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Differential Equations for Dyson Processes

Craig A. Tracy[1], **Harold Widom**[2]

[1] Department of Mathematics, University of California, Davis, CA 95616, USA.
E-mail: tracy@math.ucdavis.edu
[2] Department of Mathematics, University of California, Santa Cruz, CA 95064, USA.
E-mail: widom@math.ucsc.edu

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Dedicated to Freeman Dyson on the occasion of his eightieth birthday

Abstract: We call a *Dyson process* any process on ensembles of matrices in which the entries undergo diffusion. We are interested in the distribution of the eigenvalues (or singular values) of such matrices. In the original Dyson process it was the ensemble of $n \times n$ Hermitian matrices, and the eigenvalues describe n curves. Given sets $X_1, \dots, X_m$ the probability that for each k no curve passes through X_k at time τ_k is given by the Fredholm determinant of a certain matrix kernel, the *extended Hermite kernel*. For this reason we call this Dyson process the *Hermite process*. Similarly, when the entries of a complex matrix undergo diffusion we call the evolution of its singular values the *Laguerre process*, for which there is a corresponding *extended Laguerre kernel*. Scaling the Hermite process at the edge leads to the *Airy process* (which was introduced by Prähofer and Spohn as the limiting stationary process for a polynuclear growth model) and in the bulk to the *sine process*; scaling the Laguerre process at the edge leads to the *Bessel process*.

In earlier work the authors found a system of ordinary differential equations with independent variable ξ whose solution determined the probabilities

$$\Pr\,(A(\tau_1) < \xi_1 + \xi, \dots, A(\tau_m) < \xi_m + \xi)\,,$$

where $\tau \to A(\tau)$ denotes the top curve of the Airy process. Our first result is a generalization and strengthening of this. We assume that each X_k is a finite union of intervals and find a system of partial differential equations, with the end-points of the intervals of the X_k as independent variables, whose solution determines the probability that for each k no curve passes through X_k at time τ_k. Then we find the analogous systems for the Hermite process (which is more complicated) and also for the sine process. Finally we find an analogous system of PDEs for the Bessel process, which is the most difficult.

I. Introduction

We call a *Dyson process* any process on ensembles of matrices in which the entries undergo diffusion. In the original Dyson process [3] it was the ensemble of $n \times n$

Hermitian matrices H, where the independent coefficients of each matrix H independently executed Brownian motion subject to a harmonic restoring force. In one dimension this is the familiar Ornstein-Uhlenbeck (velocity) process. The solution to the forward (Fokker-Planck) equation generalizes to the matrix case with the result that the probability density of H at time $\tau = \tau_2$ corresponding to the initial condition $H = H'$ at $\tau = \tau_1$ is a normalization constant depending upon n and q times

$$\exp\left(-\frac{\mathrm{Tr}\ (H - qH')^2}{1-q^2}\right),$$

where $q = e^{\tau_1 - \tau_2}$. As Dyson observed, the equilibrium measure as $\tau_2 \to \infty$ is the GUE measure of random matrix theory. We refer to this particular Dyson process as the *Hermite process* for reasons that will become clear below.

With initial conditions at time τ_1 distributed according to the GUE measure, the probability that at times τ_k $(k = 2, \ldots, m)$,[1] $H(\tau_k)$ is in an infinitesimal neighborhood of H_k is a normalization constant times

$$\exp\left(-\mathrm{Tr}\ H_1^2\right) \prod_{j=2}^{m} \exp\left(-\frac{\mathrm{Tr}\ (H_j - q_{j-1}H_{j-1})^2}{1-q_{j-1}^2}\right) dH_1 \cdots dH_m\,, \qquad (1.1)$$

where $q_j = e^{\tau_j - \tau_{j+1}}$. Alternatively, (1.1) can be interpreted as the equilibrium measure for a chain of m coupled $n \times n$ Hermitian matrices H_k.

In random matrix theory, and more generally Dyson processes, one is interested in the distribution of the eigenvalues (or singular values) of H. It is a classical result of Gaudin [6] that the distribution functions for the eigenvalues in GUE are expressible in terms of the Fredholm determinant of an integral kernel called the Hermite kernel. In the process interpretation, the evolution of the eigenvalues can be thought of as consisting of n curves parametrized by time. Given $\tau_1 < \cdots < \tau_m$ and subsets X_k of $\mathbf{R}$, the quantity of interest is the probability that for all k no curve passes through X_k at time τ_k. It follows from the work of Eynard and Mehta [4] that this probability is also expressible as the Fredholm determinant of an *extended Hermite kernel*, an $m \times m$ matrix kernel related to the kernel associated with the random matrix ensemble corresponding to the equilibrium distribution.[2]

Here is how it is derived. One first diagonalizes each H_k and then employs the Harish-Chandra/Itzykson-Zuber integral (see, e.g. [18]) to integrate out the unitary parts. The result is that the induced measure on eigenvalues has a density $P(\lambda_{11}, \ldots, \lambda_{1n}; \ldots; \lambda_{m1}, \ldots, \lambda_{mn})$ given up to a normalization constant by

$$\prod_{k=1}^{m} e^{-\left(\frac{1}{1-q_{k-1}^2} + \frac{q_k^2}{1-q_k^2}\right)\sum_{i=1}^n \lambda_{k,i}^2} \prod_{k=1}^{m-1} \det\left(e^{\frac{2q_k}{1-q_k^2}\lambda_{k,i}\,\lambda_{k+1,j}}\right) \Delta(\lambda_1)\,\Delta(\lambda_m), \qquad (1.2)$$

where $q_0 = q_m = 0$ and Δ denotes Vandermonde determinant.[3]

In [4] it was shown that for a chain of coupled matrices with probability density of this type the correlation functions could be expressed as block determinants whose entries

[1] We are assuming $\tau_1 < \cdots < \tau_m$.

[2] This was described in a lecture by Kurt Johansson [8], who recently communicated to us a sketch of his derivation [9]. Matrix kernels, of a different kind, also appear in [1].

[3] This expression shows the connection with the theory of determinantal processes, in which probability densities are defined by products of determinants [5, 7, 12].

are matrix kernels evaluated at the various points, generalizing Dyson's expression for the correlation functions for a single matrix. As with the case of random matrices, one could then get a Fredholm determinant representation for the probability that for each k no curve passes through X_k at time τ_k. In the case at hand the matrix kernel

$$L(x, y) = (L_{ij}(x, y))_{i,j=1}^{m}$$

is the extended Hermite kernel and has entries

$$L_{ij}(x, y) = \begin{cases} \sum_{k=0}^{n-1} e^{k(\tau_i-\tau_j)}\, \varphi_k(x)\, \varphi_k(y) & \text{if } i \ge j, \\ -\sum_{k=n}^{\infty} e^{k(\tau_i-\tau_j)}\, \varphi_k(x)\, \varphi_k(y) & \text{if } i < j. \end{cases} \tag{1.3}$$

Here φ_k are the harmonic oscillator functions $e^{-x^2/2}\, p_k(x)$, where the p_k are the normalized Hermite polynomials. If K is the operator with matrix kernel (K_{ij}), where

$$K_{ij}(x, y) = L_{ij}(x, y)\, \chi_{X_j}(y),$$

then the probability that for each k no curve passes through X_k at time τ_k is equal to $\det(I - K)$. In the special case $X_k = (\xi_k, \infty)$ this is the probability that the largest eigenvalue at time τ_k is at most ξ_k.

It is natural to consider also the evolution of the singular values of complex matrices. This is the Dyson process on the space of $p \times n$ complex matrices. (We always take $p \ge n$.) The analogue of (1.1) here is [5]

$$\exp\left(-\text{Tr } A_1^* A_1\right) \prod_{j=2}^{m} \exp\left(-\frac{\text{Tr }\left((A_j - q_j A_{j-1})^*(A_j - q_j A_{j-1})\right)}{1-q_j^2}\right) \times dA_1 \cdots dA_m. \tag{1.4}$$

After integration over the unitary parts this becomes a normalization constant times

$$\prod_{k=1}^{m} e^{-\left(\frac{1}{1-q_{k-1}^2}+\frac{q_k^2}{1-q_k^2}\right)\sum_{i=1}^{n}\lambda_{ki}} \prod_{k=1}^{m-1} \det\left(I_\alpha\left(\frac{2q_{k+1}}{1-q_{k+1}^2}\sqrt{\lambda_{k,i}\,\lambda_{k+1,j}}\right)\right) \times \Delta(\lambda_1)\, \Delta(\lambda_m) \prod_{i=1}^{n} \lambda_{1i}^{\alpha/2} \prod_{i=1}^{n} \lambda_{mi}^{\alpha/2}\, d\lambda_{11} \cdots d\lambda_{mn}, \tag{1.5}$$

where I_α is the modified Bessel function and $\alpha = p - n$. (The λ_{ki} are the squares of the singular values.) This is of the same general form as for the Hermite process, and here also there is a corresponding matrix kernel, the *extended Laguerre kernel*. It is given by the same formulas (1.3) as before, but now $\varphi_k(x) = x^{\alpha/2} e^{-x/2}\, p_k(x)$, where the p_k are the Laguerre polynomials L_k^α, normalized.

These processes have scaling limits. If we scale the Hermite process at the edge we obtain the *Airy process* with corresponding *extended Airy kernel* [7, 13]

$$L_{ij}(x, y) = \begin{cases} \int_0^\infty e^{-z(\tau_i-\tau_j)}\, \text{Ai}(x+z)\, \text{Ai}(y+z)\, dz & \text{if } i \ge j, \\ -\int_{-\infty}^0 e^{-z(\tau_i-\tau_j)}\, \text{Ai}(x+z)\, \text{Ai}(y+z)\, dz & \text{if } i < j. \end{cases} \tag{1.6}$$

The Airy process consists of infinitely many curves and as before $\det(I-K)$ is the probability that no curve passes through X_k at time τ_k. In the case of greatest interest $X_k = (\xi_k,\ \infty)$, and then the determinant is equal to the probability

$$\Pr\,(A(\tau_1) < \xi_1, \dots, A(\tau_m) < \xi_m)\,, \tag{1.7}$$

where $A(\tau)$ is the top curve of the Airy process. (This is what has been called the Airy process in the literature. It is convenient for us to use the different terminology.)

If we scale the Hermite process in the bulk we obtain the *sine process* with the associated *extended sine kernel*

$$L_{ij}(x,y) = \begin{cases} \displaystyle\int_0^1 e^{z^2(\tau_i-\tau_j)} \cos z(x-y)\,dz & \text{if } i \ge j, \\ \displaystyle -\int_1^\infty e^{z^2(\tau_i-\tau_j)} \cos z(x-y)\,dz & \text{if } i < j. \end{cases}$$

If we scale the Laguerre process at the bottom (the "hard edge") we obtain the *Bessel process* and its associated *extended Bessel kernel*

$$L_{ij}(x,y) = \begin{cases} \displaystyle\int_0^1 e^{z^2(\tau_i-\tau_j)/2}\, \Phi_\alpha(xz)\,\Phi_\alpha(yz)\,dz & \text{if } i \ge j, \\ \displaystyle -\int_1^\infty e^{z^2(\tau_i-\tau_j)/2}\, \Phi_\alpha(xz)\,\Phi_\alpha(yz)\,dz & \text{if } i < j, \end{cases}$$

where $\Phi_\alpha(z) = \sqrt{z}\, J_\alpha(z)$.[4]

The Airy process $A(\tau)$ was introduced by Prähofer and Spohn [13] as the limiting stationary process for a polynuclear growth model. (See also [7].) It is conjectured that it is in fact the limiting process for a wide class of random growth models. Thus it is more significant than the Hermite process. It might be expected that likewise the sine process (possibly) and the Bessel process (more likely) will prove to be more significant than the unscaled processes.

For $m = 1$ the extended Airy kernel reduces to the Airy kernel and it is known [14] that then (1.7) is expressible in terms of a solution to Painlevé II. It was thus natural for the authors of [7, 13] to conjecture that the m-dimensional distribution functions (1.7) are also expressible in terms of a solution to a system of differential equations. This conjecture was established in two different forms, by the authors in [17] and by Adler and van Moerbeke for $m = 2$ in [2].[5] Specifically, in [17] we found a system of ordinary differential equations with independent variable ξ whose solution determined the probabilities

$$\Pr\left(A_{\tau_1} < \xi_1 + \xi, \dots, A_{\tau_1} < \xi_m + \xi\right).$$

The ξ_k appeared as parameters in the equations.

Our first result is a generalization and strengthening of this. We assume that each X_k is a finite union of intervals rather than a single interval, and find a total system of partial differential equations, with the end-points of the intervals of the X_k as independent

[4] Hints of this kernel for $m = 2$ appear in [10].

[5] In [1] the authors had already considered the Hermite process in the case $m = 2$, in our terminology, and found a PDE in $\tau = \tau_2 - \tau_1$ and the end-points of X_1 and X_2 for the probability that at time τ_i no curve passes through X_i. In [2] they deduced for the Airy process by a limiting argument a PDE in $\xi_1,\ \xi_2$ and $\tau = \tau_1 - \tau_2$ when $X_i = (\xi_i,\ \infty)$. These equations and those we find appear to be unrelated.

variables, whose solution determines $\det(I - K)$. (When $X_k = (\xi_k, \infty)$ it it easy to recover the system of ODEs found in [17].)

Then we find the analogous systems for the Hermite process (which is more complicated) and also for the sine process. Finally we find a system of PDEs for the Bessel process, which was the most difficult. It is possible that we could find a system for the Laguerre process also, but it would be even more complicated (since Laguerre:Bessel::Hermite:Airy) and probably of less interest.

All of these equations in a sense generalize those for the Hermite, Airy, sine and Laguerre kernels found in [15], which are the cases when $m = 1$. Although some of the ingredients are the same, the equations derived here when $m = 1$ are not the same, as those of [15]. For example, the special case of the extended Airy equations for a semi-infinite interval and $m = 1$ is the Painlevé II equation whereas in [15] one had to do a little work to get to Painlevé II from the equations.

We begin with Sect. II, where we revisit the class of probabilities for which the correlation functions were derived in [4] and give a direct derivation of the corresponding Fredholm determinant representations. The method has similarities to that of [4] (in fact we adopt much of their notation) and the results are equivalent. But we avoid some awkward combinatorics. Our derivation is analogous to that of [16] for random matrix ensembles whereas the method of [4] is more like that in [11].

In Sect. III we use the previous result to derive the extended Hermite kernel. This is of course not new. But since the derivation does not seem to have seen print before, this seems a reasonable place to present it.

In the following sections we derive the systems of PDEs for the extended Airy, Hermite and sine kernels. Presumably the other two could be obtained by scaling the equations for Hermite, but Airy is simpler and so we do it first. Moreover all the systems will have the same general form, and doing Airy first will simplify the other derivations.

In Sect. VII we derive the extended Laguerre kernel, and in Sect. VIII establish the system of PDEs for the extended Bessel kernel.

