JOURNAL OF MATHEMATICAL PHYSICS **50**, 095204 (2009)

Total current fluctuations in the asymmetric simple exclusion process

Craig A. Tracy[1,a)] and Harold Widom[2,b)]
[1]*Department of Mathematics, University of California, Davis, California 95616, USA*
[2]*Department of Mathematics, University of California, Santa Cruz, California 95064, USA*

(Received 4 February 2009; accepted 17 March 2009; published online 8 June 2009)

A limit theorem for the total current in the asymmetric simple exclusion process (ASEP) with step initial condition is proven. This extends the result of Johansson on TASEP to ASEP. © *2009 American Institute of Physics.*
[DOI: 10.1063/1.3136630]

I. INTRODUCTION

The asymmetric simple exclusion process (ASEP) is a continuous time Markov process of interacting particles on the integer lattice $\mathbb{Z}$ subject to two rules: (1) A particle at x waits an exponential time with parameter one (independently of all other particles) and then it chooses y with probability $p(x,y)$; (2) if y is vacant at that time it moves to y, while if y is occupied, it remains at x and restarts its clock. The adjective "simple" refers to the fact that the allowed jumps are one step to the right, $p(x,x+1)=p$, or one step to the left $p(x,x-1)=1-p=q$. The asymmetric condition is $p\neq q$ so there is a net drift of particles. The special cases $p=1$ (particles hop only to the right) or $q=1$ (particles hop only to the left) are called the T(totally)ASEP. The dynamics are uniquely determined once we specify the initial state, which may be either deterministic or random. A rigorous construction of this infinite particle process can be found in Ref. 7 by Liggett.

Since its introduction by Spitzer,[12] the ASEP has remained a popular model among probabilists and physicists because it is one of the simplest nontrivial processes modeling nonequilibrium phenomena. (For recent reviews, see, Refs. 4, 8, 10, and 13.) If initially the particles are located at $\mathbb{Z}^+=\{1,2,\ldots\}$, called the *step initial condition*, and if $p<q$, then there will be on average a net flow of particles, or *current*, to the left. More precisely, we introduce the *total current* $\mathcal{I}$ at position $x\leq 0$ at time t,

$$\mathcal{I}(x,t) \coloneqq \#\ \text{ of particles} \leq x \ \text{ at time } \ t.$$

With step initial condition, it has been known for some time (see, e.g., Theorem 5.12 in Ref. 7) that if we set $\gamma\coloneqq q-p>0$ and $0\leq c\leq\gamma$, then the current $\mathcal{I}$ satisfies the strong law,

$$\lim_{t\to\infty}\frac{\mathcal{I}([-ct],t)}{t}=\frac{1}{4\gamma}(\gamma-c)^2 \text{ a.s.}$$

The natural next step is to examine the current fluctuations,

$$\mathcal{I}(x,t)-\frac{1}{4\gamma}(\gamma-c)^2 t \tag{1}$$

for large x and t. Physicists conjectured,[6] and Johansson proved for TASEP,[5] that to obtain a nontrivial limiting distribution the correct normalization of (1) is cube root in t. For TASEP

[a)]Electronic mail: tracy@math.ucdavis.edu.
[b)]Electronic mail: widom@ucsc.edu.

Johansson not only proved that the fluctuations are of order $t^{1/3}$ but also found the limiting distribution function. Precisely, for $0 \le v < 1$ we have[1]

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\mathcal{I}([-vt],t) - a_1 t}{a_2 t^{1/3}} \le s\right) = 1 - F_2(-s), \tag{2}$$

where

$$a_1 = \tfrac{1}{4}(1-v)^2, \quad a_2 = 2^{-4/3}(1-v^2)^{2/3}, \tag{3}$$

and F_2 is the limiting distribution of the largest eigenvalue in the Gaussian unitary ensemble.[14]

The proof of this relied on the fact that TASEP is a determinantal process.[5,11,13] However, universality arguments suggest that (2) should extend to ASEP with step initial condition even though ASEP is not a determinantal process. When the initial state is the Bernoulli product measure, it has been recently proved, using general probabilistic arguments, that the correct normalization remains $t^{1/3}$ for a large class of stochastic models including ASEP.[1–3,9]

In this paper we show that (2) does extend to ASEP.

