})^{-1}$, we refer to (6.11) and (6.12) and find that

$$\begin{pmatrix} I & L_x \Lambda \\ L_y \Lambda & I \end{pmatrix} (I + \mathcal{K}) = \begin{pmatrix} I + L_x L_y & 0 \\ 0 & I + L_y L_x \end{pmatrix} + \begin{pmatrix} L_x \Lambda R_y^- & R_x^+ \\ R_y^+ & L_y \Lambda R_x^- \end{pmatrix},$$

where we use subscripts as before. Using (6.13), this may be written

$$\begin{pmatrix} I + L_x L_y & 0 \\ 0 & I + L_y L_x \end{pmatrix} \left[I + \begin{pmatrix} (I - H_1 H_2) L_x \Lambda R_y^- & (I - H_1 H_2) R_x^+ \\ (I - H_2 H_1) R_y^+ & (I - H_2 H_1) L_y \Lambda R_x^- \end{pmatrix} \right].$$

Hence

$$(I + \mathcal{K})^{-1} = \left[I + \begin{pmatrix} (I - H_1 H_2)L_x \Lambda R_y^- & (I - H_1 H_2)R_x^+ \\ (I - H_2 H_1)R_y^+ & (I - H_2 H_1)L_y \Lambda R_x^- \end{pmatrix} \right]^{-1} \\ \times \begin{pmatrix} I - H_1 H_2 & (I - H_1 H_2)L_x \Lambda \\ (I - H_2 H_1)L_y \Lambda & I - H_2 H_1 \end{pmatrix}.$$

To compute the entries of the matrix inside the large bracket, we show that

$$(I - H_1 H_2)L_x \Lambda q_y = H_1 H_2 e \quad \text{and} \quad (I - H_1 H_2)q_x = T_1 e, \quad (6.15)$$

where we set

$$T_1 = T(Q_x \tilde{Q}_y^{-1}) \quad \text{and} \quad T_2 = T(Q_y \tilde{Q}_x^{-1}).$$

For the first we use the fact that $\Lambda q = L e$, which gives

$$(I - H_1 H_2)L_x \Lambda q_y = (I - H_1 H_2)L_x L_y e = H_1 H_2 e.$$

To derive the second we use the fact that $q_x = T(Q_x) e$ and compute

$$H_1 H_2 T(Q_x) = H(Q_x \tilde{Q}_y^{-1}) H(Q_y \tilde{Q}_x^{-1}) T(Q_x) = H(Q_x \tilde{Q}_y^{-1}) H(Q_y) \\ = T(Q_x) - T(Q_x \tilde{Q}_y^{-1}) T(\tilde{Q}_y).$$

Since $T(\tilde{Q}_y) e = e$ (the matrix is upper-triangular with $(0, 0)$ -entry 1), this gives

$$H_1 H_2 q_x = q_x - T_1 e,$$

which is equivalent to the desired identity. Of course, the same identities hold if we make the interchanges $x \leftrightarrow y$ and $1 \leftrightarrow 2$.

With these identities and the fact that $\Lambda R^- = -(1/2) - \Lambda q \otimes (q + \omega e)$, we find that the matrix in large brackets may be written

$$I + \frac{1}{2} \begin{pmatrix} -H_1 H_2 e \otimes (q_y + \omega e) & T_1 e \otimes (q_x - \omega e) + (I - H_1 H_2) e \otimes (\omega q_x - e) \\ T_2 e \otimes (q_y - \omega e) + (I - H_2 H_1) e \otimes (\omega q_y - e) & -H_2 H_1 e \otimes (q_x + \omega e) \end{pmatrix}.$$

This in turn has the form

$$I + \sum_{i=1}^4 a_i \otimes b_i,$$

where

$$\begin{aligned}
 a_1 &= \frac{1}{2} \begin{pmatrix} -H_1 H_2 e \\ -\omega H_2 H_1 e + T_2 e + \omega e \end{pmatrix}, & a_2 &= \frac{1}{2} \begin{pmatrix} -\omega H_1 H_2 e + T_1 e + \omega e \\ -H_2 H_1 e \end{pmatrix}, \\
 a_3 &= \frac{1}{2} \begin{pmatrix} -\omega H_1 H_2 e \\ H_2 H_1 e - \omega T_2 e - e \end{pmatrix}, & a_4 &= \frac{1}{2} \begin{pmatrix} H_1 H_2 e - \omega T_1 e - e \\ -\omega H_2 H_1 e \end{pmatrix}, \\
 b_1 &= \begin{pmatrix} q_y \\ 0 \end{pmatrix}, & b_2 &= \begin{pmatrix} 0 \\ q_x \end{pmatrix}, & b_3 &= \begin{pmatrix} e \\ 0 \end{pmatrix}, & b_4 &= \begin{pmatrix} 0 \\ e \end{pmatrix}.
 \end{aligned}$$

At this stage we have shown that

$$(I + \mathcal{K})^{-1} = \left(I + \sum_{i=1}^4 a_i \otimes b_i \right)^{-1} \begin{pmatrix} I - H_1 H_2 & (I - H_1 H_2) L_x \Lambda \\ (I - H_2 H_1) L_y \Lambda & I - H_2 H_1 \end{pmatrix}. \quad (6.16)$$

If we have a finite-rank operator $\sum a_i \otimes b_i$, then

$$\left(I + \sum a_i \otimes b_i \right)^{-1} = I - \sum_{i,j} (S^{-1})_{ij} a_i \otimes b_j, \quad (6.17)$$

where S is the matrix with entries

$$S_{ij} = \delta_{ij} + (b_i, a_j).$$

In our case we have to compute 16 inner products, which is not as bad as it might seem since there are basic inner products from which the others can be derived. And if we have evaluated any inner product, then we have evaluated another with the interchanges $x \leftrightarrow y$ and $1 \leftrightarrow 2$. Two basic inner products are trivial:

$$(e, e) = 1 \quad \text{and} \quad (e, q_y) = 1.$$

Two are not evaluable in simpler terms but just notationally. We set

$$t = (T_1 e, e) = (T_2 e, e) \quad \text{and} \quad h = (H_1 H_2 e, e) = (H_2 H_1 e, e).$$

(The equality of the first two inner products follows from the fact that $T_1^* = J T_2 J$ and $J e = e$.) The nontrivial ones are

$$(T_1 e, q_y) = 1 \quad \text{and} \quad (H_1 H_2 e, q_y) = 1 - t.$$

For the first, we have $(T_1e, q_y) = (T_1e, T(Q_y)e) = (T(\tilde{Q}_y)T_1e, e)$, and this is the $(0, 0)$ -entry of $T(\tilde{Q}_y)T(Q_x\tilde{Q}_y^{-1}) = T(Q_x)$. The $(0, 0)$ -entry equals 1. For the second, we have

$$(H_1H_2e, q_y) = 1 - ((I - H_1H_2)e, q_y) = 1 - (e, (I - H_2H_1)q_y) = 1 - (e, T_2e)$$

by the second identity of (6.15).