II. Extended Kernels

For the most part we shall follow the notation in [4]. We assume the probability density for the eigenvalues λ_{ki} $(i = 1, \dots, n,\ k = 1, \dots, m)$ is given up to a normalization constant by

$$P(\lambda_{11}, \dots, \lambda_{1n}; \dots; \lambda_{m1}, \dots, \lambda_{mn}) = \prod_{k=1}^{m} e^{-\sum_{i=1}^{n} V_k(\lambda_{ki})} \prod_{k=1}^{m-1} \det(u_k(\lambda_{k,i},\ \lambda_{k+1,j}))\, \Delta(\lambda_1)\, \Delta(\lambda_m), \tag{2.1}$$

where V_k and u_k are given functions satisfying some general conditions and Δ denotes the Vandermonde determinant. (Indices $i,\ j$ in the determinants run from 1 to n, and here λ_1 resp. λ_m denotes λ_{1i} resp. λ_{mi}.) What we are interested in is the expected value of

$$\prod_{k=1}^{m} \prod_{i=1}^{n} (1 + f_k(\lambda_{ki})),$$ [6]

so we integrate this times P over all the λ_{ki}.

[6] In our applications f_k will be minus the characteristic function of X_k, so the expected value will equal the probability that $\lambda_{ki} \notin X_k$ for all k and i.

We apply the general identity

$$\int \cdots \int \det(\varphi_j(x_k))_{j,k=1}^n \cdot \det(\psi_j(x_k))_{j,k=1}^n \, d\mu(x_1) \cdots d\mu(x_n)$$
$$= n! \det\left(\int \varphi_j(x)\, \psi_k(x)\, d\mu(x)\right)_{j,k=1}^n$$

to the integral over $\lambda_{11}, \ldots, \lambda_{1n}$, with the part of the integrand containing these variables. This includes two determinants, $\Delta(\lambda_1)$ and the factor $\det(u_1(\lambda_{1i}, \lambda_{2j}))$. The result is that this n-tuple integral is replaced by the determinant

$$\det\left(\int \lambda_1^i e^{-V(\lambda_1)}\, u_1(\lambda_1, \lambda_{2j})\, (1 + f_1(\lambda_1))\, d\lambda_1\right).$$

Then we use the same identity to rewrite the integral with respect to the λ_{2i} using this determinant and the factor $\det(u_2(\lambda_{2i}, \lambda_{3j}))$. And so on. At the end we use the determinant coming from the previous use of the identity and $\Delta(\lambda_m)$. The end result is that the expected value in question is a constant times the determinant of the matrix with i, j entry

$$\int \cdots \int \lambda_1^i \lambda_m^j \, e^{-\sum_{k=1}^m V_k(\lambda_k)} \prod_{k=1}^{m-1} u_k(\lambda_k, \lambda_{k+1}) \prod_{k=1}^m (1 + f_k(\lambda_k))\, d\lambda_1 \cdots d\lambda_m. \quad (2.2)$$

By changing the normalization factor we may replace λ_1^i by any sequence of polynomials, which we call $P_{1i}(\lambda_1)$, and replace λ_m^j by any sequence of polynomials, which we call $Q_{mj}(\lambda_m)$. We choose them so that after these replacements the integral with all the f_k set equal to zero equals δ_{ij}. In particular the normalization constant is now equal to 1.

If we write

$$e^{-\sum_{k=1}^m V_k(\lambda_k)} \prod_{k=1}^{m-1} u_k(\lambda_k, \lambda_{k+1}) = E_{12}(\lambda_1, \lambda_2)\, E_{23}(\lambda_2, \lambda_3) \cdots E_{m-1,m}(\lambda_{m-1}, \lambda_m) \quad (2.3)$$

(there is some choice in the factors on the right), we see that the matrix in question equals the identity matrix plus the matrix with i, j entry

$$\int \cdots \int P_{1i}(\lambda_1) \prod_{k=1}^{m-1} E_{k,k+1}(\lambda_k, \lambda_{k+1}) \left[\prod_{k=1}^m (1 + f_k(\lambda_k)) - 1\right] Q_{mj}(\lambda_m)\, d\lambda_1 \cdots d\lambda_m.$$

The bracketed expression may be written as a sum of products,

$$\sum_{r \geq 1} \sum_{k_1 < \cdots < k_r} f_{k_1}(\lambda_1) \cdots f_{k_r}(\lambda_{k_r}).$$

Correspondingly the integral is a sum of integrals. Consider the integral corresponding to the above-displayed summand. For $k > j$ we define

$$E_{jk}(\lambda_j, \lambda_k) = E_{j,j+1}(\lambda_j, \lambda_{j+1}) * \cdots * E_{k-1,k}(\lambda_{k-1}, \lambda_k),$$

where the asterisk denotes kernel composition, and set

$$P_{ki}(\lambda_k) = \int P_{1i}(\lambda_1)\; E_{1k}(\lambda_1, \lambda_k)\, d\lambda_1, \quad Q_{kj}(\lambda_k) = \int E_{km}(\lambda_k, \lambda_m)\, Q_{mj}(\lambda_m)\, d\lambda_m.$$

By integrating first with respect to the λ_k with $k \neq k_1, \dots, k_r$, we see that the corresponding integral is equal to

$$\int \cdots \int f_{k_1}(\lambda_{k_1})\, P_{k_1,i}(\lambda_{k_1})\, E_{k_1,k_2}(\lambda_{k_1}, \lambda_{k_2}) f_{k_2}(\lambda_{k_2}) \cdots \\ \cdots E_{k_{r-1},k_r}(\lambda_{k_{r-1}}, \lambda_{k_r}) f_{k_r}(\lambda_{k_r})\, Q_{k_r,j}(\lambda_{k_r})\, d\lambda_{k_1} \cdots d\lambda_{k_r}.$$

We deliberately distributed the f factors as we did since if we let $A_{k,\ell}$ be the operator with kernel $A_{k\ell}(\lambda_k, \lambda_\ell) = E_{k\ell}(\lambda_k, \lambda_\ell) f(\lambda_\ell)$ then the above may be written as the single integral

$$\int f_{k_1}(\lambda)\, P_{k_1,i}(\lambda)\, A_{k_1,k_2} \cdots A_{k_{r-1},k_r} Q_{k_r,j}(\lambda)\, d\lambda.$$

(If $r = 1$ we interpret the operator product to be the identity.) Replacing the index k_1 by k and changing notation, we see that the sum of all of these equals

$$\int \sum_k f_k(\lambda)\, P_{k,i}(\lambda) \left(\sum_{r \geq 0} \sum_{k_1,\dots,k_r} A_{k,k_1} A_{k_1,k_2} \cdots A_{k_{r-1},k_r} Q_{k_r,j}(\lambda) \right) d\lambda,$$

where the inner sum runs over all $k_r > \cdots > k_1 > k$. (If $r = 0$ the inner sum is interpreted to be $Q_{k,j}(\lambda)$.)

We think of $f_k(\lambda)\, P_{k,i}(\lambda)$ as the k^{th} entry of a row matrix and the inner sum

$$\sum_{r \geq 0} \sum_{k_1,\dots,k_r} A_{k,k_1} A_{k_1,k_2} \cdots A_{k_{r-1},k_r} Q_{k_r,j}(\lambda)$$

as the k^{th} entry of a column matrix. The integrand is the product of these matrices. If we use the general fact that $\det(I + ST) = \det(I + TS)$ we see that the determinant of I plus the matrix with the above i, j entry is equal to the determinant of I plus the operator with matrix kernel having k, ℓ entry

$$\sum_{j=0}^{n-1} \left(\sum_{r \geq 0} \sum_{k_1,\dots,k_r} A_{k,k_1} A_{k_1,k_2} \cdots A_{k_{r-1},k_r} Q_{k_r,j}(\lambda) \right) P_{\ell,j}(\mu) f_\ell(\mu),$$

where in the inner sum $k_r > \cdots > k_1 > k$.

This is the k, ℓ entry of a certain operator matrix acting from the left on the matrix with k, ℓ entry

$$\sum_{j=0}^{n-1} Q_{k,j}(\lambda)\, P_{\ell,j}(\mu) f_\ell(\mu).$$

That matrix is upper-triangular, all diagonal entries are I, and for $k < \ell$ the k, ℓ entry equals

$$\sum_{k<k_1<\cdots<k_r<\ell} A_{k,k_1} A_{k_1,k_2} \cdots A_{k_r,\ell}.$$

Elementary algebra shows (even for non-commuting variables $A_{k\ell}$) that this is the inverse of the upper-triangular matrix with diagonal entries I and k, ℓ entry $-A_{k\ell}$ otherwise.

If we recall that $A_{k\ell}(\lambda, \mu) = E_{k\ell}(\lambda\, \mu)\, f_\ell(\mu)$ then we see that we have shown the following: Let $H(\lambda, \mu)$ be the matrix kernel given by

$$H_{k\ell}(\lambda, \mu) = \sum_{j=0}^{n-1} Q_{k,j}(\lambda)\, P_{\ell,j}(\mu),$$

let E be the matrix kernel with k, ℓ entry $E_{k\ell}(\lambda, \mu)$ (thought of as 0 when $k \geq l$), and let $f(\mu) = \text{diag}\,(f_k(\mu))$. Then the expected value equals the determinant of

$$I + (I - Ef)^{-1} Hf = (I - Ef)^{-1}\, [I + (H - E) f].$$

The factor on the left equals I plus a strictly upper-triangular matrix, so its determinant equals one. Therefore the expected value equals

$$\det\, [I + (H - E) f],$$

and $H - E$ is the extended kernel.

III. The Extended Hermite Kernel

We have times $\tau_1 < \cdots < \tau_m$ and we set $q_k = e^{\tau_k - \tau_{k+1}}$, with the conventions $\tau_0 = -\infty,\ \tau_{m+1} = +\infty$ so that $q_0 = q_m = 0$. For the Hermite process the probability density is given by (1.2) so we are in the case where

$$V_k(\lambda) = \left(\frac{1}{1 - q_{k-1}^2} + \frac{q_k^2}{1 - q_k^2} \right) \lambda^2, \quad u_k(\lambda,\ \mu) = \exp \left\{ \frac{2q_k}{1 - q_k^2} \lambda\mu \right\},$$

and we want to compute the kernel $H - E$ of Sect. II.

We define the Mehler kernel

$$K(q;\ \lambda, \mu) = (\pi(1 - q^2))^{-1/2}\, e^{-\frac{q^2}{1-q^2}\lambda^2 - \frac{1}{1-q^2}\mu^2 + \frac{2q}{1-q^2}\lambda\mu},$$

which has the representation

$$K(q;\ \lambda, \mu) = \sum_{i=0}^{\infty} q^i\ p_i(\lambda)\, p_i(\mu)\, e^{-\mu^2}, \tag{3.1}$$

so

$$\int K(q;\ \lambda, \mu)\, p_i(\mu)\, d\mu = q^i\ p_i(\lambda). \tag{3.2}$$

Here p_i are the normalized Hermite polynomials.

We can write the exponent on the left side of (2.3) as

$$-\sum_{k=1}^{m-1} \frac{q_k^2}{1-q_k^2}\lambda_k^2 - \sum_{k=0}^{m-1} \frac{1}{1-q_k^2}\lambda_{k+1}^2,$$

and so, aside from a normalization constant, the left side of (2.3) is equal to

$$e^{-\lambda_1^2} \prod_{k=1}^{m-1} K(q_k;\ \lambda_k, \lambda_{k+1}).$$

Thus we may take in (2.3)

$$E_{12}(\lambda_1, \lambda_2) = e^{-\lambda_1^2}\, K(q_1;\ \lambda_1, \lambda_2),$$
$$E_{k,k+1}(\lambda_k, \lambda_{k+1}) = K(q_k;\ \lambda_k, \lambda_{k+1}), \quad (k > 1).$$

It follows from (3.1) that $K(q) * K(q') = K(qq')$ when $q,\ q' > 0$ and so

$$E_{1k}(\lambda, \mu) = e^{-\lambda^2}\, K(q_1 \cdots q_{k-1};\ \lambda, \mu).$$

In particular we deduce from (3.2) that

$$\int\int p_i(\lambda)\, E_{1m}(\lambda,\mu)\, p_j(\mu)\, d\mu d\lambda = (q_1 \cdots q_{m-1})^j \int p_i(\lambda)\, e^{-\lambda^2}\, p_j(\lambda)\, d\lambda$$
$$= (q_1 \cdots q_{m-1})^j\, \delta_{ij}.$$

Hence we may take

$$P_{1i} = p_i, \quad Q_{mj} = (q_1 \cdots q_{m-1})^{-j}\, p_j$$

as the polynomials in the previous discussion. We see that

$$P_{ki}(\mu) = \int p_i(\lambda)\, e^{-\lambda^2}\, K(q_1 \cdots q_{k-1};\ \lambda, \mu)\, d\lambda$$
$$= (q_1 \cdots q_{k-1})^i\, p_i(\mu)\, e^{-\mu^2}, \quad (k > 1),$$
$$Q_{kj}(\lambda) = \int K(q_k \cdots q_{m-1};\ \lambda, \mu)\, Q_{mj}(\mu) d\mu = (q_1 \cdots q_{k-1})^{-j}\, p_j(\lambda), \quad (k > 1),$$
$$Q_{1j}(\lambda) = \int e^{-\lambda^2} K(q_1 \cdots q_{m-1};\ \lambda, \mu)\, Q_{mj}(\mu) d\mu = e^{-\lambda^2}\, p_j(\lambda).$$

It follows that H is the matrix with k, ℓ entry

$$\sum_{j=0}^{n-1} \left(\frac{q_1 \cdots q_{\ell-1}}{q_1 \cdots q_{k-1}}\right)^j\, p_j(\lambda)\, p_j(\mu)$$

left-multiplied by the matrix $\mathrm{diag}\,(e^{-\lambda^2}\ 1\ \cdots\ 1)$ and right-multiplied by the matrix $\mathrm{diag}\,(1\ e^{-\mu^2}\ \cdots\ e^{-\mu^2})$. Similarly E is the strictly upper-triangular matrix with k, ℓ entry $K(q_k \cdots q_{\ell-1};\ \lambda, \mu)$ left-multiplied by the matrix $\mathrm{diag}\,(e^{-\lambda^2}\ 1\ \cdots\ 1)$.

Thus we have computed $H - E$. The actual extended Hermite kernel will be a modification of this. The determinant is unchanged if we multiply $H - E$ on the left by diag $(e^{\lambda^2/2}\ e^{-\lambda^2/2}\ \cdots\ e^{-\lambda^2/2})$ and on the right by diag $(e^{-\mu^2/2}\ e^{\mu^2/2}\ \cdots\ e^{\mu^2/2})$. Recalling that φ_i are the harmonic oscillator functions and recalling the definition of the q_k in terms of the τ_k we see that the expected value in question is equal to

$$\det\,[I + (\hat{H} - \hat{E})f],$$

where

$$\hat{H}_{k\ell}(\lambda,\mu) = \sum_{j=0}^{n-1} e^{j\,(\tau_k-\tau_\ell)}\,\varphi_j(\lambda)\,\varphi_j(\mu),$$

and $\hat{E}$ is the strictly upper-triangular matrix with k,ℓ entry

$$e^{(\mu^2-\lambda^2)/2}\,K(e^{\tau_k-\tau_\ell}\,;\,\lambda,\mu).$$

If we observe that by (3.1)

$$e^{(\mu^2-\lambda^2)/2}\,K(e^{\tau_k-\tau_\ell}\,;\,\lambda,\mu) = \sum_{j=0}^{\infty} e^{j\,(\tau_k-\tau_\ell)}\,\varphi_j(\lambda)\,\varphi_j(\mu)$$

when $k < \ell$ we see that $\hat{H} - \hat{E}$ has k,ℓ entry

$$(\hat{H} - \hat{E})_{k\ell} = \begin{cases} \sum\limits_{j=0}^{n-1} e^{j\,(\tau_k-\tau_\ell)}\,\varphi_j(\lambda)\,\varphi_j(\mu) & \text{if } k \ge \ell, \\ -\sum\limits_{j=n}^{\infty} e^{j\,(\tau_k-\tau_\ell)}\,\varphi_j(\lambda)\,\varphi_j(\mu) & \text{if } k < \ell, \end{cases}$$

which is the extended Hermite kernel (1.3).

IV. PDEs for the Extended Airy Kernel

We consider first the case $X_k = (\xi_k, \infty)$, so that

$$\det\,(I - K) = \Pr\,\left(A(\tau_1) < \xi_1, \ldots, A(\tau_m) < \xi_m\right).$$

The derivation is simplest here but it will also give the main ideas for all the derivations.

Observe first that

$$\partial_k\,K = -L\,\delta_k, \tag{4.1}$$

where δ_k denotes multiplication by the diagonal matrix with all entries zero except for the k^{th}, which equals $\delta(y - \xi_k)$. It follows that if we let $R = K\,(I - K)^{-1}$, then

$$\partial_k \log\,\det(I - K) = -\mathrm{Tr}\,(I - K)^{-1}\,\partial_k K = R_{kk}(\xi_k, \xi_k).$$

The matrix entries on the right will be among the unknowns. To explain the others, let $A(x)$ denote the $m \times m$ diagonal matrix diag $(\mathrm{Ai}(x))$ and $\chi(x)$ the diagonal matrix diag $(\chi_k(x))$, where $\chi_k = \chi_{(\xi_k,\infty)}$. Then we define the matrix functions $Q(x)$ and $\tilde{Q}(x)$ by

$$Q = (I - K)^{-1} A, \quad \tilde{Q} = A\,\chi\,(I - K)^{-1}$$

(where for $\tilde{Q}$ the operators act on the right). These and $R(x, y)$ are functions of the ξ_k as well as x and y. We define the matrix functions q, $\tilde{q}$ and r of the ξ_j only by

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j), \quad r_{ij} = R_{ij}(\xi_i, \xi_j).$$ [7]

Our unknown functions will be these and the matrix functions q' and $\tilde{q}'$ defined by

$$q'_{ij} = Q'_{ij}(\xi_i), \quad \tilde{q}'_{ij} = \tilde{Q}'_{ij}(\xi_j).$$

We shall also write r_x and r_y for the matrices $(R_{xij}(\xi_i, \xi_j))$ and $(R_{yij}(\xi_i, \xi_j))$.

The ξ_k are the independent variables in our equations. We denote by ξ the matrix diag (ξ_k) and by $d\xi$ the matrix of differentials diag $(d\xi_k)$. With these notations our system of equations is

$$dr = -r\,d\xi\,r + d\xi\,r_x + r_y\,d\xi, \tag{4.2}$$
$$dq = d\xi\,q' - r\,d\xi\,q, \tag{4.3}$$
$$d\tilde{q} = \tilde{q}'\,d\xi - \tilde{q}\,d\xi\,r, \tag{4.4}$$
$$dq' = d\xi\,\xi\,q - (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q', \tag{4.5}$$
$$d\tilde{q}' = \tilde{q}\,\xi\,d\xi - \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi. \tag{4.6}$$

One sees that the right sides involve the diagonal entries of $r_x + r_y$ and the off-diagonal entries of r_x and r_y. We shall show below that these are "known" in the sense that they are expressible algebraically in terms of our unknown functions, so the above is a closed system of PDEs.

We begin by establishing the assertions about r_x and r_y.

In the following $D = d/dx$, we set $\rho = (I - K)^{-1}$ and $\delta = \sum_k \delta_k$, and τ is the diagonal matrix diag (τ_k). We denote by Θ the matrix with all entries equal to one. For clarity we sometimes write the kernel of an operator in place of the operator itself.

Lemma 1. *We have the commutator relation*

$$[D,\ R] = -Q(x)\,\Theta\,\tilde{Q}(y) + R\,\delta\,\rho + [\tau,\ R]. \tag{4.7}$$

Proof. Integrating by parts in (1.6) gives

$$[D,\ K]_{ij} = -\mathrm{Ai}(x)\,\mathrm{Ai}(y)\,\chi_j(y) + L_{ij}(x, \xi_j)\,\delta(y - \xi_j) + (\tau_i - \tau_j)\,K_{ij}(x,\ y).$$

Equivalently,

$$[D,\ K] = -A(x)\,\Theta\,A(y)\,\chi(y) + L\,\delta + [\tau,\ K].$$

To obtain $[D,\ R]$ we replace K by $K - I$ in the commutators and left- and right-multiply by ρ. The result is (4.7).[8] □

[7] At points of discontinuity we always take limits from the right. For example we interpret $R_{ij}(x,\ \xi_j)$ as the limit $R_{ij}(x,\ \xi_j+)$.

[8] Because of the fact $\rho\,L\,\chi = R$ and our interpretation of $R_{ij}(x,\ \xi_j)$ as $R_{ij}(x,\ \xi_j+)$ we are able to write $R\,\delta\,\rho$ in place of $\rho\,L\,\delta\,\rho$.

If we take the $i,\ j$ entry of both sides of (4.7) and set $x = \xi_i,\ \ y = \xi_j$ we obtain

$$r_x + r_y = -q\,\Theta\,\tilde{q} + r^2 + [\tau,\ r]. \tag{4.8}$$

Thus all entries of $r_x + r_y$ are known.

For the off-diagonal entries of r_x and r_y we need a second commutator identity. Here M is multiplication by x.

Lemma 2. *We have*

$$[D^2 - M,\ \rho] = R\,\delta\,\rho_x - R_y\,\delta\,\rho,$$

where $R_y(x,\,y)$ is interpreted as not containing a delta-function summand.

Proof. We use the facts that $D^2 - M$ commutes with L and that M commutes with χ. These give

$$[D^2 - M,\ K] = [D^2 - M,\ L\,\chi] = L\,[D^2 - M,\ \chi] = L\,[D^2,\ \chi] = L\,(\delta\,D + D\,\delta).$$

Using the commutator identity

$$[T,\ (I - K)^{-1}] = (I - K)^{-1}\,[T,\ K]\,(I - K)^{-1},$$

valid for any operators T and K, we deduce

$$[D^2 - M,\ \rho] = \rho\,L\,\delta\,D\,\rho + \rho\,L\,D\,\delta\,\rho.$$

The first term on the right equals $R\,\delta\,\rho_x$. The second term equals $-R_y\,\delta\,\rho$ where R_y is interpreted as not containing the delta-function summand. This establishes the lemma. □

Lemma 1 says

$$R_x + R_y = -Q(x)\,\Theta\,\tilde{Q}(y) + R\delta\rho + [\tau,\ R],$$

and applying $\partial_x - \partial_y$ to both sides gives

$$R_{xx} - R_{yy} = -Q'(x)\,\Theta\,\tilde{Q}(y) + Q(x)\,\Theta\,\tilde{Q}'(y) + R_x\delta\rho - R\delta\rho_y + [\tau,\ R_x - R_y].$$

Lemma 2 says

$$R_{xx} - R_{yy} - (x - y)\,R = R\delta\rho_x - R_y\delta\rho.$$

Equating the two expressions for $R_{xx} - R_{yy}$ gives

$$\begin{aligned}(x - y)\,R(x,\,y) = -Q'(x)\,\Theta\,\tilde{Q}(y) + Q(x)\,\Theta\,\tilde{Q}'(y) + (R_x + R_y)\delta\rho \\ -R\delta(\rho_x + \rho_y) + [\tau,\ R_x - R_y].\end{aligned} \tag{4.9}$$

Taking the $i,\,j$ entries and setting $x = \xi_i,\,y = \xi_j$ give

$$[\xi,\ r] + r\,r_x - r_y\,r = -q'\,\Theta\,\tilde{q} + q\,\Theta\,\tilde{q}' + r_x\,r - r\,r_y + [\tau,\ r_x - r_y],$$

or

$$[\tau,\ r_x - r_y] = q'\,\Theta\tilde{q} - q\,\Theta\,\tilde{q}' + [r,\ r_x + r_y] + [\xi,\ r]. \tag{4.10}$$

The left side has i, j entry $(\tau_i - \tau_j)\,(r_{xij} - r_{yij})$ and the right side is known.[9] Therefore the off-diagonal entries of $r_x - r_y$ are known, and therefore so also are the off-diagonal entries of r_x and r_y individually.

To be more explicit we define matrices U and V by

$$U = -q\,\Theta\,\tilde{q} + r^2 + [\tau,\, r]$$

and

$$V_{ij} = \frac{(q'\,\Theta\,\tilde{q} - q\,\Theta\,\tilde{q}')_{ij} + [r,\, -q\,\Theta\,\tilde{q} + [\tau,\, r] - \xi]_{ij}}{\tau_i - \tau_j}$$

when $i \neq j$. Then (4.8) says

$$r_{xij} + r_{yij} = U_{ij}$$

and (4.10) gives

$$r_{xij} - r_{yij} = V_{ij}$$

when $i \neq j$. It follows that for such i, j we have

$$\begin{aligned} d\xi_i\, r_{xij} + d\xi_j\, r_{yij} &= \frac{1}{2}(d\xi_i + d\xi_j)U_{ij} + \frac{1}{2}(d\xi_i - d\xi_j)V_{ij}, \\ d\xi_i\, r_{yij} + d\xi_j\, r_{xij} &= \frac{1}{2}(d\xi_i + d\xi_j)U_{ij} - \frac{1}{2}(d\xi_i - d\xi_j)V_{ij}. \end{aligned}$$

The same hold when $i = j$ if we interpret the second terms to be zero then. More succinctly,

$$d\xi\, r_x + r_y\, d\xi = \frac{1}{2}\{d\xi,\, U\} + \frac{1}{2}[d\xi,\, V], \quad d\xi\, r_y + r_x\, d\xi = \frac{1}{2}\{d\xi,\, U\} - \frac{1}{2}[d\xi,\, V],$$

where the curly brackets indicate anticommutator. These give the explicit representations for the terms involving r_x and r_y in the equations.

With our assertions concerning r_x and r_y established we proceed to derive the equations. It follows from the general identity

$$\partial_k\,(I - K)^{-1} = (I - K)^{-1}\,\partial_k K\,(I - K)^{-1},$$

relation (4.1) and the remark in footnote 7 that

$$\partial_k\,\rho = -R\,\delta_k\,\rho. \tag{4.11}$$

From this we obtain (since $\partial_k\,R = \partial_k\,\rho$)

$$\begin{aligned} \partial_k\, r_{ij} &= \partial_k\,(R_{ij}(\xi_i, \xi_j)) = (\partial_k\, R_{ij})(\xi_i, \xi_j) + R_{xij}(\xi_i, \xi_j)\,\delta_{ik} + R_{yij}(\xi_i, \xi_j)\,\delta_{jk} \\ &= -r_{ik}\, r_{kj} + R_{xij}(\xi_i, \xi_j)\,\delta_{ik} + R_{yij}(\xi_i, \xi_j)\,\delta_{jk}. \end{aligned}$$

Multiplying by $d\xi_k$ and summing over k give (4.2).

Using (4.11) applied to A we obtain

$$\partial_k\, q_{ij} = Q'_{ij}(\xi_i)\,\delta_{ik} - (R\,\delta_k\,Q)_{ij}(\xi_i) = Q'_{ij}(\xi_i)\,\delta_{ik} - r_{ik}\, q_{kj}\,. \tag{4.12}$$

[9] Here r_{xij} is notational shorthand for $(r_x)_{ij}$ and r_{yij} for $(r_y)_{ij}$.

Now multiplying by $d\xi_k$ and summing over k give (4.3).

It follows from (4.11) that $\partial_k\,\rho_x = -R_x\,\delta_k\,\rho$. Applying this to A gives $\partial_k\,Q' = -R_x\,\delta_k\,Q$, whose $i,\,j$ entry evaluated at $x=\xi_i$ equals $-r_{xik}\,q_{kj}$. Hence

$$\partial_k\,q'_{ij} = \partial_k\,Q'_{ij}(\xi_i) = -r_{xik}\,q_{kj} + \delta_{ik}\,Q''_{ij}(\xi_i). \tag{4.13}$$

Now we use Lemma 2 again. Applying both sides to A and using the fact that $(D^2-M)A=0$ we obtain

$$Q''(x) - x\,Q(x) = R\,\delta\,Q' - R_y\,\delta\,Q. \tag{4.14}$$

Taking the $i,\ j$ entry and evaluating at $x=\xi_i$ gives

$$Q''_{ij}(\xi_i) - \xi_i\,q_{ij} = (rq' - r_y\,q)_{ij}.$$

Substituting this into (4.13) we obtain

$$\partial_k\,q'_{ij} = -r_{xik}\,q_{kj} + \delta_{ik}\,[\xi_i\,q_{ij} + (rq' - r_y\,q)_{ij}].$$

Multiplying by $d\xi_k$ and summing over k give (4.5).

To obtain the other equations, we point out that identities such as these occur in dual pairs. Observe that the function $\chi_j(y)\,\rho_{jk}(y,x)$ is equal to $\chi_k(x)$ times $\tilde\rho_{kj}(x,y)$, where $\tilde\rho$ is the resolvent kernel for the matrix kernel with $i,\,j$ entry $L_{ji}(x,y)\,\chi_j(y)$. Hence $\tilde Q_{jk}(x)$ is equal to $\chi_k(x)$ times the $Q_{kj}(x)$ associated with L_{ji}. The upshot is that for any formula involving q or $\tilde q$ there is another. We replace q by $\tilde q^t$ and $\tilde q$ with q^t. (If a formula involves r we replace it by r^t and subscripts x and y appearing in r are interchanged.) In this way Eqs. (4.4) and (4.6) are consequences of (4.3) and (4.5).