We can now write down all 16 inner products. For convenience, we multiply them by 2:

$$\begin{aligned} 2(a_1, b_1) &= -1 + t, & 2(a_2, b_1) &= \omega t + 1, \\ 2(a_3, b_1) &= -\omega(1 - t), & 2(a_4, b_1) &= -t - \omega - 2, \\ 2(a_1, b_2) &= \omega t + 1, & 2(a_2, b_2) &= -1 + t, \\ 2(a_3, b_2) &= -t - \omega - 2, & 2(a_4, b_2) &= -\omega(1 - t), \\ 2(a_1, b_3) &= -h, & 2(a_2, b_3) &= -\omega h + t + \omega, \\ 2(a_3, b_3) &= -\omega h, & 2(a_4, b_3) &= h - \omega t - 1, \\ 2(a_1, b_4) &= -\omega h + t + \omega, & 2(a_2, b_4) &= -h, \\ 2(a_3, b_4) &= h - \omega t - 1, & 2(a_4, b_4) &= -\omega h. \end{aligned}$$

Let us see which vectors arise in the end. From (6.16) and (6.17),

$$(I + \mathcal{K})^{-1} = \begin{pmatrix} I - H_1H_2 & (I - H_1H_2)L_x\Lambda \\ (I - H_2H_1)L_y\Lambda & I - H_2H_1 \end{pmatrix} - \sum_{i,j=1}^4 s_{ij} a_i \otimes b'_j, \quad (6.18)$$

where $s_{ij} = (S^{-1})_{ij}$ and

$$b'_j = \begin{pmatrix} I - H_2H_1 & \Lambda^*L_y(I - H_1H_2) \\ \Lambda^*L_x(I - H_2H_1) & I - H_1H_2 \end{pmatrix} b_j.$$

The quantities that appear in the a_i , other than e , which we can ignore since $(I - P_h)e = 0$, are H_1H_2e and T_1e in the first component and H_2H_1e and T_2e in the second. Those in the b_j are q_y and e in the first component and q_x and e in the second. For b'_j we use (6.15) to see that T_2e , H_2H_1e , $\Lambda^*L_yT_1e$, and $\Lambda^*L_y(I - H_1H_2)e$ appear in the first component and T_1e , H_1H_2e , $\Lambda^*L_xT_2e$, and $\Lambda^*L_x(I - H_2H_1)e$ in the second. (The e that appear once again drop out in the end.)

Two new vectors appear here (as well as those obtained by the usual interchanges). We claim that

$$\Lambda^*L_yT_1e = T_2e - te \quad \text{and} \quad \Lambda^*L_y(I - H_1H_2)e = H_2H_1e + tT_2e - (1 - h)e. \quad (6.19)$$

For the first, we have

$$\begin{aligned} L_y T_1 &= -JH(Q_y^{-1})T(\tilde{Q}_y)T(Q_x \tilde{Q}_y^{-1}) = -JH(Q_y^{-1})T(Q_x) \\ &= -JH(\tilde{Q}_x Q_y^{-1}) = H(Q_y \tilde{Q}_x^{-1})J, \end{aligned}$$

so

$$\Lambda^* L_y T_1 e = \Lambda^* H(Q_y \tilde{Q}_x^{-1})e = T_2 e - t e.$$

For the second, we take transposes and interchange x and y in (6.14) to obtain

$$\begin{aligned} L_y T_1 &= -JH(Q_y^{-1})L_y(1 - H_1 H_2)e = JT(\tilde{Q}_x Q_y^{-1})H(Q_y \tilde{Q}_x^{-1})e \\ &= -T(Q_y \tilde{Q}_x^{-1})H(\tilde{Q}_x Q_y^{-1})e = H(Q_y \tilde{Q}_x^{-1})T(Q_x \tilde{Q}_y^{-1})e. \end{aligned}$$

So

$$\begin{aligned} \Lambda^* L_y(1 - H_1 H_2)e &= [H(z Q_y \tilde{Q}_x^{-1}) - e \otimes T_2 e]T(Q_x \tilde{Q}_y^{-1})e \\ &= H(z Q_y \tilde{Q}_x^{-1})H(z Q_x \tilde{Q}_y^{-1})e - (T_1 e, T_2 e)e \\ &= H_2 H_1 e + (T_2 e \otimes T_1 e)e - (T_1 e, T_2 e)e. \end{aligned}$$

The next-to-last term equals $t T_2 e$, while the last inner product equals

$$(JT_2^* J J T_1 J e, e) = (T(Q_x \tilde{Q}_y^{-1})T(\tilde{Q}_y Q_x^{-1})e, e) = ((I - H_1 H_2)e, e).$$

This establishes the second claim.

It follows from the above that the only vectors that arise in the b'_j are $T_2 e$ and $H_2 H_1 e$ in the first component and $T_1 e$ and $H_1 H_2 e$ in the second.

6.3. Specialization

At this point we impose the α -specialization. Thus the first m x_i and the first n y_i are equal to α and the rest are equal to zero. We assume that $\tau = m/n$ is a constant satisfying $\alpha^2 < \tau < \alpha^{-2}$, and we first determine the asymptotics as $n \rightarrow \infty$ of the quantities appearing in the inner products. We claim that

$$\lim_{n \rightarrow \infty} t = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} h = \frac{1}{2}.$$

For the first, we have

$$t = (T_1)_{0,0} = \frac{1}{2\pi i} \int \left(\frac{1 + \alpha z}{1 - \alpha z} \right)^m \left(\frac{z - \alpha}{z + \alpha} \right)^n \frac{dz}{z}. \quad (6.20)$$

If we apply steepest descent, we see that the saddle points should satisfy

$$\frac{\tau}{1 - \alpha^2 z^2} + \frac{1}{z^2 - \alpha^2} = 0$$

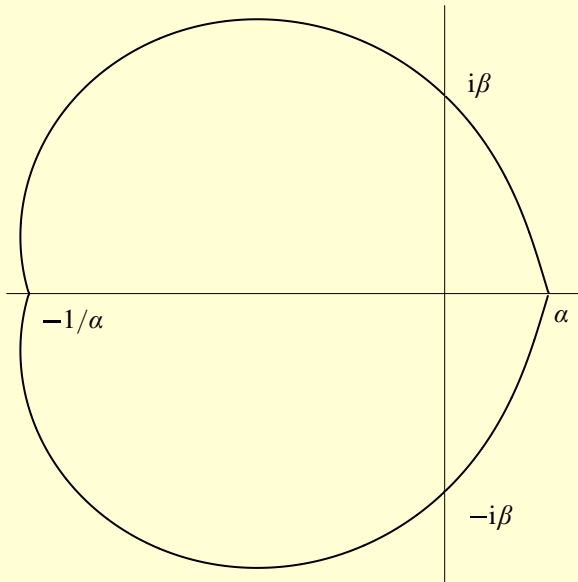


Figure 1. Steepest descent curve

and so are given by

$$z = \pm i \sqrt{\frac{1 - \alpha^2 \tau}{\tau - \alpha^2}} = \pm i\beta.$$

These are purely imaginary under our assumption on τ . The steepest descent curve passes through these points and closes at α and $-\alpha^{-1}$ (see Fig. 1). The integral is $O(n^{-1/2})$. For the second, we have