Let us derive the system of equations found in [17]. We introduce the differential operator $\mathcal{D} = \sum_k \partial_k$. The system of equations is

$$\mathcal{D}^2\,q = \xi\,q + 2\,q\,\Theta\,\tilde q\,q - 2\,[\tau,\,r]\,q, \tag{4.15}$$
$$\mathcal{D}^2\,\tilde q = \tilde q\,\xi + 2\,\tilde q\,q\,\Theta\,\tilde q - 2\,\tilde q\,[\tau,\,r], \tag{4.16}$$
$$\mathcal{D}\,r = -q\,\Theta\,\tilde q + [\tau,\,r]. \tag{4.17}$$

This can in fact be thought of as a system of ODEs since if we replace $\xi_1,\cdots,\xi_m$ by $\xi_1+\xi,\cdots,\xi_m+\xi$ then $\mathcal{D}=d/d\xi$ and the ξ_j are parameters in the equations.

Equation (4.17) follows upon summing over k the coefficients of the $d\xi_k$ in (4.2) and using (4.8). Similarly (4.3) gives $\mathcal{D}\,q = q' - r\,q$, so

$$\mathcal{D}^2\,q = \mathcal{D}\,q' + (q\,\Theta\,\tilde q - [\tau,\,r])\,q - r\,(q' - r\,q). \tag{4.18}$$

Finally, (4.5) gives

$$\mathcal{D}\,q' = -(r_x + r_y)\,q + \xi\,q + r\,q'.$$

Substituting this into (4.18) and using (4.8) again give (4.15). We derive (4.16) similarly.

When $m=1$ (4.15) is the Painlevé II equation $q'' = \xi q + 2\,q^3$.

We now consider the more general case where each X_k is a finite union of intervals,

$$X_k = (\xi_{k1},\,\xi_{k2}) \cup (\xi_{k3},\,\xi_{k4}) \cup \cdots.$$

We write ∂_{kw} for $\partial/\partial\xi_{kw}$. We have

$$\partial_{kw}\,K = (-1)^w L\,\delta_{kw}(y), \tag{4.19}$$

where $\delta_{kw}(y)$ is the $m \times m$ diagonal matrix all of whose entries are 0 except for the k^{th}, which equals $\delta(y - \xi_{kw})$. It follows that

$$\partial_{kw} \log \det(I - K) = -\text{Tr } (I - K)^{-1} \partial_{kw} K = (-1)^{w+1} R_{kk}(\xi_{kw}, \xi_{kw}).$$

The various ξ_{kw} are the independent variables. (We shall systematically use u, v and w as indices to order the end-points of the intervals of X_i, X_j and X_k, respectively.) We now define the matrix functions r, q, $\tilde{q}$, q' and $\tilde{q}'$ of the ξ_{kw} by

$$r_{iu,\, jv} = R_{ij}(\xi_{iu}, \xi_{jv}), \quad q_{iu,\, j} = Q_{ij}(\xi_{iu}), \quad \tilde{q}_{i,\, jv} = \tilde{Q}_{ij}(\xi_{jv}),$$

and

$$q'_{iu,\, j} = Q'_{ij}(\xi_{iu}), \quad \tilde{q}'_{i,\, jv} = \tilde{Q}'_{ij}(\xi_{jv}).$$ [10]

These will be the unknown functions in our PDEs. We also define r_x and r_y by

$$r_{x,\, iu,\, jv} = R_{xij}(\xi_{iu}, \xi_{jv}), \quad r_{y,\, iu,\, jv} = R_{yij}(\xi_{iu}, \xi_{jv}).$$

Observe that r, r_x and r_y are square matrices with rows and columns indexed by the end-points kw of the X_k while q, q', $\tilde{q}$ and $\tilde{q}'$ are rectangular matrices. Further notation is

$$\begin{aligned} \xi &= \text{diag}\,(\xi_{kw}), \quad d\xi = \text{diag}\,((-1)^{w+1}\, d\xi_{kw}), \\ \widehat{d}\xi &= \text{diag}\,(d\xi_{kw}), \quad \delta = \sum_{k,w} (-1)^{w+1}\, \delta_{kw}. \end{aligned} \tag{4.20}$$

These are all square matrices but ξ, $d\xi$ and $\hat{d}\xi$ are indexed by the end-points of the X_k while δ is $m \times m$.

With these notations our system of equations is

$$dr = -r\, d\xi\, r + \widehat{d}\xi\, r_x + r_y\, \widehat{d}\xi, \tag{4.21}$$

$$dq = \widehat{d}\xi\, q' - r\, d\xi\, q, \tag{4.22}$$

$$d\tilde{q} = \tilde{q}'\, \widehat{d}\xi - \tilde{q}\, d\xi\, r, \tag{4.23}$$

$$dq' = \widehat{d}\xi\, \xi\, q - (r_x\, d\xi + d\xi\, r_y)\, q + d\xi\, r\, q', \tag{4.24}$$

$$d\tilde{q}' = \tilde{q}\, \xi\, \widehat{d}\xi - \tilde{q}\, (d\xi\, r_y + r_x\, d\xi) + \tilde{q}'\, r\, d\xi. \tag{4.25}$$

As before the right sides involve the diagonal entries of $r_x + r_y$ and the off-diagonal entries of r_x and r_y, and we must show that these are known.

It is easy to see that Lemmas 1 and 2 still hold with the new definition of δ. Lemma 1 gives

$$r_{x,\, iu,\, jv} + r_{y,\, iu,\, jv} = -\sum_{k,\ell} q_{iu,\, k}\, \tilde{q}_{\ell,\, jv} + \sum_{k,w} (-1)^w r_{iu,\, kw}\, r_{kw,\, jv} + (\tau_i - \tau_j)\, r_{iu,\, jv}.$$

In matrix terms,

$$r_x + r_y = -q\, \Theta\, \tilde{q} - r\, s\, r + [\tau,\, r],$$

where $s = \text{diag}\,((-1)^{w+1})$. Thus $r_x + r_y$ is known.

[10] At points of discontinuity we always take limits from inside X_k.

What remains is to show that $r_{x,iu,\,jv}$ and $r_{y,iu,\,jv}$ are known when $iu \neq jv$.

From (4.9) we have, using (4.7) again,

$$(x-y)\,R(x,y) = -Q'(x) \ominus \tilde{Q}(y) + Q(x) \ominus \tilde{Q}'(y) + (R_x + R_y)\delta\rho \\ -R\delta(\rho_x + \rho_y) + [\tau,\ R_x - R_y],$$

so

$$[M,\ R] = -Q'(x) \ominus \tilde{Q}(y) + Q(x) \ominus \tilde{Q}'(y) - Q(x) \ominus \tilde{Q}\delta\rho(y) + R\delta Q(x) \ominus \tilde{Q}(y) \\ +[\tau,\ R]\,\delta\rho - R\delta\,[\tau,\ R] + [\tau,\ R_x - R_y].$$

It follows, as before, that $r_{x,\,iu,\,jv} - r_{y,\,iu,\,jv}$ is known when $i \neq j$ and so also are $r_{x,\,iu,\,jv}$ and $r_{y,\,iu,\,jv}$ individually. It remains to determine these when $i = j$ but $u \neq v$.

To do this we use the identity $[DM,\ R] = D\,[M,\ R] + [D,\ R]\,M$ to compute

$$\begin{aligned}[DM,\ R] = &-Q''(x) \ominus \tilde{Q}(y) + Q'(x) \ominus \tilde{Q}'(y) \\ &-Q'(x) \ominus \tilde{Q}\delta\rho(y) + R_x\delta Q(x) \ominus \tilde{Q}(y) \\ &+[\tau,\ R_x]\,\delta\rho - R_x\delta\,[\tau,\ R] + [\tau,\ R_{xx} - R_{xy}] \\ &-yQ(x) \ominus \tilde{Q}(y) + yR\delta\rho + y[\tau,\ R].\end{aligned}$$

Next we use (4.14), which is the same here. This gives an expression for $Q''(x)$ which we substitute into the first term above to obtain

$$\begin{aligned}[DM,\ R] = &-(x+y)\,Q(x) \ominus \tilde{Q}(y) + Q'(x) \ominus \tilde{Q}'(y) \\ &-Q'(x) \ominus \tilde{Q}\delta\rho(y) + (R_x + R_y)\delta Q(x) \ominus \tilde{Q}(y) \\ &+R\,\delta\,Q'(x) \ominus \tilde{Q}(y) + [\tau,\ R_x]\,\delta\rho - R_x\delta\,[\tau,\ R] \\ &+[\tau,\ R_{xx} - R_{xy}] + yR\delta\rho + y[\tau,\ R].\end{aligned}$$

The left side equals $xR_x + yR_y + R$ and its i, i entry evaluated at $(\xi_{iu},\ \xi_{iv})$ equals $\xi_{iu}\, r_{x,\,iu,\,iv} + \xi_{iv}\, r_{y,\,iu,\,iv} + r_{iu,iv}$. If we can compute this sum then we know $r_{x,\,iu,\,iv}$ and $r_{y,\,iu,\,iv}$ individually since we know $r_{x,\,iu,\,iv} + r_{y,\,iu,\,iv}$ and $\xi_{iu} \neq \xi_{iv}$. To see that the corresponding right side is computable observe that the term arising from $R_x + R_y$ is known because of Lemma 1, and the diagonal entries of $[\tau,\ R_{xx} - R_{xy}]$ are zero. Everything else is easily seen to be computable except possibly the terms arising from the sum $[\tau,\ R_x]\,\delta\rho - R_x\delta\,[\tau,\ R]$. Its i, i entry equals two times

$$\tau_i \sum_{k,w} (-1)^w R_{xik}(x,\ \xi_{kw})\, R_{ki}(\xi_{kw},\ y) - \sum_{k,w} (-1)^w R_{xik}(x,\ \xi_{kw})\, \tau_k\, R_{ki}(\xi_{kw},\ y).$$

The two summands corresponding to $k = i$ cancel. The remaining terms evaluated at $(\xi_{iu},\ \xi_{iv})$ involve $r_{kw,\,iv}$ and $r_{x,\,iu,\,kw}$ with $k \neq i$, all of which are known.

This completes the demonstration that all terms on the right sides of our equations are known. This was the hard part. With (4.11) replaced by

$$\partial_{kw}\,\rho = (-1)^w\, R\,\delta_{kw}\,\rho,$$

the derivation of the equations proceeds exactly as before, and need not be repeated.

Remark 1. One might wonder whether the systems of Eqs. (4.2)–(4.6) and (4.21)–(4.25) are integrable in the sense that one can derive from the equations themselves that the differentials of the right sides are zero. Because of the complicated expressions for r_x and r_y we have not attempted to show this in general. For Eqs. (4.2)–(4.6), where we have relatively simple expressions for the right sides, we verified that this is so when $m = 2$ or 3.

Remark 2. We point out how little the equations depend on the operator L, as long as we still define $K = L\chi$ with $\chi = \operatorname{diag}(\chi_{X_k})$. Equation (4.21) holds for any integral operator L. So does (4.22) if q is defined as before in terms of $Q = (I - K)^{-1}\varphi$, where φ can be any function whatsoever. Similarly for $\tilde{q}$ and (4.23). Similarly also for the right-hand sides of (4.24) and (4.25) except for the first terms $\widehat{d\xi}\,\xi\,q$ and $\tilde{q}\,\xi\,\widehat{d\xi}$. What does depend on the specifics of L are the following:

(i) The expressions for r_x and r_y in terms of the unknowns. We do not see these explicitly in the equations. This is where the choice of φ arises.
(ii) The first terms on the right sides of (4.24) and (4.25), which arise from the computation of Q''. (See (4.14).) All our systems will have the same form as these, most of the equations being universal, i.e., independent of the specific L or φ.[11] In most cases there will be two functions such as φ. That will add to the number of equations but not their complexity. The main difficulty in all cases will be (i).

V. PDEs for the Extended Hermite Kernel

We modify (1.3) by setting

$$L_{ij}(x, y) = \begin{cases} \sum_{k=0}^{n-1} e^{(k-n)\,(\tau_i-\tau_j)}\,\varphi_k(x)\,\varphi_k(y) & \text{if } i \geq j, \\ -\sum_{k=n}^{\infty} e^{(k-n)\,(\tau_i-\tau_j)}\,\varphi_k(x)\,\varphi_k(y) & \text{if } i < j. \end{cases}$$

The extra factors $e^{-n\,(\tau_i-\tau_j)}$ do not change the determinant.

Again we consider first the case where $X_k = (\xi_k, \infty)$. We define R and ρ as before, and again

$$\partial_k \log \det(I - K) = R_{kk}(\xi_k, \xi_k).$$

Now we shall have more unknown functions. We set

$$\varphi = (2n)^{1/4}\,\varphi_n, \quad \psi = (2n)^{1/4}\,\varphi_{n-1},$$

and define

$$Q = \rho\,\varphi, \quad P = \rho\,\psi, \quad \tilde{Q} = \varphi\chi\,\rho, \quad \tilde{P} = \psi\chi\,\rho.$$

Our unknowns will be, in addition to $r_{ij} = R_{ij}(\xi_i, \xi_j)$, the matrix functions q, $\tilde{q}$, p and $\tilde{p}$ given by

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j), \quad p_{ij} = P_{ij}(\xi_i), \quad \tilde{p}_{ij} = \tilde{P}_{ij}(\xi_j),$$

[11] This splitting into universal and nonuniversal equations was also a feature of [15].

and

$$q'_{ij} = Q'_{ij}(\xi_i), \quad \tilde{q}'_{ij} = \tilde{Q}_{ij}(\xi_j), \quad p'_{ij} = P'_{ij}(\xi_i), \quad \tilde{p}'_{ij} = \tilde{P}'_{ij}(\xi_j).$$

Again ξ denotes the matrix diag (ξ_k) and $d\xi$ denotes diag $(d\xi_k)$.

With these notations our system of equations is

$$dr = -r\,d\xi\,r + d\xi\,r_x + r_y\,d\xi, \tag{5.1}$$
$$dq = d\xi\,q' - r\,d\xi\,q, \tag{5.2}$$
$$d\tilde{q} = \tilde{q}'\,d\xi - \tilde{q}\,d\xi\,r, \tag{5.3}$$
$$dq' = d\xi\,(\xi^2 - 2n - 1)\,q - (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q', \tag{5.4}$$
$$d\tilde{q}' = \tilde{q}\,(\xi^2 - 2n - 1)\,d\xi - \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi, \tag{5.5}$$
$$dp = d\xi\,p' - r\,d\xi\,p, \tag{5.6}$$
$$d\tilde{p} = \tilde{p}'\,d\xi - \tilde{p}\,d\xi\,r, \tag{5.7}$$
$$dp' = d\xi\,(\xi^2 - 2n + 1)\,p - (r_x\,d\xi + d\xi\,r_y)\,p + d\xi\,r\,p', \tag{5.8}$$
$$d\tilde{p}' = \tilde{p}\,(\xi^2 - 2n + 1)\,d\xi - \tilde{p}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{p}'\,r\,d\xi. \tag{5.9}$$

By Remark 2 and duality (each equation for q or p giving rise to one for $\tilde{q}$ or $\tilde{p}$) all we have to show is that the diagonal entries of $r_x + r_y$ and the off-diagonal entries of r_x and r_y are known (i.e., expressible in terms of the unknowns) and to derive (5.4) and (5.8).

We begin by finding a substitute for Lemma 1. We write $D^{\pm}$ for $D \pm M$.

Lemma 3. *We have*

$$\begin{aligned} D^+ L_{ij} - e^{\tau_i - \tau_j}\,L_{ij}\,D^+ &= -\psi(x)\,\varphi(y), \\ e^{\tau_i - \tau_j}\,D^- L_{ij} - L_{ij}\,D^- &= -\varphi(x)\,\psi(y). \end{aligned} \tag{5.10}$$

Proof. Let J be the operator on $L^2(\mathbf{R})$ with kernel

$$J(x, y) = \sum_{k=0}^{n-1} \sigma^k\,\varphi_k(x)\,\varphi_k(y),$$

and set $a_k = \sqrt{k/2}$. We have the formulas

$$x\varphi_k = a_{k+1}\,\varphi_{k+1} + a_k\,\varphi_{k-1}, \quad \varphi_k' = -a_{k+1}\,\varphi_{k+1} + a_k\,\varphi_{k-1}.$$

Therefore

$$(x + \partial_x)\,J(x, y) = 2\sum_{k=0}^{n-1} \sigma^k\,a_k\,\varphi_{k-1}(x)\,\varphi_k(y),$$
$$(y - \partial_y)\,J(x, y) = 2\sum_{k=0}^{n-1} \sigma^k\,a_{k+1}\,\varphi_k(x)\,\varphi_{k+1}(y).$$

This gives

$$[\,(x + \partial_x) - \sigma\,(y - \partial_y)\,]\,J(x, y) = -2\,\sigma^n\,a_n\,\varphi_{n-1}(x)\,\varphi_n(y).$$

If we take $\sigma = e^{\tau_i - \tau_j}$ and multiply by $e^{-n(\tau_i - \tau_j)}$ we obtain the first identity of (5.10) when $i \geq j$. If $\sigma < 1$ and one takes $n \to \infty$ in the last identity for J one gets zero for the right sides. It follows that replacing $\sum_{k=0}^{n-1}$ by $-\sum_{k=n}^{\infty}$ in its definition does not change the right side. Thus we obtain the identity for $i < j$ as well. The second identity of (5.10) is obtained from the first by taking adjoints and using the fact that L_{ij} is self-adjoint. □

We can now find the analogue (actually, analogues) of Lemma 1. Observe that since $\tau = \text{diag}\,(\tau_i)$ we have $e^{\tau} = \text{diag}\,(e^{\tau_i})$.

Lemma 4. *We have*

$$e^{-\tau} D^+ R - R\, e^{-\tau} D^+ = -P(x)\, e^{-\tau} \,\Theta\, \tilde{Q}(y) + R\delta e^{-\tau} \rho, \tag{5.11}$$

$$e^{\tau} D^- R - R\, e^{\tau} D^- = -Q(x)\, e^{\tau} \,\Theta\, \tilde{P}(y) + R\delta e^{\tau} \rho, \tag{5.12}$$

Proof. If we multiply the relations (5.10) on the right by χ and use the fact $[D^{\pm},\ \chi] = \delta$ we obtain

$$e^{-\tau} D^+ K - K\, e^{-\tau} D^+ = -e^{-\tau}\, \psi(x) \,\Theta\, \chi(y)\, \varphi(y) + L\delta e^{-\tau},$$
$$e^{\tau} D^- K - K\, e^{\tau}\, D^- = -e^{\tau}\, \varphi(x) \,\Theta\, \chi(y)\, \psi(y) + L\delta e^{\tau}.$$

We replace K on the left by $K - I$ and left- and right-multiply by ρ, and the result follows. (We used the fact that $e^{\pm\tau}$ commutes with the matrix functions φ and ψ.) □

If we take $i,\ j$ entries in (5.11) and (5.12) and set $x = \xi_i,\ y = \xi_j$ we obtain

$$e^{-\tau} r_x + r_y\, e^{-\tau} = -e^{-\tau}\, \xi\, r + r\, e^{-\tau}\, \xi - p e^{-\tau} \,\Theta\, \tilde{q} + r\, e^{-\tau}\, r,$$
$$e^{\tau}\, r_x + r_y\, e^{\tau} = e^{\tau}\, \xi\, r - r\, e^{\tau}\, \xi - q e^{\tau} \,\Theta\, \tilde{p} + r\, e^{\tau}\, r. \tag{5.13}$$

The right sides here are known. If we add and subtract these identities and take $i,\ j$ entries we obtain

$$2\,(\cosh \tau_i\, r_{xij} + \cosh \tau_j\, r_{yij}) = \cdots, \tag{5.14}$$

$$2\,(\sinh \tau_i\, r_{xij} + \sinh \tau_j\, r_{yij}) = \cdots, \tag{5.15}$$

where the dots on the right represent known quantities. The first relation with $j = i$ gives $r_{xii} + r_{yii}$. If the two relations are thought of as a system of equations for r_{xij} and r_{yij} the determinant of the system is nonzero when $i \neq j$. Therefore we can solve for r_{xij} and r_{yij} individually then.

What remains is to derive (5.4) and (5.8). For this we need the analogue of Lemma 2.

Lemma 5. *We have*

$$[D^2 - M^2,\ \rho] = R\,\delta\,\rho_x - R_y\,\delta\,\rho, \tag{5.16}$$

where $R_y(x, y)$ is interpreted as not containing a delta-function summand.

The proof is analogous to that of Lemma 2. Here we use the fact that $D^2 - M^2$ commutes with L, a consequence of the fact that each φ_k is an eigenfunction of $D^2 - M^2$.

Since φ is an eigenfunction of $D^2 - M^2$ with eigenvalue $-2n - 1$ and ψ an eigenfunction with eigenvalue $-2n + 1$ applying both sides of (5.16) to φ and to ψ gives

$$Q'' - x^2\, Q + (2n + 1)\, Q = R\delta Q' - R_y \delta Q, \tag{5.17}$$

$$P'' - x^2\, P + (2n - 1)\, P = R\delta P' - R_y \delta P. \tag{5.18}$$

We have (4.13) here just as before. Taking the i, j entry in (5.17) and evaluating at $x = \xi_i$ gives

$$Q''_{ij}(\xi_i) - (\xi_i^2 - 2n - 1)\, q_{ij} = (r\, q' - r_y\, q)_{ij}.$$

Substituting this into (4.13) we obtain

$$\partial_k\, q'_{ij} = -r_{xik}\, q_{kj} + \delta_{ik}\, [(\xi_i^2 - 2n - 1)\, q_{ij} + (r\, q' - r_y\, q)_{ij}],$$

which is (5.4). Equation (5.8) is established in exactly the same way using (5.18).

We can also derive a system analogous to Eqs. (4.15)–(4.17):

$$\mathcal{D}^2\, q = (\xi^2 - 2n - 1)\, q - 2\, \mathcal{D} r \cdot q, \tag{5.19}$$
$$\mathcal{D}^2\, \tilde{q} = \tilde{q}\, (\xi^2 - 2n - 1) - 2\, \tilde{q} \cdot \mathcal{D} r, \tag{5.20}$$
$$\mathcal{D}^2\, p = (\xi^2 - 2n + 1)\, p - 2\, \mathcal{D} r \cdot p, \tag{5.21}$$
$$\mathcal{D}^2\, \tilde{p} = \tilde{p}\, (\xi^2 - 2n + 1) - 2\, \tilde{p} \cdot \mathcal{D} r, \tag{5.22}$$
$$\mathcal{D} r = -r^2 + r_x + r_y. \tag{5.23}$$

These equations are not as simple as (4.15)–(4.17) since the expressions for the entries of $r_x + r_y$ are messy. The last equation we already know. The other equations are derived as for Airy: Summing the coefficients of $d\xi_k$ in (5.2) gives $\mathcal{D}\, q = q' - r\, q$, so

$$\mathcal{D}^2\, q = \mathcal{D}\, q' - \mathcal{D}\; r \cdot q - r\, (q' - r\, q). \tag{5.24}$$

Similarly (5.4) gives

$$\mathcal{D}\, q' = (\xi^2 - 2n - 1)\, q + r\, q' - (r_x + r_y)\, q.$$

Substituting this into (5.24) and using (5.23) again give (5.19). We derive (5.21) similarly, and (5.20) and (5.22) are obtained by duality.

In case $m = 1$ (5.13) gives $r_x + r_y = r^2 - pq$, and our system of equations becomes

$$r' = -pq, \quad q'' = (\xi^2 - 2n - 1)\, q + 2\, q^2\, p, \quad p'' = (\xi^2 - 2n + 1)\, p + 2\, p^2\, q.$$

From the last two we find $(pq' - qp')' = pq'' - qp'' = -2pq$, and by the first equation this is $2r'$. Thus $pq' - qp' = 2r$. Using this, and successively computing r'', r''' and r'''' using the differentiation formulas, we arrive at

$$r'''' = 4(\xi^2 - 2n)r'' + 4\xi r' - 12r'r'' - 4r = (4(\xi^2 - 2n)r')' - 4(\xi r)' - 6(r'^{\,2})',$$

and so

$$r''' = 4(\xi^2 - 2n)r' - 4\xi r - 6r'^{\,2}.$$

This is the third-order equation found in [15] which integrates to Painlevé IV.

We turn to the more general case where each X_k is a finite union of intervals, and will again use the notations (4.20). The equations are

$$dr = -r\,d\xi\,r + \widehat{d\xi}\,r_x + r_y\,\widehat{d\xi}, \tag{5.25}$$

$$dq = \widehat{d\xi}\,q' - r\,d\xi\,q, \tag{5.26}$$

$$d\tilde{q} = \tilde{q}'\,\widehat{d\xi} - \tilde{q}\,d\xi\,r, \tag{5.27}$$

$$dq' = \widehat{d\xi}\,(\xi^2 - 2n - 1)\,q - (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q', \tag{5.28}$$

$$d\tilde{q}' = \tilde{q}\,(\xi^2 - 2n - 1)\,\widehat{d\xi} - \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi, \tag{5.29}$$

$$dp = \widehat{d\xi}\,p' - r\,d\xi\,p, \tag{5.30}$$

$$d\tilde{p} = \tilde{p}'\,\widehat{d\xi} - \tilde{p}\,d\xi\,r, \tag{5.31}$$

$$dp' = \widehat{d\xi}\,(\xi^2 - 2n + 1)\,p - (r_x\,d\xi + d\xi\,r_y)\,p + d\xi\,r\,p', \tag{5.32}$$

$$d\tilde{p}' = \tilde{p}\,(\xi^2 - 2n + 1)\,\widehat{d\xi} - \tilde{p}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{p}'\,r\,d\xi. \tag{5.33}$$

Nothing is new here except to establish that the terms involving r_x and r_y on the right are known. As usual those that occur are the diagonal entries of $r_x + r_y$ and the off-diagonal entries of r_x and r_y. In our case the terms r_{xij} and r_{yij} in (5.14) and (5.15) are replaced by $r_{x,\,iu,\,jv}$ and $r_{y,\,iu,\,jv}$ and the relations show that these are known when $i \neq j$ and that the $r_{x,\,iu,\,iv} + r_{y,\,iu,\,iv}$ are known. It remains to show that $r_{x,\,iu,\,iv}$ and $r_{y,\,iu,\,iv}$ are known when $u \neq v$.

From (5.12), which says

$$[e^\tau D^-,\ R] = -Q(x)\,e^\tau\,\Theta\,\tilde{P}(y) + R\delta e^\tau \rho,$$

we deduce

$$[(e^\tau D^-)^2,\ R] = e^\tau D^-\,(-Q(x)\,e^\tau\,\Theta\,\tilde{P}(y) + R\delta e^\tau \rho) \\ +(-Q(x)\,e^\tau\,\Theta\,\tilde{P}(y) + R\delta e^\tau \rho)\,e^\tau D^-.$$

We use $S \equiv T$ for matrix functions S and T to denote that the differences $S_{iu,iv}(\xi_{iu},\ \xi_{iv}) - T_{iu,iv}(\xi_{iu},\ \xi_{iv})$ are known. If we keep in mind that $q,\ q',\ \tilde{p}$ and $\tilde{p}'$ are among our unknowns, we see that it follows from the above, after multiplying by $e^{-2\tau}$, that

$$[D^2 - 2MD,\ R] \equiv R_x \delta e^\tau \rho e^{-\tau} - R\delta e^\tau \rho_y e^{-\tau}.$$

If we subtract this from (5.16) we obtain (since $[M^2,\ R]$ is known)

$$2[MD,\ R] \equiv (R\delta\rho_x - R_x\delta e^\tau \rho e^{-\tau}) + (R\delta e^\tau \rho_y e^{-\tau} - R_y \delta\rho). \tag{5.34}$$

Consider the first term on the right. Its $iu,\ iv$ entry evaluated at $(\xi_{iu},\ \xi_{iv})$ equals

$$\sum_{kw} r_{iu,\,kw}(-1)^{w+1} r_{x,\,kw,\,iv} - \sum_{kw} r_{x,iu,\,kw}(-1)^{w+1} e^{\tau_k}\, r_{kw,\,iv}\, e^{-\tau_i}.$$

The terms of both sums corresponding to $k \neq i$ are known. So remaining as unknown is the sum

$$\sum_{w} (-1)^{w+1} (r_{iu,\,iw}\, r_{x,\,iw,\,iv} - r_{x,iu,\,iw}\, r_{iw,\,iv}).$$

Analogously the second term on the right of (5.34) is a known quantity plus

$$\sum_{w}(-1)^{w+1}(r_{iu,\,iw}\,r_{y,\,iw,\,iv}-r_{y,iu,\,iw}\,r_{iw,\,iv}).$$

Adding this to the last sum gives

$$\sum_{kw}(-1)^{w+1}[r_{iu,\,iw}\,(r_{x,\,iw,\,iv}+r_{y,\,iw,\,iv})-(r_{x,\,iu,\,iw}+r_{y,iu,\,iw})\,r_{iw,\,iv})].$$

But this is known since, as we saw at the beginning, the $r_{x,\,iu,\,iv}+r_{y,iu,\,iv}$ are known.

We have shown that $[MD,\ R]\equiv$ a known matrix function. Its $iu,\ jv$ entry evaluated at $(\xi_{iu},\ \xi_{iv})$ equals

$$\xi_{iu}\,r_{x,\,iu,\,jv}+\xi_{iv}\,r_{y,\,iu,\,jv}+r_{y,\,iu,\,jv},$$

so $\xi_{iu}\,r_{x,\,iu,\,jv}+\xi_{iv}\,r_{y,\,iu,\,jv}$ is known. But so is $r_{x,\,iu,\,jv}+r_{y,\,iu,\,jv}$. Therefore $r_{x,\,iu,\,jv}$ and $r_{y,\,iu,\,jv}$ are both known when $u\neq v$.