$$\begin{aligned} (H_1 H_2)_{i,j} &= \left(\frac{1}{2\pi i}\right)^2 \sum_{k=0}^{\infty} \iint \left(\frac{1+\alpha z}{1-\alpha z}\right)^m \left(\frac{z-\alpha}{z+\alpha}\right)^n \left(\frac{1+\alpha \zeta}{1-\alpha \zeta}\right)^n \left(\frac{\zeta-\alpha}{\zeta+\alpha}\right)^m \\ &\quad \times z^{-i-k-2} \zeta^{-k-j-2} dz d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^2 \iint \left(\frac{1+\alpha z}{1-\alpha z}\right)^m \left(\frac{z-\alpha}{z+\alpha}\right)^n \left(\frac{1+\alpha \zeta}{1-\alpha \zeta}\right)^n \left(\frac{\zeta-\alpha}{\zeta+\alpha}\right)^m z^{-i-1} \\ &\quad \times \zeta^{-j-1} \frac{dz d\zeta}{z\zeta-1}, \end{aligned} \tag{6.21}$$

where the contours are such that $|z\zeta| > 1$. Setting $i = j = 0$ and making the substitution $\zeta \rightarrow \zeta^{-1}$ gives

$$\begin{aligned} h &= (H_1 H_2)_{0,0} \\ &= \left(\frac{1}{2\pi i}\right)^2 \iint \left(\frac{1+\alpha z}{1-\alpha z}\right)^m \left(\frac{z-\alpha}{z+\alpha}\right)^n \left(\frac{\zeta+\alpha}{\zeta-\alpha}\right)^n \left(\frac{1-\alpha \zeta}{1+\alpha \zeta}\right)^m \frac{dz d\zeta}{z(z-\zeta)}, \end{aligned}$$

where now on the contours (both still described counterclockwise) $|z| > |\zeta|$. If we ignore the $z - \zeta$ in the denominator, we have two integrals, to each of which we apply steepest descent. The saddle points are $\pm i\beta$ for both. The new z -contour is as before, but the new ζ -contour closes at $-\alpha$ and α^{-1} . We first deform the original ζ -contour to this, always remaining inside the original z -contour. Then we deform the z -contour to its steepest descent curve. In the process we pass through the points of the ζ -contour from $-i\beta$ to $i\beta$ in the right half-plane. The z residues at these points are $1/\zeta$, and so the deformations lead to the double integral over the steepest descent contours, which is $O(n^{-1/2})$, plus $(2\pi i)^{-1} \int_{-i\beta}^{i\beta} d\zeta/\zeta = 1/2$. This establishes the second limit.

When $\omega = 1 - \sqrt{2}$, we find that $\det S$ has the limit $(5 - 3\sqrt{2})/8 \neq 0$ as $t \rightarrow 0$, $h \rightarrow 1/2$.^{*} Hence the entries of S^{-1} are all bounded. (The invertibility of S for large n implies in turn the invertibility of $I + \mathcal{K}$ and hence the validity of the determinant identities we have been using.)

6.4. Scaling

If we write P for $I - P_h$, then we see from (6.18) that $P(I + \mathcal{K})^{-1}P$ equals the identity operator I minus

$$\begin{aligned} & \begin{pmatrix} PH_1H_2P & 0 \\ 0 & PH_2H_1P \end{pmatrix} - \begin{pmatrix} 0 & P(I - H_1H_2)L_x\Lambda P \\ P(I - H_2H_1)L_y\Lambda P & 0 \end{pmatrix} \\ & \quad + \sum_{i,j=1}^4 s_{ij} Pa_i \otimes Pb'_j. \end{aligned} \tag{6.22}$$

Eventually we set $i = h + n^{1/3}x$, $j = h + n^{1/3}y$, where $h = cn + n^{1/3}s$ with c to be determined. The operators PH_1 and H_1P give rise to integrals like

$$\int \left(\frac{1 + \alpha z}{1 - \alpha z}\right)^m \left(\frac{z - \alpha}{z + \alpha}\right)^n z^{-cn - n^{1/3}x} dz$$

(with a different x), and PH_2 and H_2P give rise to integrals like

$$\int \left(\frac{1 + \alpha z}{1 - \alpha z}\right)^n \left(\frac{z - \alpha}{z + \alpha}\right)^m z^{-cn - n^{1/3}x} dz.$$

If we make the substitution $z \rightarrow z^{-1}$ in the latter, we get an integral like

$$\int \left(\frac{z + \alpha}{z - \alpha}\right)^n \left(\frac{1 - \alpha z}{1 + \alpha z}\right)^m z^{cn + n^{1/3}x} dz.$$

^{*}Computations verify that when $\tau < \alpha^2$, the limit of h is zero and $t = (-1)^m + o(1)$, and that the limit of $\det S$ is $(9 - 4\sqrt{2})/2$ as $m \rightarrow \infty$ through even values and $-1/2$ as $m \rightarrow \infty$ through odd values. For the limits when $\tau > \alpha^{-2}$, we replace m by n . The proofs should be similar to what we have already done, except that the saddle points will now be real.

If we set

$$\psi(z) = \left(\frac{1 + \alpha z}{1 - \alpha z}\right)^m \left(\frac{z - \alpha}{z + \alpha}\right)^n z^{-cn},$$

our integrals become

$$\int \psi(z) z^{-n^{1/3}x} dz \quad \text{and} \quad \int \psi(z)^{-1} z^{n^{1/3}x} dz.$$

If we think of the factors $\psi(z)^{\pm 1}$ as the dominant ones and apply steepest descent, there are in general two saddle points for the two integrals, and the product of the critical values is exponentially small or large. If c is chosen so that the critical points coincide,* then the product of the critical values is 1 and the product of the operators has nontrivial scaling.

To determine c , let $\sigma(z) = n^{-1} \log \psi(z)$, so that

$$\sigma'(z) = \frac{2\alpha\tau}{1 - \alpha^2 z^2} + \frac{2\alpha}{z^2 - \alpha^2} - \frac{c}{z}.$$

If we eliminate c from $\sigma'(z) = \sigma''(z) = 0$, we obtain

$$\frac{\tau(1 + \alpha^2 z^2)}{(1 - \alpha^2 z^2)^2} - \frac{\alpha^2 + z^2}{(z^2 - \alpha^2)^2} = 0. \tag{6.23}$$

The function on the left is strictly increasing from $-\infty$ to $+\infty$ on the interval (α, α^{-1}) . It follows that there is a unique point z_0 in this interval where the function vanishes. This is our saddle point, and we set

$$c = 2\alpha z_0 \left(\frac{\tau}{1 - \alpha^2 z_0^2} + \frac{1}{z_0^2 - \alpha^2} \right) > 0. \tag{6.24}$$

From the behavior of $\sigma'(z)$ for large negative z and near $-\alpha^{-1}$, α , and zero, we see that σ' has a zero in $(-\infty, -\alpha^{-1})$ and a zero in $(-\alpha, 0)$. Since it has a double zero at z_0 , this accounts for all four of its finite zeros. Since $\sigma'(z)$ tends to $+\infty$ at the endpoints of (α, α^{-1}) , it follows that it is positive everywhere there except at z_0 , and this implies that $\sigma'''(z_0) > 0$. (Well, this only shows that $\sigma'''(z_0) \geq 0$. We find in Section 6.6 an explicit expression for $\sigma'''(z_0)$ in terms of z_0 , from which it is clear that it is positive.)

The two steepest descent curves, which we call Γ^+ for the first integral and Γ^- for the second, together form a single contour. The first emanates from z_0 at angles $\pm\pi/3$ with branches going to ∞ in two directions. The second emanates from z_0 at angles $\pm 2\pi/3$ and closes at $z = 0$ (see Figure 2).

It is convenient to replace $\ell^2([h, \infty))$ by $\ell^2(\mathbb{Z}_+)$, and to do that we change our meaning of the operator P . A P appearing on the left is to be interpreted as Λ^h ,

*This c is the $c_1(\alpha, \tau)$ of the main theorem.

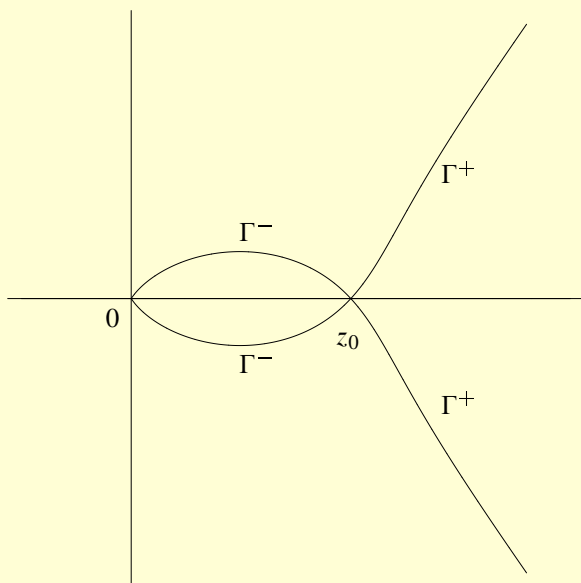


Figure 2. Steepest descent curves Γ^\pm

and a P appearing on the right is to be interpreted as Λ^{*h} . So, for example, the i th component of PT_1e is $(T_1e)_{h+i}$ and the (i, j) -entry of PH_1H_2 is $(H_1H_2)_{h+i, h+j}$. With these reinterpretations of P , the operators in (6.22) all act on $\ell^2(\mathbb{Z}_+)$ and the determinant has not changed. (Recall that $h = cn + n^{1/3}s$.)

Let D be the diagonal matrix with i th diagonal element equal to $\psi(z_0)^{-1}z_0^{n^{1/3}s+i}$, and multiply (6.22) by $\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}$ on the left and by $\begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix}$ on the right. This does not affect the determinant. (The reason is that all the operators in (6.22) have (i, j) -entry $O(r^{i+j})$ for fixed n , where r can be any number larger than α , and $\alpha < z_0 < \alpha^{-1}$.) We show that after these multiplications the first operator scales to the direct sum of two Airy operators,* the second operator has trace norm $o(1)$, and the vectors in the sum all have norm $o(1)$.

*If a matrix acting on $\ell^2(\mathbb{Z}_+)$ has entries $M(i, j)$ and if the kernel $n^{1/3}M([n^{1/3}x], [n^{1/3}y])$ acting on $L^2(0, \infty)$, which is unitarily equivalent to the matrix operator, converges in trace norm to a limiting kernel, then we say that the matrix scales in trace norm to the limiting kernel. The Fredholm determinant of $M(i, j)$ then converges to the Fredholm determinant of the limiting kernel. This is what happens here, with the limiting kernel being the direct sum of two Airy kernels.

6.4.1. *The first operator in (6.22)*

The upper left corner of the first operator becomes, aside from the identity operator, $DPH_1H_2PD^{-1}$. We write this as

$$(DPH_1D_0)(D_0^{-1}H_2PD^{-1}),$$

where D_0 is the diagonal matrix with i th diagonal element equal to z_0^i , and we scale each factor. We have

$$\begin{aligned} z_0^{n^{1/3}s+i+j}(PH_1)_{i,j} &= \frac{z_0^{n^{1/3}s+i+j}}{2\pi i} \int \left(\frac{1+\alpha z}{1-\alpha z}\right)^m \left(\frac{z-\alpha}{z+\alpha}\right)^n z^{-h-i-j-2} dz \\ &= \frac{z_0^{n^{1/3}s+i+j}}{2\pi i} \int \psi(z) z^{-n^{1/3}s-i-j} \frac{dz}{z^2}. \end{aligned} \tag{6.25}$$

The main fact is the following. Define

$$\psi(z, \gamma) = \left(\frac{1+\alpha z}{1-\alpha z}\right)^m \left(\frac{z-\alpha}{z+\alpha}\right)^n z^{-\gamma n},$$

and set

$$I_n(x) = \frac{1}{2\pi i} \int \psi(z, c+n^{-2/3}x) \frac{dz}{z^2},$$

the contour being the unit circle. Then

$$\psi(z_0)^{-1} z_0^{n^{1/3}x} n^{1/3} |I_n(x)| \leq e^{-\delta x}, \tag{6.26}$$

valid for some δ and all n if x is bounded from below, and

$$\lim_{n \rightarrow \infty} \psi(z_0)^{-1} z_0^{n^{1/3}x} n^{1/3} I_n(x) = z_0^{-1} g \operatorname{Ai}(gx) \tag{6.27}$$

pointwise, with g the constant given by (6.30). (The limit in (6.27) is uniform for x in a bounded set.)