VI. PDEs for the Extended sine Kernel

If we make the substitutions $\tau_i\to\tau_i/2n,\ x\to x/\sqrt{2n},\ y\to y/\sqrt{2n}$ in the extended Hermite kernel (1.3) and let $n\to\infty$ we obtain the *extended sine kernel*

$$L_{ij}(x,y)=\begin{cases}\displaystyle\int_0^1 e^{z^2\,(\tau_i-\tau_j)}\,\cos z(x-y)\,dz & \text{if } i\geq j,\\ \displaystyle-\int_1^\infty e^{z^2\,(\tau_i-\tau_j)}\,\cos z(x-y)\,dz & \text{if } i<j.\end{cases}$$

Here we set

$$\varphi(x)=\sin x,\quad \psi(x)=\cos x,$$

and then the other definitions are exactly as in Hermite with the above replacements. The unknowns now will be only $r,\ q,\ \tilde{q},\ p$ and $\tilde{p}$ and the equations for general X_k are

$$dr=-r\,d\xi\,r+\widehat{d\xi}\,r_x+r_y\,\widehat{d\xi},\tag{6.1}$$
$$dq=\widehat{d\xi}\,(p+rsq)-r\,d\xi\,q,\tag{6.2}$$
$$d\tilde{q}=(\tilde{p}+\tilde{q}sr)\,\widehat{d\xi}-\tilde{q}\,d\xi\,r,\tag{6.3}$$
$$dp=\widehat{d\xi}\,(-q+rsp)-r\,d\xi\,p,\tag{6.4}$$
$$d\tilde{p}=(-\tilde{q}+\tilde{p}sr)\,\widehat{d\xi}-\tilde{p}\,d\xi\,r.\tag{6.5}$$

(Recall that $s=\mathrm{diag}\,((-1)^{w+1})$.)

We know that Eq. (6.1) is completely general, as are the equations

$$dq=\widehat{d\xi}\,q'-r\,d\xi\,q,$$
$$d\tilde{q}=\tilde{q}'\,\widehat{d\xi}-\tilde{q}\,d\xi\,r,$$
$$dp=\widehat{d\xi}\,p'-r\,d\xi\,p,$$
$$d\tilde{p}=\tilde{p}'\,\widehat{d\xi}-\tilde{p}\,d\xi\,r.$$

To derive (6.2)–(6.5) from these we establish the formulas

$$q' = p + rsq, \quad p' = -q + rsp, \quad \tilde{q}' = \tilde{p} + \tilde{q}sr, \quad \tilde{p}' = -\tilde{q} + \tilde{p}sr. \tag{6.6}$$

We have $[D,\ L] = 0$, whence $[D,\ K] = L\delta$, whence

$$R_x + R_y = [D,\ \rho] = R\,\delta\,\rho. \tag{6.7}$$

Applying (6.7) on the left to φ and ψ, using $\varphi' = \psi,\ \psi' = -\varphi$, we obtain

$$Q'(x) - P(x) = (R\delta\, Q)(x), \quad P'(x) + Q(x) = (R\delta\, P)(x).$$

Since $q'_{iu,\,j} = Q'_{ij}(\xi_{iu})$ and $p'_{iu,\,j} = P'_{ij}(\xi_{iu})$, the first two relations of (6.6) follow, and the others are analogous.

So Eqs. (6.1)–(6.5) hold, and it remains to deal with the entries of r_x and r_y appearing on the right side of (6.1). We have to show that the diagonal entries of $r_x + r_y$ are known and that $r_{x,\,iu,\,jv}$ and $r_{y,\,iu,\,jv}$ are known when $iu \neq jv$.

It follows from (6.7) that $r_x + r_y = rsr$, and so all entries of the sum are known.

Next, for $i \geq j$ we have

$$\begin{aligned} 2\,(\tau_i - \tau_j)\,L_{xij} &= -2\,(\tau_i - \tau_j) \int_0^1 e^{z^2\,(\tau_i - \tau_j)}\, z\, \sin z(x-y)\,dz \\ &= -e^{\tau_i - \tau_j}\, \sin(x-y) + (x-y)\,L_{ij}. \end{aligned}$$

The same holds when $i < j$. Since $[\tau D,\ L]_{ij} = \tau_i\,L_{xij} + \tau_j\,L_{yij}$ and $L_y = -L_x$, this gives

$$2\,[\tau D,\ L] = -e^{\tau} \otimes e^{-\tau}\, \sin(x-y)\, + [M,\ L],$$

where $e^{\tau} \otimes e^{-\tau}$ is the matrix with i, j entry $e^{\tau_i - \tau_j}$. (This is not a tensor product.) Hence

$$2\,[\tau D,\ K] = -e^{\tau} \otimes e^{-\tau}\, \sin(x-y)\,\chi(y) + [M,\ K] + 2\,K\delta\tau.$$

Replacing K by $K - I$ in the commutators and applying ρ left and right give

$$2\,[\tau D,\ R] = P(x)\,e^{\tau} \otimes e^{-\tau}\,\tilde{Q}(y) - Q(x)\,e^{\tau} \otimes e^{-\tau}\,\tilde{P}(y) + [M,\ R] + 2\,R\delta\tau\rho.$$

The i, j entry of the left side evaluated at $(\xi_{iu},\ \xi_{jv})$ equals twice $\tau_i\, r_{x,\,iu,\,jv} + \tau_j\, r_{y,\,iu,\,jv}$, so these are known. We deduce, since $r_{x,\,iu,\,jv} + r_{y,\,iu,\,jv}$ is known, that $r_{x,\,iu,\,jv}$ and $r_{y,\,iu,\,jv}$ are both known when $i \neq j$. Just as before, the trickier part is to show that $r_{x,\,iu,\,iv}$ and $r_{y,\,iu,\,iv}$ are known when $u \neq v$.

We compute

$$\begin{aligned} [DM,\ R] &= [D,\ R]\,M + D\,[M,\ R] \\ &= R\delta\rho\, y + 2\,(\partial_x\,[\tau D,\ R] - R_x\delta\tau\rho) \\ &\quad -P'(x)\,e^{\tau} \otimes e^{-\tau}\,\tilde{Q}(y) + Q'(x)\,e^{\tau} \otimes e^{-\tau}\,\tilde{P}(y). \end{aligned} \tag{6.8}$$

The i, i entry of $[\tau D,\ R]$ evaluated at $(\xi_{iu},\ \xi_{jv})$ equals $\tau_i\ (R_x + R_y)(\xi_{iu},\ \xi_{iv})$. Hence, since $\partial_x\,(R_x + R_y) = R_x\delta\rho$ by (6.7), the i, i entry of $\partial_x\,[\tau D,\ R]$ evaluated at $(\xi_{iu},\ \xi_{jv})$ equals $\tau_i\ (R_x\delta\rho)_{ii}(\xi_{iu},\ \xi_{iv})$. It follows that i, i entry of $\partial_x\,[\tau D,\ R] - R_x\delta\tau\rho$ evaluated at $(\xi_{iu},\ \xi_{jv})$ equals

$$\sum_{kw} (-1)^{w+1}(\tau_i - \tau_k)\, r_{x,\,iu,\,kw}\, r_{kw,\,iv}.$$

Since we need only sum over $k \neq i$ all these terms are known. So are the other terms of (6.8) evaluated at $(\xi_{iu},\ \xi_{jv})$.

We have shown that the i, i entry of $[DM,\ R]$ evaluated at $(\xi_{iu},\ \xi_{jv})$ is known. This equals $r_{iu,\,iv}+\xi_{iu}\,r_{x,\,iu,\,jv}+\xi_{iv}\,r_{y,\,iu,\,jv}$. Thus $\xi_{iu}\,r_{x,\,iu,\,jv}+\xi_{iv}\,r_{y,\,iu,\,jv}$ is known. Since $r_{x,\,iu,\,jv}+r_{y,\,iu,\,jv}$ is known and $\xi_{iu}\neq\xi_{iv}$ so also are $r_{x,\,iu,\,jv}$ and $r_{y,\,iu,\,jv}$ known.

Let us see what these give in the case $m = 1$ for a single interval $(-t,\ t)$. Here $\xi_1 = -t,\ \xi_2 = t$. If we use the fact that $K(-x,\ -y) = K(x,\ y)$ and the evenness of cosine and the oddness of sine we get $q_2 = -q_1,\ p_2 = p_1$ and if we use the fact that $R(x,\ y) = R(y,\ x)$ for $x,\ y \in (-t,\ t)$ we get $r_{ij} = r_{ji}$.

We use the notations $r = r_{11},\ \bar r = r_{12}$. If we observe that $d/dt = \partial_2 - \partial_1$ then (6.6) gives

$$\frac{dq_1}{dt} = -p_1 - 2\,\bar r\,q_1, \quad \frac{dp_1}{dt} = q_1 + 2\,\bar r\,p_1,$$

and (6.1) gives

$$\frac{dr}{dt} = r^2 + \bar r^2 - r_x - r_y,$$

and the trivial relation $d\bar r/dt = -\bar r_x + \bar r_y$. The general relation $r_x + r_y = rsr$ gives in the present notation $r_x + r_y = r^2 - \bar r^2$, and so

$$\frac{dr}{dt} = 2\,\bar r^2.$$

Finally (6.8) gives

$$\bar r - t\,\bar r_x + t\,\bar r_y = -P_1'(-t)\,Q_2(t) + Q_1'(-t)\,P_2(t) = -\frac{d}{dt}(Q_1(-t)\,P_1(-t)).$$

Thus

$$\frac{d}{dt}(t\,\bar r) = -\frac{d}{dt}(q_1\,p_1),$$

which gives

$$\bar r = -\frac{q_1\,p_1}{t}.$$

VII. The Laguerre Process

The Dyson process $\tau \to A(\tau)$ on the space of $p \times n$ complex matrices (we assume $p \geq n$) is specified by its finite-dimensional distribution functions. The probability measure on $A_k = A(\tau_k)$ $(k = 1, \ldots, m)$ is a normalization constant times (1.4), which may be written

$$\prod_{j=1}^{m} \exp\left(-\left(\frac{1}{1-q_j^2} + \frac{q_{j+1}^2}{1-q_{j+1}^2}\right) \mathrm{Tr}\ A_j^* A_j\right) \times \prod_{j=2}^{m} \exp\left(\frac{q_j}{1-q_j^2}\,\mathrm{Tr}\ (A_j^* A_{j-1} + \mathrm{hc})\right) dA_1 \cdots dA_m. \tag{7.1}$$

(Here “hc” is an abbreviation for “Hermitian conjugate”.) We show how to derive (1.5) from this.

Any complex matrix $p \times n$ complex matrix A can, by the singular value decomposition (SVD) theorem, be written as

$$A = UDV^*,$$

where U is a $p \times p$ unitary matrix, V is an $n \times n$ unitary matrix and D is a $p \times n$ matrix all of whose entries are zero except for the diagonal consisting of the singular values of A. Thus we write each A_j as

$$A_j = U_j D_j V_j^*$$

with the goal of eventually integrating over the unitaries U_j and V_j. Of course,

$$\mathrm{Tr}\ (A_j^* A_j) = \mathrm{Tr}\ (D_j^* D_j) = \sum_{j=1}^{n} \lambda_j,$$

where $\lambda_j = d_{jj}^2$.

Let us examine one term

$$\mathrm{Tr}\ \left(A_j^* A_{j-1} + \mathrm{hc}\right)$$

appearing in the exponential of the second product in (7.1). Using the SVD representation we have terms

$$\mathrm{Tr}\ \left(V_j\, D_j^*\, U_j^*\, U_{j-1}\, D_{j-1}\, V_{j-1}^* + \mathrm{hc}\right).$$

The integrals over the unitary group (Haar measure) are both left- and right-invariant. Thus in the V_{j-1} integration we let

$$V_{j-1}^* \to V_{j-1}^* V_j^*$$

so that the trace term becomes

$$\mathrm{Tr}\ \left(D_j^*\, U_j^*\, U_{j-1}\, D_{j-1}\, V_{j-1}^* + \mathrm{hc}\right).$$

In the U_{j-1} integration we let $U_{j-1} \to U_j U_{j-1}$ and the trace becomes

$$\mathrm{Tr}\ \left(D_j^*\, U_{j-1}\, D_{j-1}\, V_{j-1}^* + \mathrm{hc}\right).$$

Thus, we have integrals of the form

$$\int\!\!\int \exp\left(\frac{q_j}{1-q_j^2}\ \mathrm{Tr}\ (D_j^*\, U_{j-1}\, D_{j-1}\, V_{j-1}^* + \mathrm{hc})\right)\ d\mu(V_{j-1})\, d\mu(U_{j-1}).$$

Let S denote an $n \times p$ complex matrix, T a $p \times n$ complex matrix and U (resp. V) elements of the unitary group of $p \times p$ (resp. $n \times n$) matrices. We assume $p \geq n$

and set $\alpha = p - n$. We let s_i resp. t_i denote the eigenvalues of SS^* resp. T^*T. The Harish-Chandra/Itzykson-Zuber integral for rectangular matrices (see, e.g., [18]) is

$$\int\int \exp\left(c\,\mathrm{Tr}\,(SUTV^* + \mathrm{hc})\right)\, d\mu(U)\, d\mu(V) = \frac{C_{p,n,c}}{\Delta(a)\Delta(b)}\, \frac{\det\left(I_\alpha(2c\sqrt{a_i b_j}\,)\right)}{\prod_{i=1}^n (s_i\, t_i)^{\alpha/2}}.$$

Here c can be any constant, $\alpha = p - n,\ I_\alpha$ is the modified Bessel function and $C_{p.n,c}$ is a known constant.

When the $q_j = 0$ the measure (7.1) must reduce, after integration over the unitary parts, to the well-known Laguerre measure. It follows that (7.1) becomes after integration over the unitary parts a normalization constant times

$$\prod_{k=1}^m e^{-\left(\frac{1}{1-q_{k-1}^2}+\frac{q_k^2}{1-q_k^2}\right)\sum_{i=1}^n \lambda_{ki}} \prod_{k=1}^{m-1} \det\left(I_\alpha\left(\frac{2q_{k+1}}{1-q_{k+1}^2}\sqrt{\lambda_{i,\,k}\,\lambda_{j,k+1}}\right)\right)$$
$$\times \Delta(\lambda_1)\,\Delta(\lambda_m) \prod_{i=1}^n \lambda_{1i}^{\alpha/2} \prod_{i=1}^n \lambda_{mi}^{\alpha/2}\, d\lambda_{11}\cdots d\lambda_{mn},$$

which is (1.5).

We shall now compute the extended kernel using the method of Sect. II. This density is not quite of the form (2.1) because of the last factors in the integrand here. Consequently in (2.2) there are extra factors $\lambda_i^{\alpha/2}$ and $\lambda_m^{\alpha/2}$, and so in the discussion that follows $P_{1i}(\lambda)$ and $Q_{mi}(\lambda)$ are no longer polynomials of degree i but $\lambda^{\alpha/2}$ times polynomials of degree i.