We first show that (6.26) holds if $n^{1/3}x > \eta n$ for some $\eta > 0$. For this we set $\gamma = c + n^{-2/3}x$ so that $\gamma - c > \eta$, write our main integrand as $\psi(z, \gamma)$, and do steepest descent. With

$$\sigma(z, \gamma) = n^{-1} \log \psi(z, \gamma) = \sigma(z) - (\gamma - c) \log z, \tag{6.28}$$

our saddle points z_γ^\pm (there are two of them when $\gamma > c$) are solutions of $\sigma'(z_\gamma^\pm, \gamma) = 0$. Differentiating this with respect to γ gives

$$\sigma''(z_\gamma^\pm, \gamma) \frac{dz_\gamma^\pm}{d\gamma} - \frac{1}{z_\gamma^\pm} = 0.$$

Since $\sigma''(z_\gamma^\pm, \gamma) \neq 0$ (since c is the only value of γ for which there is a double saddle point), we find that if z_γ^\pm are chosen so that $z_\gamma^+ > z_\gamma^-$ for γ near c , then z_γ^+ increases always as γ increases and z_γ^- decreases, and $\sigma_1''(z_\gamma^+, \gamma) > 0$ and $\sigma_1''(z_\gamma^-, \gamma) < 0$. In particular, z_γ^+ is the saddle point we take for our steepest descent, and $z_\gamma^+ > z_0$ when $\gamma > c$ since $z_c^+ = z_0$. For the critical value, we have to see how $\sigma(z_\gamma^+, \gamma)$ behaves as a function of γ . From (6.28) and the fact $\sigma'(z_\gamma^\pm, \gamma) = 0$, we obtain

$$\frac{d\sigma(z_\gamma^\pm, \gamma)}{d\gamma} = -\log z_\gamma^\pm,$$

and so

$$\frac{d}{d\gamma} [\sigma(z_\gamma^\pm, \gamma) + \gamma \log z_0] = \log \frac{z_0}{z_\gamma^\pm}. \tag{6.29}$$

Since z_γ^+ is an increasing function of γ , this shows that $\sigma(z_\gamma^+, \gamma) + \gamma \log z_0$ is a decreasing concave function of γ . It follows that for some $\delta > 0$,

$$\sigma(z_\gamma^+, \gamma) + \gamma \log z_0 < \sigma(z_0) + c \log z_0 - \delta(\gamma - c)$$

for $\gamma \geq c + \eta$. Thus for these γ ,

$$\psi(z_0)^{-1} z_0^{n(\gamma-c)} \psi(z_\gamma^+, \gamma) \leq e^{-\delta n(\gamma-c)}.$$

Since $|\psi(z, \gamma)|$ achieves its maximum on the steepest descent contour at z_γ and since the contour is bounded away from zero (this follows from the fact that $z_\gamma \rightarrow a$ as $\gamma \rightarrow \infty$), we see that in this case $I_n(x)$ is at most a constant times $\psi(z_\gamma^+, \gamma)$, where $\gamma = c + n^{-2/3}x$. Hence

$$\psi(z_0)^{-1} z_0^{n^{1/3}x} |I_n(x)| = O(e^{-\delta n^{1/3}x})$$

when $n^{1/3}x > \eta n$. This is an even better estimate than (6.26).

Since we have shown that (6.26) holds if $n^{1/3}x > \eta n$, where η can be as small as we please, we may assume $n^{1/3}x = o(n)$ when $x > 0$.

Write

$$I_n(x) = \frac{1}{2\pi i} \int \psi(z) z^{-n^{1/3}x} \frac{dz}{z^2},$$

and use the steepest descent curve Γ^+ . It emanates from z_0 at angles $\pm\pi/3$. Clearly it is bounded away from zero. Choose ε small, and let

$$\Gamma^{(1)} = \{z \in \Gamma^+ : |z - z_0| > \varepsilon\} \quad \text{and} \quad \Gamma^{(2)} = \{z \in \Gamma^+ : |z - z_0| < \varepsilon\}$$

with corresponding $I_n^{(1)}(x)$ and $I_n^{(2)}(x)$. We show that both of these satisfy the uniform estimate (6.26) and

$$\lim_{n \rightarrow \infty} \psi(z_0)^{-1} n^{1/3} I_n^{(1)}(x) = 0$$

and

$$\lim_{n \rightarrow \infty} \psi(z_0)^{-1} z_0^{n^{1/3}x} n^{1/3} I_n^{(2)}(x) = z_0^{-1} g \operatorname{Ai}(gx).$$

Consider $I_n^{(1)}(x)$ first. Since $\Re \sigma(z)$ is strictly decreasing as we move away from z_0 on Γ , we know that $|\psi(z)| < \psi(z_0) e^{-\delta n}$ on $\Gamma^{(1)}$ for some $\delta > 0$. The assertions follow from this since we are in the case $n^{1/3}x = o(n)$. (This also holds also for $x < 0$ since x is bounded below.)

Making the variable change $z \rightarrow z_0(1 + \zeta)$, we can write, since $z = z_0 e^{\zeta(1+O(\zeta))}$ near $z = z_0$,

$$I_n^{(2)}(x) = \psi(z_0) z_0^{-n^{1/3}x} \frac{1}{2} \pi i \int_{|\zeta| < \varepsilon/z_0} e^{nbz_0^3 \zeta^3(1+O(\zeta)) - n^{1/3}x \zeta(1+O(\zeta))} (z_0^{-1} + O(\zeta)) d\zeta,$$

where $b = \sigma'''(z_0)/6$. The path of integration here is the portion of the contour $\Gamma^{(2)}$ satisfying the indicated inequality. It consists of two little arcs emanating from $\zeta = 0$ tangent to the line segments making angles $\pm\pi/3$ with the positive axis. If ε is small enough and we replace the integral by the line segments themselves, we introduce an error of the form $O(e^{-\delta n})$ with a different δ since $n^{1/3}x = o(n)$. With the variable change $\zeta \rightarrow n^{-1/3}\zeta$, we obtain

$$\begin{aligned} & \psi(z_0)^{-1} z_0^{n^{1/3}x} n^{1/3} I_n^{(2)}(x) \\ &= \frac{1}{2\pi i} \int_{|\zeta| < n^{1/3}\varepsilon} e^{bz_0^3 \varepsilon^3(1+O(n^{-1/3}\zeta)) - x \zeta(1+O(n^{-1/3}\zeta))} (z_0^{-1} + O(n^{-1/3}\zeta)) d\zeta \\ & \quad + O(e^{-\delta n}), \end{aligned}$$

where now the integration is taken over line segments of length of the order $n^{1/3}$.

On the path of integration, we have $\Re(\zeta^3) \leq -\delta|\zeta|^3$ and $\Re(\zeta) \geq \delta|\zeta|$ for some $\delta > 0$, and so for the above we have an estimate of the form

$$\int_0^\infty e^{-\delta(t^3 - xt)} dt + O(e^{-\delta n}).$$

Since $n \gg x$, this gives the required uniform bound. The limit of the integral, with its factor $1/2\pi i$, equals $z_0^{-1} g \operatorname{Ai}(gx)$, where

$$g = (3bz_0^3)^{-1/3} = z_0^{-1} \left(\frac{2}{\sigma'''(z_0)} \right)^{1/3}. \tag{6.30}$$

That the limit is as stated follows by taking the limit under the integral sign, which is justified by dominated convergence.

To obtain the scaling of the matrix $DPH_1 D_0$, we need only observe that by (6.25),

$$\begin{aligned} & n^{1/3} (DPH_1 D_0)_{[n^{1/3}x], [n^{1/3}y]} \\ &= \psi(z_0)^{-1} z_0^{n^{1/3}s + [n^{1/3}x] + [n^{1/3}x]} n^{1/3} I_n(s + n^{-1/3}([n^{1/3}x] + [n^{1/3}y])). \end{aligned}$$

It follows from (6.26) and (6.27) that this kernel on $(0, \infty)$ converges in Hilbert-Schmidt norm to $z_0^{-1} g \operatorname{Ai}(g(s+x+y))$.

To scale $D_0^{-1} H_2 P D^{-1}$, we write

$$z_0^{-n^{1/3}s-i-j} (P H_2)_{i,j} = \frac{z_0^{-n^{1/3}s-i-j}}{2\pi i} \int \left(\frac{1+\alpha z}{1-\alpha z}\right)^n \left(\frac{z-\alpha}{z+\alpha}\right)^m z^{-cn-n^{1/3}s-i-j-2} dz.$$

If we make the substitution $z \rightarrow z^{-1}$, this becomes

$$\frac{z_0^{-n^{1/3}s-i-j}}{2\pi i} \int \psi(z)^{-1} z^{n^{1/3}s+i+j} dz.$$

This is completely analogous to (6.25). To use the analogous argument we mention only that we use (6.29) with the minus signs to see that both $-\sigma(z_\gamma^-, \gamma) - \gamma \log z_0$ and its derivative are decreasing functions of γ . The steepest descent curve now is Γ^- . We need not go through the details again. We find that

$$n^{1/3} (D_0^{-1} P H_1 D^{-1})_{[n^{1/3}x], [n^{1/3}y]}$$

converges in Hilbert-Schmidt norm to $z_0 g \operatorname{Ai}(g(s+x+y))$. Hence

$$n^{1/3} (D P H_1 H_2 P D^{-1})_{[n^{1/3}x], [n^{1/3}y]}$$

converges in trace norm to $g K_{\operatorname{Airy}}(g(s+x), g(x+y))$ on $L^2(0, \infty)$.

By taking transposes, we see that $D^{-1} P H_2 H_1 P D^{-1}$ has the same scaling limit, which takes care of the lower right corner of the first operator in (6.22).

6.4.2. The second operator in (6.22)

Next, we have to look at

$$D P (I - H_1 H_2) L_x \Lambda P D \quad \text{and} \quad D^{-1} P (I - H_2 H_1) L_y \Lambda P D^{-1}.$$

We find, using (6.14),

$$\begin{aligned} & z_0^{2n^{1/3}s+i+j} ((I - H_1 H_2) L_x \Lambda)_{h+i, h+j} \\ &= \left(\frac{1}{2\pi i}\right)^2 \iint \left(\frac{1+\alpha z}{1-\alpha z}\right)^m \left(\frac{z-\alpha}{z+\alpha}\right)^n \left(\frac{1+\alpha \zeta}{1-\alpha \zeta}\right)^n \left(\frac{\zeta-\alpha}{\zeta+\alpha}\right)^m \\ & \quad \times z_0^{2n^{1/3}s+i+j} \frac{z^{-h-i-1} \zeta^{h+j-1}}{z\zeta-1} dz d\zeta (-1)^{h+j}. \end{aligned}$$

After the substitution of $\zeta \rightarrow -\zeta^{-1}$, this becomes

$$\left(\frac{1}{2\pi i}\right)^2 \iint \psi(z) \psi(\zeta) \left(\frac{z}{z_0}\right)^{-n^{1/3}s-i} \left(\frac{\zeta}{z_0}\right)^{-n^{1/3}s-j} \frac{dz d\zeta}{z(z+\zeta)}.$$

The integrals here are initially taken over circles close to the unit circle, with $|z| > |\zeta|$. We first deform the ζ -contour to Γ^+ while always having $z + \zeta$ nonzero. Then if we deform the z -contour, we pass through a pole at $\zeta = -z$ for every $z \in \Gamma^+$. The residue at the pole equals a constant times $z^{-2h-i-j-2}$, and integrating this over Γ^+ gives zero. So both integrals may be taken over Γ^+ . Since now the denominator does not vanish at $z = \zeta = z_0$, the same sort of argument we already gave shows that this operator equals a constant times $\psi(z_0)^2 n^{-1/3}$ times an operator that scales to the trace class operator $g \text{Ai}(g(s+x)) \otimes g \text{Ai}(g(s+y))$. In particular, its trace norm is $O(\psi(z_0)^2 n^{-1/3})$. This shows that $D P(I - H_1 H_2) L_x \Lambda P D$ has trace norm $O(n^{-1/3})$, and a similar argument applies to $D^{-1} P(I - H_2 H_1) L_y \Lambda P D^{-1}$.

6.4.3. *The last operator in (6.22)*

Finally, we consider the vectors $P a_i$ and $P b'_j$ and look at their constituents $P T_1 e$, $P T_2 e$, $H_1 H_2 e$, and $H_2 H_1 e$. We show that

$$\begin{aligned} \|D P T_1 e\| &= O(n^{-1/6}) & \text{and} & & \|D^{-1} P T_2 e\| &= O(n^{-1/6}), \\ \|D H_1 H_2 e\| &= O(n^{-2/3}) & \text{and} & & \|D^{-1} H_2 H_1 e\| &= O(n^{-2/3}). \end{aligned}$$

For the first, we have

$$(T_1 e)_{h+i} = \frac{1}{2} \pi i \int \psi(z) z^{-n^{1/3}s-i-1} dz = I_n(s + n^{-1/3}(i-1)),$$

and from (6.26) and (6.27) we deduce now that the function

$$n^{1/3} (D P T_1 e)_{[n^{1/3}x]}$$

converges in $L^2(0, \infty)$. In particular, its norm is $O(1)$. But then

$$\|\{(D P T_1 e)\}_i\|_{\ell^2} = n^{1/6} \|(D P T_1 e)_{[n^{1/3}x]}\|_{L^2} = O(n^{-1/6}).$$

Similarly, $\|D^{-1} P T_2 e\| = O(n^{-1/6})$.

For $D H_1 H_2 e$, we have from (6.21),

$$\begin{aligned} & z_0^{n^{1/3}s+i} (H_1 H_2 e)_{h+i} \\ &= \left(\frac{1}{2\pi i}\right)^2 \iint \left(\frac{1+\alpha z}{1-\alpha z}\right)^m \left(\frac{z-\alpha}{z+\alpha}\right)^n \left(\frac{1+\alpha \zeta}{1-\alpha \zeta}\right)^n \left(\frac{\zeta-\alpha}{\zeta+\alpha}\right)^m \\ & \quad \times z^{-h-i-1} z_0^{n^{1/3}s+i} \zeta^{-1} \frac{dz d\zeta}{z\zeta-1}, \end{aligned}$$

and with the substitution $\zeta \rightarrow \zeta^{-1}$, this becomes

$$\left(\frac{1}{2\pi i}\right)^2 \iint \psi(z) \left(\frac{\zeta+\alpha}{\zeta-\alpha}\right)^n \left(\frac{1-\alpha \zeta}{1+\alpha \zeta}\right)^m z^{-n^{1/3}s-i-1} z_0^{n^{1/3}s+i} \frac{dz d\zeta}{z-\zeta}. \tag{6.31}$$

As before, the integrals here are initially taken over circles close to the unit circle, with $|z| > |\zeta|$. Now we want to deform the z -contour to Γ^+ and the ζ -contour to its steepest descent contour C , a curve passing through the saddle points $\pm i\beta^{-1}$ and closing at $-\alpha$ and α^{-1} .