We have now

$$V_k(\lambda) = \left(\frac{1}{1-q_{k-1}^2} + \frac{q_k^2}{1-q_k^2}\right)\lambda, \quad u_k(\lambda,\mu) = I_\alpha\left(\frac{2q_k}{1-q_k^2}\sqrt{\lambda\mu}\right).$$

We introduce the Hille-Hardy kernel (the analogue of the Mehler kernel)

$$K(q;\lambda,\mu) = \frac{q^{-\alpha}}{1-q^2}\, e^{-\frac{q^2}{1-q^2}\lambda - \frac{1}{1-q^2}\mu} \left(\frac{\mu}{\lambda}\right)^{\alpha/2} I_\alpha\left(\frac{2q}{1-q^2}\sqrt{\lambda\mu}\right)$$

which has the representation

$$K(q;\lambda,\mu) = \sum_{i=0}^\infty q^{2i}\, p_i^\alpha(\lambda)\, p_i^\alpha(\mu)\, \mu^\alpha e^{-\mu},$$

where p_i^α are the Laguerre polynomials L_i^α, normalized. It follows that

$$\int_0^\infty K(q;\lambda,\mu)\, p_i^\alpha(\mu)\, d\mu = q^{2i}\, p_i^\alpha(\lambda) \tag{7.2}$$

and so again $K(q) * K(q') = K(qq')$.

Now we may take in (2.3),

$$E_{12}(\lambda_1,\lambda_2) = e^{-\lambda_1}\,(\lambda_1/\lambda_2)^{\alpha/2}\, K(q_1;\ \lambda_1,\lambda_2),$$
$$E_{k,k+1}(\lambda_k,\lambda_{k+1}) = (\lambda_k/\lambda_{k+1})^{\alpha/2}\, K(q_k;\ \lambda_k,\lambda_{k+1}), \quad (k>1),$$

and so

$$E_{1k}(\lambda,\mu) = e^{-\lambda}\,(\lambda/\mu)^{\alpha/2}\,K(q_1\cdots q_{k-1};\ \lambda,\mu).$$

We deduce from (7.2) that

$$\int\!\!\int \lambda^{\alpha/2}\,p_i^\alpha(\lambda)\,E_{1m}(\lambda,\mu)\,\mu^{\alpha/2}\,p_j^\alpha(\mu)\,d\mu d\lambda$$
$$= (q_1\cdots q_{m-1})^{2j}\int \lambda^\alpha\,p_i^\alpha(\lambda)\,e^{-\lambda}\,p_j^\alpha(\lambda)\,d\lambda = (q_1\cdots q_{m-1})^{2j}\,\delta_{ij}.$$

Hence we may take

$$P_{1i}(\lambda_1) = \lambda_1^{\alpha/2}\,p_i^\alpha(\lambda_1),\quad Q_{mj}(\lambda_m) = (q_1\cdots q_{m-1})^{-2j}\,\lambda_m^{\alpha/2}\,p_j^\alpha(\lambda_m).$$

We see that

$$P_{ki}(\mu) = \int \lambda^{\alpha/2}\,P_{1i}(\lambda)\,e^{-\lambda}\,(\lambda/\mu)^{\alpha/2}\,K(q_1\cdots q_{k-1};\ \lambda,\mu)\,d\lambda$$
$$= (q_1\cdots q_{k-1})^{2i}\,p_i^\alpha(\mu)\,\mu^{\alpha/2}\,e^{-\mu},\quad (k>1),$$
$$Q_{kj}(\lambda) = \int (\lambda/\mu)^{\alpha/2}\,K(q_k\cdots q_{m-1};\ \lambda,\mu)\,Q_{mj}(\mu)\,\mu^{\alpha/2}\,d\mu$$
$$= (q_1\cdots q_{k-1})^{-2j}\,\lambda^{\alpha/2}\,p_j^\alpha(\lambda),\quad (k>1),$$
$$Q_{1j}(\lambda) = \int e^{-\lambda}(\lambda/\mu)^{\alpha/2}\,K(q_1\cdots q_{m-1};\ \lambda,\mu)\,Q_{mj}(\mu)d\mu = e^{-\lambda}\,\lambda^{\alpha/2}\,p_j^\alpha(\lambda).$$

It follows that H is the matrix with k,ℓ entry

$$\sum_{j=0}^{n-1}\left(\frac{q_1\cdots q_{\ell-1}}{q_1\cdots q_{k-1}}\right)^{2j}\ p_j(\lambda)\,p_j(\mu)$$

left-multiplied by the matrix diag $(\lambda^{\alpha/2}e^{-\lambda}\ \ \lambda^{\alpha/2}\ \cdots\ \lambda^{\alpha/2})$ and right-multiplied by the matrix diag $(\mu^{\alpha/2}\ \ \mu^{\alpha/2}e^{-\mu}\ \cdots\ \mu^{\alpha/2}e^{-\mu})$. Similarly E is the strictly upper-triangular matrix with k,ℓ entry $(\lambda/\mu)^{\alpha/2}\,K(q_k\cdots q_{\ell-1};\ \lambda,\mu)$ left-multiplied by the matrix diag $(\lambda^{\alpha/2}e^{-\lambda}\ \ \lambda^{\alpha/2}\ \cdots\ \lambda^{\alpha/2})$. Now we use the fact that the determinant is unchanged if we multiply on the left by diag $(e^{\lambda/2}\ \ e^{-\lambda/2}\ \cdots\ e^{-\lambda/2})$ and on the right by diag $(e^{-\mu/2}\ \ e^{\mu/2}\ \cdots\ e^{\mu/2})$.

In this way we find the analogue of the kernel which was denoted by $\hat H-\hat E$ in Sect. III. It is now given by (1.3) but with coefficients $e^{2k\,(\tau_i-\tau_j)}$ and with $\varphi_k(x)$ equal to $x^{\alpha/2}\,e^{-x/2}\,p_k^\alpha(x)$. This is the extended Laguerre kernel.

VIII. PDEs for the Extended Bessel Kernel

If we make the substitutions $\tau_i\to\tau_i/2n,\ x\to x^2/4n,\ y\to y^2/4n$ in the extended Laguerre kernel and then let $n\to\infty$ we obtain the *extended Bessel kernel*

$$L_{ij}(x,y) = \begin{cases} \displaystyle\int_0^1 e^{z^2\,(\tau_i-\tau_j)/2}\,\Phi_\alpha(xz)\,\Phi_\alpha(yz)\,dz & \text{if } i\ge j,\\ \displaystyle -\int_1^\infty e^{z^2\,(\tau_i-\tau_j)/2}\,\Phi_\alpha(xz)\,\Phi_\alpha(yz)\,dz & \text{if } i<j,\end{cases}$$

where

$$\Phi_\alpha(z) = \sqrt{z}\, J_\alpha(z).$$

Let us immediately explain the difficulty. In the previous cases we were able to find one commutator for L involving D and another involving D^2, the latter arising from the differential operator whose eigenfunctions appear in the integrand or summand of the expression for the kernel. (For the extended Airy kernel these were given in Lemmas 1 and 2.) These enabled us to express r_x and r_y in terms of the unknown functions.

Here there does not seem to be a commutator involving the first power of D. We are able to find two relations involving the first power of D, but each involves both a commutator and an anticommutator. Fortunately we are able to deduce from these relations three commutator relations involving D^2, and these relations will enable us to show that the derivatives of r_x and r_y are expressible in terms of r_x and r_y and the other unknown functions. The upshot is that we are able to find a system of PDEs in which r_x and r_y are now among the unknowns. Although the system of equations seems no more complicated than those we have already derived (just larger) it is actually much more so because of the expressions for the derivatives of r_x and r_y in terms of the unknown functions.

To state the equations, we define φ and ψ by

$$\varphi = \Phi_\alpha, \quad \psi = \Phi_{\alpha+1}.$$

From these we define q in the usual way. But now we set

$$P = (I-K)^{-1} M\psi, \quad \tilde{P} = M\psi\chi\,(I-K)^{-1},$$

and from these we define p and $\tilde{p}$ in the usual way. (The reason we do this is that eventually it is these p and $\tilde{p}$ which will arise in the expressions for the derivatives of r_x and r_y.) With these notations our system of equations, in the general case where each X_k is a finite union of intervals, is

$$dr = -r\,d\xi\,r + \widehat{d\xi}\,r_x + r_y\,\widehat{d\xi}, \tag{7.1}$$
$$dr_x = -r_x\,d\xi\,r + \widehat{d\xi}\,r_{xx} + r_{xy}\,\widehat{d\xi}, \tag{7.2}$$
$$dr_y = -r\,d\xi\,r_y + \widehat{d\xi}\,r_{xy} + r_{yy}\,\widehat{d\xi}, \tag{7.3}$$
$$dq = \widehat{d\xi}\,q' - r\,d\xi\,q, \tag{7.4}$$
$$d\tilde{q} = \tilde{q}'\,\widehat{d\xi} - \tilde{q}\,d\xi\,r, \tag{7.5}$$
$$dq' = \widehat{d\xi}\,((\alpha^2 - \tfrac{1}{4})\,\xi^{-2}\,q - q) - (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q', \tag{7.6}$$
$$d\tilde{q}' = ((\alpha^2 - \tfrac{1}{4})\,\tilde{q}\,\xi^{-2} - \tilde{q})\,\widehat{d\xi} - \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi, \tag{7.7}$$
$$dp = \widehat{d\xi}\,p' - r\,d\xi\,p, \tag{7.8}$$
$$d\tilde{p} = \tilde{p}'\,\widehat{d\xi} - \tilde{p}\,d\xi\,r, \tag{7.9}$$
$$dp' = \widehat{d\xi}\,((\alpha^2 - \tfrac{1}{4})\,\xi^{-2}\,p + 2q - p) - (r_x\,d\xi + d\xi\,r_y)\,p + d\xi\,r\,p', \tag{7.10}$$
$$d\tilde{p}' = ((\alpha^2 - \tfrac{1}{4})\,\tilde{p}\,\xi^{-2} + 2\tilde{q} - \tilde{p})\,\widehat{d\xi} - \tilde{p}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{p}'\,r\,d\xi. \tag{7.11}$$

Equations (7.2) and (7.1) are obtained in the same way as (4.21). We have

$$\partial_{kw} r_{x,iu,jv} = \partial_{kw} R_{xij}(\xi_{iu}, \xi_{jv}) = (-1)^w R_{xik}(\xi_{iu}, \xi_{kw}) R_{kj}(\xi_{kw}, \xi_{jv}) \\ + R_{xxij}(\xi_{iu}, \xi_{jv}) \delta_{iu,kw} + R_{xyij}(\xi_{iu}, \xi_{jv}) \delta_{jv,kw}.$$

This gives (7.2) and (7.3) is analogous.

So all the equations are universal except for (7.6) and (7.10) and their duals. What we have to do is show that the diagonal entries of $r_{xx} + r_{xy}$ and $r_{xy} + r_{yy}$, and the off-diagonal entries of $r_{xx},\ r_{xy}$ and r_{yy} are all known, and to establish Eqs. (7.6) and (7.10).

To begin, we denote by $L^{\pm}$ the kernels where $\Phi_\alpha(xz)\,\Phi_\alpha(yz)$ in the integrand is replaced by

$$\Phi_\alpha(xz)\,\Phi_\alpha(yz) \pm \Phi_{\alpha+1}(xz)\,\Phi_{\alpha+1}(yz).$$

When $\alpha = -1/2,\ L^+$ is essentially the extended sine kernel and some of the formulas we derive here will specialize to those obtained in Sect. VI. We use the notations $\beta = \frac{1}{2} + \alpha$ and

$$\zeta(x, y) = \varphi(x)\,\psi(y) - \psi(x)\varphi(y), \quad \eta(x, y) = \varphi(x)\,\psi(y) + \psi(x)\,\varphi(y),$$

$$\Omega = (e^{(\tau_i - \tau_j)/2}).$$

After integration by parts and some computation using the differentiation formulas

$$\Phi_\alpha'(z) = -\Phi_{\alpha+1}(z) + \beta\, z^{-1} \Phi_\alpha(z), \quad \Phi_{\alpha+1}'(z) = \Phi_\alpha(z) - \beta\, z^{-1} \Phi_{\alpha+1}(z)$$

we find that

$$\begin{aligned}
L_x^+ &= \frac{1}{\tau_i - \tau_j} \Omega\, \zeta(x, y) + \frac{\beta}{x}\, L^- + \frac{1}{\tau_i - \tau_j} (x - y)\, L^+, \\
L_y^+ &= -\frac{1}{\tau_i - \tau_j} \Omega\, \zeta(x, y) + \frac{\beta}{y}\, L^- + \frac{1}{\tau_i - \tau_j} (y - x)\, L^+, \\
L_x^- &= -\frac{1}{\tau_i - \tau_j} \Omega\, \eta(x, y) + \frac{\beta}{x}\, L^+ + \frac{1}{\tau_i - \tau_j} (x + y)\, L^-, \\
L_y^- &= -\frac{1}{\tau_i - \tau_j} \Omega\, \eta(x, y) + \frac{\beta}{y}\, L^+ + \frac{1}{\tau_i - \tau_j} (x + y)\, L^-.
\end{aligned}$$

Here the i, j entries of the matrices $L^{\pm}$ and Ω are to be understood.

If we add the first two identities and subtract the last two we obtain the commutator-anticommutator pair

$$[D, L^+] = \beta\{M^{-1}, L^-\}, \quad \{D, L^-\} = \beta[M^{-1}, L^+]. \tag{7.12}$$

To obtain another pair, first multiply the first two identities by $\tau_i - \tau_j$ and subtract, getting

$$(\tau_i - \tau_j)\,(L_x^+ - L_y^+) = 2\,\Omega\,\zeta(x, y) + \left(\frac{\beta}{x} - \frac{\beta}{y}\right)(\tau_i - \tau_j)\, L^- + 2(x - y)\, L^+.$$

Using the first two identities again we can write the left side as

$$\tau_i\, L_x^+ + \tau_j\, L_y^+ - \tau_i \left(-\frac{1}{\tau_i - \tau_j} \Omega\, \zeta(x,y) + \frac{\beta}{y}\, L^- + \frac{1}{\tau_i - \tau_j}\,(y-x)\, L^+ \right)$$

$$-\tau_j \left(\frac{1}{\tau_i - \tau_j} \Omega\, \zeta(x,y) + \frac{\beta}{x}\, L^- + \frac{1}{\tau_i - \tau_j}\,(x-y)\, L^+ \right)$$

$$= \tau_i\, L_x^+ + \tau_j\, L_y^+ - \left(\frac{\beta}{y}\tau_i + \frac{\beta}{x}\tau_j \right) L^- + (x-y)\, L^+ + \Omega\, \zeta(x,y).$$

Thus

$$\tau_i\, L_x^+ + \tau_j\, L_y^+ = \left(\frac{\beta}{x}\tau_i + \frac{\beta}{y}\tau_j \right) L^- + (x-y)\, L^+ + \Omega\, \zeta(x,y).$$

In other words

$$[\tau D - M,\ L^+] = \beta\{M^{-1}\tau,\ L^-\} + \Omega\, \zeta(x,y).$$

Next multiply the last two identities by $\tau_i - \tau_j$ and add, getting

$$(\tau_i - \tau_j)\,(L_x^- + L_y^-) = -2\, \Omega\, \eta(x,y) + \left(\frac{\beta}{x} + \frac{\beta}{y} \right) (\tau_i - \tau_j)\, L^+ + 2(x+y)\, L^-.$$

The left side may be rewritten

$$\tau_i\, L_x^- - \tau_j\, L_y^- + \tau_i \left(-\frac{1}{\tau_i - \tau_j} \Omega\, \eta(x,y) + \frac{\beta}{y}\, L^+ + \frac{1}{\tau_i - \tau_j}\,(x+y)\, L^- \right)$$

$$-\tau_j \left(-\frac{1}{\tau_i - \tau_j} \Omega\, \eta(x,y) + \frac{\beta}{x}\, L^+ + \frac{1}{\tau_i - \tau_j}\,(x+y)\, L^- \right)$$

$$= \tau_i\, L_x^- - \tau_j\, L_y^- + \left(\frac{\beta}{y}\tau_i - \frac{\beta}{x}\tau_j \right) L^+ + (x+y)\, L^- - \Omega\, \eta(x,y).$$

Thus

$$\tau_i\, L_x^- - \tau_j\, L_y^- = \left(\frac{\beta}{x}\tau_i - \frac{\beta}{y}\tau_j \right) L^+ + (x+y)\, L^- - \Omega\, \eta(x,y).$$

In other words

$$\{\tau D - M,\ L^-\} = \beta[M^{-1}\tau,\ L^+] - \Omega\, \eta(x,y).$$

Thus we have our second commutator-anticommutator pair

$$[\tau D - M,\ L^+] = \beta\{M^{-1}\tau,\ L^-\} + \Omega\ \zeta(x,y), \tag{7.13}$$

$$\{\tau D - M,\ L^-\} = \beta[M^{-1}\tau,\ L^+] - \Omega\ \eta(x,y). \tag{7.14}$$

Now we have the following.

Lemma. *Suppose A and B are such that*

$$[A,\ L^+] = \{B,\ L^-\} + F,\ \ \{A,\ L^-\} = [B,\ L^+] + G.$$

Then

$$[A^2 - B^2,\ L^+] = [[A,\ B],\ L^-] + \{A,\ F\} + \{B,\ G\},$$

$$[A^2 - B^2,\ L^-] = [[A,\ B],\ L^+] + [A,\ G] + [B,\ F].$$

Proof. We have

$$[A^2,\ L^+] = \{A,\ [A,\ L^+]\} = \{A, \{B,\ L^-\}\} + \{A,\ F\}.$$

By the general identity

$$\{A, \{B, C\}\} = [[A, B], C] + \{B, \{A, C\}\},$$

the first term on the right side above may be written

$$\begin{aligned} [[A, B], L^-] + \{B, \{A, L^-\}\} &= [[A, B], L^-] + \{B, [B,\ L^+]\} + \{B,\ G\} \\ &= [[A, B], L^-] + [B^2,\ L^+] + \{B,\ G\}. \end{aligned}$$

This establishes the first stated identity.

For the second we write

$$[A^2,\ L^-] = [A,\ \{A,\ L^-\}] = [A,\ [B,\ L^+]] + [A,\ G].$$

By the general identity

$$[A,\ [B,\ C]] + [B,\ [C,\ A]] + [C,\ [A,\ B]] = 0,$$

the first term on the right side above may be written

$$\begin{aligned} -[B,\ [L^+,\ A]] - [L^+,\ [A,\ B]] &= [B,\ \{B,\ L^-\}] + [B,\ F] - [L^+,\ [A,\ B]] \\ &= [B^2,\ L^-] + [[A,\ B],\ L^+] + [B,\ F]. \end{aligned}$$

This gives the second identity.

We have obtained in (7.12) and (7.13)–(7.14) two quadruples $(A_1,\ B_1,\ F_1,\ G_1)$ and $(A_2,\ B_2,\ F_2,\ G_2)$ satisfying the hypothesis of the lemma. Each gives commutator relations involving L^+ and L^-. However $(A_1 + A_2,\ B_1 + B_2,\ F_1 + F_2,\ G_1 + G_2)$ will also satisfy the hypothesis of the lemma and so gives commutator relations involving L^+ and L^-. If we subtract from these the relations resulting from the other two we obtain

$$\begin{aligned} [AA' + A'A - BB' - B'B,\ L^+] &= [[A,\ B'] + [A',\ B],\ L^-] + \{A,\ F'\} \\ &\quad + \{A',\ F\} + \{B,\ G'\} + \{B',\ G\}, \\ [AA' + A'A - BB' - B'B,\ L^-] &= [[A,\ B'] + [A',\ B],\ L^+] + [A,\ G'] \\ &\quad + [A',\ G] + [B,\ F'] + [B',\ F]. \end{aligned}$$

So in the end we will obtain three pairs of commutator relations involving L^+ and L^-. If we add the identities in each pair and divide by 2 we obtain three commutator identities for L. For the explicit computations we have to keep in mind that all matrices and operators commute with φ and ψ, and D and M commute with τ and Ω. We write down the results, sparing the reader the details:

$$\begin{aligned}
&[D^2 + \beta\,(1-\beta)\,M^{-2},\ L] = 0,\\
&2[\tau D^2 - MD + \beta(1-\beta)\tau\,M^{-2},\ L]\\
&\quad = \Omega\left(\varphi\otimes\psi\,(D - \beta\,M^{-1}) - (D + \beta\,M^{-1})\,\psi\otimes\varphi\right),\\
&[(\tau\,D - M)^2 + \beta\,(1-\beta)\,\tau^2\,M^{-2},\ L]\\
&\quad = \Omega\,\varphi\otimes\psi\,(\tau\,D - M - \beta\,\tau\,M^{-1}) - (\tau\,D - M + \beta\,\tau\,M^{-1})\,\Omega\,\psi\otimes\varphi.
\end{aligned}$$

The differentiation formula for $\Phi_{\alpha+1}$ is in our notation $(D + \beta\,M^{-1})\,\psi = \varphi$. Also, an operator acts on $\varphi\otimes\psi$ from the right by applying its transpose to ψ. Using these facts we see that the last two identities simplify to

$$\begin{aligned}
&[\tau D^2 - MD + \beta(1-\beta)\tau\,M^{-2},\ L] = -2\,\Omega\,\varphi\otimes\varphi,\\
&[(\tau\,D - M)^2 + \beta\,(1-\beta)\,\tau^2\,M^{-2},\ L] = -2\Omega\,\tau\,\varphi\otimes\varphi - \Omega(\varphi\otimes M\psi - M\psi\otimes\varphi).
\end{aligned}$$

The commutator identities for L lead as before to commutator identities for $K = L\chi$. They are

$$\begin{aligned}
[D^2 + \beta\,(1-\beta)\,M^{-2},\ K] &= L\,(\delta D + D\delta),\\
[\tau D^2 - MD + \beta(1-\beta)\tau\,M^{-2},\ K] &= -2\,\Omega\,\varphi\otimes\varphi\chi + L\ \left(\tau\,(\delta D + D\delta) - M\delta\right),\\
[(\tau\,D - M)^2 + \beta\,(1-\beta)\,\tau^2\,M^{-2},\ K] &= -2\Omega\,\tau\,\varphi\otimes\varphi\chi\\
&\quad -\Omega(\varphi\otimes M\psi\chi - M\psi\otimes\varphi\chi)\\
&\quad +L\ \left(\tau^2\,(\delta D + D\delta) - 2\tau\,M\delta\right).
\end{aligned}$$

We are ready to apply $\rho = (I - K)^{-1}$ to both sides. The only functions that appear on the right sides are φ and $M\psi$, which is why we define

$$\begin{aligned}
Q &= (I-K)^{-1}\varphi, \quad \tilde{Q} = \varphi\,\chi\,(I-K)^{-1},\\
P &= (I-K)^{-1}M\psi, \quad \tilde{P} = M\psi\,\chi\,(I-K)^{-1}.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
[D^2 + \beta\,(1-\beta)\,M^{-2},\ R] &= R\,\delta\,\rho_x - R_y\,\delta\,\rho,\\
[\tau D^2 - MD + \beta(1-\beta)\tau\,M^{-2},\ R] &= -2\,Q(x)\,\Omega\,\tilde{Q}(y) + R\,\tau\delta\,\rho_x\\
&\quad -R_y\,\tau\delta\,\rho - R\,\xi\,\rho,\\
[(\tau\,D - M)^2 + \beta(1-\beta)\tau^2\,M^{-2},\ R] &= -2Q(x)\,\Omega\tau\,\tilde{Q}(y) - Q(x)\,\Omega\,\tilde{P}(y)\\
&\quad +P(x)\,\Omega\,\tilde{Q}(y) + R\,\tau^2\delta\,\rho_x\\
&\quad -R_y\,\tau^2\delta\,\rho - R\,\tau\xi\,\rho.
\end{aligned} \tag{7.15}$$

We now show that the diagonal entries of $r_{xx} + r_{xy}$ and $r_{xy} + r_{yy}$, and the off-diagonal entries of $r_{xx},\ r_{xy}$ and r_{yy} can all be expressed in terms of the unknowns.

We use the symbol $\equiv$ here to mean that the difference of the quantities on its left and right is expressible in terms of $Q,\ P,\ \tilde{Q},\ \tilde{P}$ and R, but no derivatives of these functions. The three commutator identities above yield in this notation the relations

$$R_{xx} - R_{yy} \equiv R\delta\rho_x - R_y\delta\rho, \tag{7.16}$$
$$\tau_i\, R_{xx} - \tau_j\, R_{yy} \equiv x\, R_x + y\, R_y + R\,\tau\delta\,\rho_x - R_y\,\tau\delta\,\rho, \tag{7.17}$$
$$\tau_i^2\, R_{xx} - \tau_j^2\, R_{yy} \equiv 2\,\tau\, x\, R_x + 2\, y\, R_y\,\tau + R\,\tau^2\delta\,\rho_x - R_y\,\tau^2\delta\,\rho. \tag{7.18}$$

Consider first the case $i \neq j$. It follows from any pair of the above equations (everything now is to be evaluated at (ξ_{iu}, ξ_{jv})) that both R_{xx} and R_{yy} are known. If we call the right sides above $A,\ B$ and C then

$$\begin{vmatrix} 1 & 1 & A \\ \tau_i & \tau_j & B \\ \tau_i^2 & \tau_j^2 & C \end{vmatrix} \equiv 0.$$

If we differentiate with respect to x we deduce that the sum of all terms involving R_{xy} is known. (Since our unknowns involved up to one derivative, this is why in our definition of $\equiv$ we required that no derivatives were involved in the difference.) This sum is

$$-\tau_i\tau_j(\tau_j - \tau_i)\, R_{xy}\delta R - (\tau_j^2 - \tau_i^2)\,(yR_{xy} - R_{xy}\tau\delta R)$$
$$+(\tau_j - \tau_i)\,(2\tau_j y\, R_{xy} - R_{xy}\tau^2\delta R).$$

Dividing this by $\tau_j - \tau_i$, evaluating at (ξ_{iu}, ξ_{jv}) and expanding we obtain

$$\sum_{k,w}(-1)^w(\tau_i - \tau_k)\,(\tau_j - \tau_k)\, r_{kw,jv}\, r_{xy,iu,kw} + (\tau_j - \tau_i)\,\xi_{jv}\, r_{xy,iu,jv}.$$

The terms involving $k = i$ vanish, so equating the above with the known quantity it is equal to gives a system of equations (with iu fixed) for the $r_{xy,iu,kw}$ with $k \neq i$. The jv, kw entry of the matrix for the system is

$$(-1)^w\,(\tau_i - \tau_k)\,(\tau_j - \tau_k)\, r_{kw,jv} + (\tau_j - \tau_i)\,\xi_{jv}\,\delta_{jv,kw}.$$

The determinant of this matrix is a polynomial in the entries of r and ξ. (We think of the τ_j as fixed.) In the expansion of the determinant one summand is $\prod_{jv}(\tau_j - \tau_i)\,\xi_{jv}$. Every other summand will contain at least one $r_{kw,jv}$ factor. If we look at the series expansions for these other summands valid for small ξ_{jv} (coming from the series for the Bessel functions and the Neumann series for the resolvent), every term will be a product of powers of the ξ_{jv} and have as coefficient a negative integral power of $\Gamma(\alpha)$ times a rational function of α. It follows that in the series expansion of the determinant the coefficient of $\prod_{jv}\xi_{jv}$ is nonzero. Thus the determinant cannot be identically zero.

We have shown that if $i \neq j$ then $r_{xy,iu,jv}$ is expressible in terms of the unknown functions. It remains to consider the cases where $i = j$, and we always evaluate at (ξ_{iu}, ξ_{iv}). In this case (7.16) shows that $R_{xx} - R_{yy}$ is known. Subtracting τ_i times (7.16) from (7.17) gives

$$0 \equiv x\, R_x + y\, R_y + R\tau\delta R_x - R_y\tau\delta R - \tau_i(R\delta R_x - R_y\delta R).$$

All terms here involving δ are sums over k. The terms involving $k \neq i$, even after taking ∂_x or ∂_y, are known, as we have shown. Those involving $k = i$ cancel, just as before. Hence applying ∂_x and ∂_y to the above and evaluating at (ξ_{iu}, ξ_{iv}) shows that

$$\xi_{iu}\, r_{xx,iu,iv} + \xi_{iv}\, r_{xy,iu,iv} \text{ and } \xi_{iu}\, r_{xy,iu,iv} + \xi_{iv}\, r_{yy,iu,iv}$$

are known. Taking $v = u$ shows that both $r_{xx,iu,iu} + r_{xy,iu,iu}$ and $r_{xy,iu,iu} + r_{yy,iu,iu}$ are known. If $u \neq v$, using the fact that $r_{yy,iu,iv} - r_{xx,iu,iv}$ is known, we see also that $r_{xx,iu,iv}$ and $r_{xy,iu,iv}$ are individually known.

All that we have left to show are (7.6) and (7.10). For these we use (7.15) (the analogue here of Lemma 2) and the facts

$$(D^2 + \beta\,(1-\beta)\,M^{-2})\,\varphi = -\varphi, \quad (D^2 + \beta\,(1-\beta)\,M^{-2})\,M\psi = 2\varphi - M\psi,$$

which follow from the differentiation formulas. (The first is just the differential equation satisfied by Φ_α; the second is a miracle.) We use these to compute $Q''_{iu,j}(\xi_{iu})$ and $P''_{iu,j}(\xi_{iu})$ as for previous equations. Thus, for example, to obtain (7.6) we replace the term $\widehat{d\xi}\,(\xi^2 - 2n - 1)q$ in (5.28) by $\widehat{d\xi}\,(-\beta\,(1-\beta)\,\xi^{-2}\,q - q)$ and to obtain (7.10) we replace the term $\widehat{d\xi}\,(\xi^2 - 2n + 1)p$ in (5.32) by $\widehat{d\xi}\,(-\beta\,(1-\beta)\,\xi^{-2}\,p + 2q - p)$. Any reader who has come this far can easily supply the details.

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