To do this, we show first that, except for z_0 , all points of Γ^+ satisfy $|z| > z_0$. This follows if we can show that on the circle $z = z_0 e^{i\theta}$ the absolute minimum of

$$\log \left| \left(\frac{1 + \alpha z}{1 - \alpha z} \right)^\tau \left(\frac{z - \alpha}{z + \alpha} \right) \right| \tag{6.32}$$

occurs at $\theta = 0$. (For then $|\psi(z)|$ is larger than $\psi(z_0)$ everywhere on the circle except for $z = z_0$, so no other point on the circle could be on Γ^+ . Locally, Γ^+ is outside the circle, and so it has to be everywhere outside.) Using

$$\frac{d}{d\theta} = iz \frac{d}{dz},$$

we find that the derivative with respect to θ of (6.32) equals -2α times the imaginary part of

$$\frac{\tau}{z^{-1} - \alpha^2 z} + \frac{1}{z - \alpha^2 z^{-1}} = \frac{(\tau - \alpha^2)z + (1 - \tau\alpha^2)z^{-1}}{(z^{-1} - \alpha^2 z)(z - \alpha^2 z^{-1})}.$$

This vanishes exactly when the imaginary part of

$$((\tau - \alpha^2)z + (1 - \alpha^2)z^{-1})(\bar{z}^{-1} - \alpha^2 \bar{z})(\bar{z} - \alpha^2 \bar{z}^{-1})$$

does. This is a trigonometric polynomial in θ of degree 3. It is an odd function of θ and so is of the form $\sin\theta$ times a polynomial of degree two in $\cos\theta$. Since it has at least a double zero at $\theta = 0$ (by the choice of c and z_0), the polynomial must have a factor $\cos\theta - 1$. Since it is an odd function of z , it must also have a double zero at $\theta = \pi$, so there must also be a factor $\cos\theta + 1$. Thus it must be equal to a constant times $\sin\theta (\cos^2\theta - 1)$. In particular, there can be no other zeros. Thus (6.32), which we know has a local minimum on the circle at $z = z_0$, must have its absolute minimum there (and its absolute maximum at $z = -z_0$). Thus, as claimed, all points of Γ^+ except for z_0 satisfy $|z| > z_0$.

In particular, all points of Γ^+ satisfy $|z| \geq z_0$. Since $z_0 > \alpha$, we can first take the integrals in (6.31) over the circles $|z| = z_0$ and $|\zeta| = \alpha + \varepsilon$ with ε small and positive. Then we can deform the z -contour to Γ^+ without crossing the circle $|\zeta| = \alpha + \varepsilon$. Next, we want to deform the ζ -contour to C . This curve closes on the right at α^{-1} , and so since $z_0 < \alpha^{-1}$, it intersects Γ^+ at two points, z' and z'' , say. (In principle there could be finitely many other points; the following argument could be easily modified in this case.) Hence upon deforming the ζ -contour to C , we pass through a pole for those z on the arc of Γ^+ passing through $z = z_0$ with end points z' and z'' . For each

z on this arc, the residue at $\zeta = z$ equals $z^{-h-i-1} z_0^{n^{1/3}s+i}$, and then integrating with respect to z gives

$$z_0^{n^{1/3}s} (h+i)^{-1} \left[\frac{z_0^i}{z''^{h+i}} - \frac{z_0^i}{z'^{h+i}} \right].$$

We claim that $\psi(z_0)^{-1}$ (which is the factor contained in D) times this vector is exponentially small, that is, $O(e^{-\delta n})$ for some $\delta > 0$.

Because all points of Γ^+ except z_0 satisfy $|z| > z_0$, the vectors $\{(z_0/z'')^i\}$ and $\{(z_0/z')^i\}$ belong to $\ell^2(\mathbb{Z}_+)$. So we need only show that $\psi(z_0)^{-1} |z'|^{-h}$ is exponentially small (and the same for z''). In fact, on the part of C in the right half-plane,

$$\left| \left(\frac{z+\alpha}{z-\alpha} \right) \left(\frac{1-\alpha z}{1+\alpha z} \right)^\tau \right|$$

is at most 1 and is strictly less than 1 outside a neighborhood of the critical points $\pm i\beta$. This shows that $|z'|^{-c} \leq (1-\delta) |\psi(z')|^{1/n}$ for some $\delta > 0$. Since $|\psi(z)| < |\psi(z_0)|$ on Γ^+ , this shows that $|z'|^{-c} \leq (1-\delta) |\psi(z_0)|^{1/n}$, and it follows that $\psi(z_0)^{-1} |z'|^{-h}$ is exponentially small.

We can now say that, with error $\psi(z_0)^2$ times an exponentially small quantity, the square of the norm of the vector (6.31) equals a quadruple integral in which every term in the integrand except $z^{-h-i-1} z_0^i$ has an analogous term with variables z', ζ' , and the $z^{-h-i-1} z_0^i$ -term becomes $(zz')^{-h}/(z\zeta' - z_0^2)$. For the z - and z' -integrals we integrate over Γ^+ , and for the ζ - and ζ' -integrals we integrate over C . (Here we use again the fact that $|z|, |z'| > z_0$, so we can sum under the integral signs.) The (ζ, ζ') -integrals contribute $O(n^{-1})$, while the (z, z') -integrals contribute $O(\psi(z_0)^2 n^{-1/3})$. Thus $\|DPH_1H_2e\|^2 = O(n^{-4/3})$.*

For H_2H_1e we interchange m and n and make the variable change $z \rightarrow z^{-1}$ but not the variable change $\zeta \rightarrow \zeta^{-1}$. Thus

$$\begin{aligned} & z_0^{-n^{1/3}s-i} (H_2H_1e)_{h+i} \\ &= \left(\frac{1}{2\pi i} \right)^2 \iint \psi(z)^{-1} \left(\frac{1+\alpha\zeta}{1-\alpha\zeta} \right)^n \left(\frac{\zeta-\alpha}{\zeta+\alpha} \right)^m z^{n^{1/3}s+i} z_0^{-n^{1/3}s-i} \frac{dz d\zeta}{\zeta-z}. \end{aligned}$$

Here originally we must have $|\zeta| > |z|$ on the contours, and we want to deform them so that the z -contour becomes Γ^- and the ζ -contour becomes what we again call C . Now in the deformation, we pass through a pole in the ζ -integration for those z on an arc of Γ^- passing through $z = z_0$ with end points that we again call z' and z'' . The

*This was under the basic assumption $\alpha^2 < \tau < \alpha^{-2}$. Otherwise, the ζ - and ζ' -integrals are only $O(1)$, with the result that $\|DH_1H_2e\|$ is only $O(n^{-1/6})$, and the same holds for $\|D^{-1}H_2H_1e\|$. These are still good enough since $o(1)$ is all that is needed.

residue equals $z^{h+i} z_0^{-n^{1/3}s-i}$, and integration with respect to z gives

$$z_0^{-n^{1/3}s} (h + i + 1)^{-1} \left[\frac{z''^{h+i+1}}{z_0^i} - \frac{z'^{h+i+1}}{z_0^i} \right].$$

This is completely analogous to what went before. Now $|z'|, |z''| < z_0$ and $\psi(z_0) |z'|^h$ is exponentially small. We continue as with $H_1 H_2 e$ and find that $\|D^{-1} P H_2 H_1 e\|^2 = O(n^{-4/3})$.

If we go back to the forms of the a_i and b'_j described earlier, we see that the vectors that arise after multiplying by the diagonal matrices D and D^{-1} are exactly the four whose norms we just estimated.

6.5. Recapitulation

We have shown that the matrix (6.22) acting on $\ell^2(\mathbb{Z}_+)$ scales in trace norm to the kernel

$$\begin{pmatrix} g K_{\text{Airy}}(g(s+x), g(x+y)) & 0 \\ 0 & g K_{\text{Airy}}(g(s+x), g(x+y)) \end{pmatrix}$$

acting on $L^2(0, \infty)$. It follows that its Fredholm determinant converges to $F(g s)^2$. In view of (5.1) and (6.10), this establishes that for fixed s ,

$$\lim_{n \rightarrow \infty} P_\sigma(L \leq c n + n^{1/3}s) = F_2(g s),$$

where c is determined by (6.23) and (6.24) and g by (6.30). This gives the statement of the main theorem, where the constants $c_1(\alpha, \tau)$ and $c_2(\alpha, \tau)$ of the introduction are, respectively, c and g^{-1} .

6.6. Computation of $\sigma'''(z_0)$

Think of c and z_0 as functions of τ , which they are. We have

$$\sigma''(z) = 4\alpha z \left[\frac{\alpha^2 \tau}{(1 - \alpha^2 z^2)^2} - \frac{1}{(z^2 - \alpha^2)^2} \right] + \frac{c}{z^2}.$$

Differentiating the identity $0 = \sigma''(z_0)$ with respect to τ and using the above gives

$$0 = \sigma'''(z_0) z'_0 + \frac{4\alpha^3 z_0}{(1 - \alpha^2 z_0^2)^2} + \frac{c'}{z_0^2}, \tag{6.33}$$

where z'_0 and c' denote $dz_0/d\tau$ and $dc/d\tau$, respectively.

From (6.23) and (6.24) we find that c is given in terms of z_0 by the relation

$$c = \frac{4\alpha(1 - \alpha^4) z_0^3}{(1 + \alpha^2 z_0^2)(z_0^2 - \alpha^2)^2}.$$

We compute that

$$\frac{dc}{dz_0} = -4 \frac{\alpha (1 - \alpha^4) z_0^2 (z_0^2 + 3\alpha^2 + 3\alpha^2 z_0^4 + \alpha^4 z_0^2)}{(1 + \alpha^2 z_0^2)^2 (z_0^2 - \alpha^2)^3}. \tag{6.34}$$

From (6.23), τ is given in terms of z_0 by

$$\tau = \frac{(1 - \alpha^2 z_0^2)^2 (\alpha^2 + z_0^2)}{(1 + \alpha^2 z_0^2) (z_0^2 - \alpha^2)^2}.$$

We compute that

$$\frac{d\tau}{dz_0} = - \frac{2(1 - \alpha^4) z_0 (1 - \alpha^2 z_0^2) (\alpha^4 z_0^2 + 3\alpha^2 z_0^4 + 3\alpha^2 + z_0^2)}{(z_0^2 - \alpha^2)^3 (1 + \alpha^2 z_0^2)^2}. \tag{6.35}$$

We first solve (6.33) for $\sigma'''(z_0)$ in terms of z_0' and c' (and z_0). Then we use (6.34) and (6.35) and the relations $z_0' = (d\tau/dz_0)^{-1}$ and $c' = (dc/dz_0) z_0'$. We find that

$$\sigma'''(z_0) = 4 \frac{\alpha (1 - \alpha^4) [(1 + \alpha^4) z_0^2 + 3\alpha^2 (1 + z_0^4)]}{(1 - \alpha^4 z_0^4) (z_0^2 - \alpha^2)^3}.$$

This is positive since $0 < \alpha < 1$ and $\alpha < z_0 < \alpha^{-1}$.

7. Poisson limit of the shifted Schur measure

For Schur measure there are two interesting limiting cases: the exponential limit and the Poisson limit. The exponential limit of shifted Schur measure is supported on standard shifted tableaux, and hence it is expressible in terms of f_s^λ (recall (2.2)). The Poisson limit of shifted Schur measure yields a natural interpretation of the maximizing rule as a symmetry condition of the process.* Namely, if one sets

$$m = n, \quad \alpha = \frac{t}{n}$$

and takes $n \rightarrow \infty$, the percolation-type model described in §3 becomes the following. Consider two Poisson processes of rate t^2 , both in the (same) square $[0, 1] \times [0, 1]$. Hence one can imagine two types of points, marked and unmarked, in the square. Now with probability one, no two points, whether marked or unmarked, have the same x or y coordinates, and hence the strictly increasing conditions (2) and (3) of §3 for the rule of the maximizing path are not necessary. Therefore we do not need to distinguish the unmarked and marked points. Since the union of two Poisson processes has the rate $2t^2$, the resulting process is as follows. In the square $[0, 1] \times [0, 1]$, select Poisson points of rate $2t^2$. Then take the longest path, starting from the lower right corner

*The following remarks are due to the referee.

$(1, 0)$, that follows an up/left path, turns in direction once and only once, and follows an up/right path ending at the upper right corner $(1, 1)$.

It is clear that this length is also equal to the following symmetric version. Take a realization of the Poisson process. Take the mirror image of the points about the left side. Adjoin the mirror image on the left and the original points on the right. Hence the resulting rectangle has sides of lengths 2 and 1, and there are twice as many points of the original configuration that are symmetric about the center vertical line. The (usual) longest up/right path from the left lower corner to the right upper corner is precisely the longest maximizing path from the lower right corner to the upper right corner in the above description.

The (formal) limit $m = n \rightarrow \infty$ with $\alpha = t/n$ (t fixed) in the main theorem of §1 is

$$\lim P\left(\frac{L - 4t}{(2t)^{1/3}} < s\right) = F_2(s). \quad (7.1)$$

The consequence is that the vertically symmetric Poisson process has the same fluctuation as the usual Poisson process with no symmetry condition. Also, the scaling in the above result is consistent with this intuition. There are Poisson points of rate $4t^2$ (double of $2t^2$) in the square of sides 2 and 1. For such the case the limit (7.1) is also valid for the case of no symmetry condition.

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