Theorem: For ASEP with step initial condition we have, for $0 \le v < 1$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\mathcal{I}([-vt],t/\gamma) - a_1 t}{a_2 t^{1/3}} \le s\right) = 1 - F_2(-s),$$

where $\gamma = q - p$ and a_1 and a_2 are given by (3).[2]

This theorem is a corollary, as we show below, of earlier work by the authors.[15]

II. PROOF OF THE THEOREM

We denote by $x_m(t)$ the position of the mth leftmost particle [thus $x_m(0) = m \in \mathbb{Z}^+$]. We are interested in the probability of the event,

$$\{\mathcal{I}(x,t) = m\} = \{x_m(t) \le x, x_{m+1}(t) > x\}. \tag{4}$$

The sample space consists of the four disjoint events $\{x_m(t) \le x, x_{m+1}(t) > x\}$, $\{x_m(t) \le x, x_{m+1}(t) \le x\}$, $\{x_m(t) > x, x_{m+1}(t) > x\}$, and $\{x_m(t) \le x, x_{m+1}(t) \le x\}$, and because of the exclusion property, we have

$$\{x_m(t) \le x, x_{m+1}(t) \le x\} = \{x_{m+1}(t) \le x\},$$

$$\{x_m(t) > x, x_{m+1}(t) > x\} = \{x_m(t) > x\},$$

$$\{x_m(t) > x, x_{m+1}(t) \le x\} = \emptyset .$$

These observations and (4) give (the intuitively obvious)

$$\mathbb{P}(\mathcal{I}(x,t) = m) = \mathbb{P}(x_m(t) \le x) - \mathbb{P}(x_{m+1}(t) \le x).$$

Since $\mathbb{P}(\mathcal{I}(x,t) = 0) = \mathbb{P}(x_1(t) > x)$, we have

$$\mathbb{P}(\mathcal{I}(x,t) \le m) = 1 - \mathbb{P}(x_{m+1}(t) \le x).$$

Thus, since x and $x_{m+1}(t)$ are integers, the statement of the Theorem is equivalent to the statement that

[1]The value of a_2 given in (3) corrects a misprint in Corollary 1.7 of Ref. 5.

[2]With step initial condition and $x > 0$ the total current equals the number of particles to the left of x at time t minus x. In what follows we shall require only that $|v| < 1$. Therefore the statement of the Theorem holds for all such v if when $v < 0$ the value of a_1 is decreased by $|v|$.

$$\lim_{t\to\infty} \mathbb{P}(x_{m+1}(t/\gamma) \le -vt) = F_2(s),$$

when $m=[a_1t-a_2st^{1/3}]$. In fact, we shall show that

$$\lim_{t\to\infty} \mathbb{P}(x_m(t/\gamma) \le -vt) = F_2(s), \tag{5}$$

when

$$m = a_1t - a_2st^{1/3} + o(t^{1/3}). \tag{6}$$

Let

$$\sigma = \frac{m}{t}, \quad c_1 = -1 + 2\sqrt{\sigma}, \quad c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}.$$

It was proved in Ref. 15 that when $0 \le p < q$,

$$\lim_{t\to\infty} \mathbb{P}(x_m(t/\gamma) \le c_1t + sc_2t^{1/3}) = F_2(s) \tag{7}$$

uniformly for σ in a compact subset of (0,1).

To obtain (5) from this we determine σ so that

$$-vt = c_1t + sc_2t^{1/3}.$$

Thus,

$$v = 1 - 2\sqrt{\sigma} - s\sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}t^{-2/3}.$$

Solving, we get

$$\left(\frac{1-v}{2}\right)^2 = \sigma + s\sigma^{1/3}(1-\sqrt{\sigma})^{2/3}t^{-2/3} + O(t^{-4/3}),$$

from which we deduce[3]

$$\sigma = \left(\frac{1-v}{2}\right)^2 - s\left(\frac{1-v}{2}\right)^{2/3}\left(\frac{1+v}{2}\right)^{2/3}t^{-2/3} + O(t^{-4/3}) = \left(\frac{1-v}{2}\right)^2 - s2^{-4/3}(1-v^2)^{2/3}t^{-2/3} + O(t^{-4/3}).$$

By the uniformity of (7) in σ, we get the same asymptotics if we replace the σ we just computed by any σ satisfying

$$\sigma = \left(\frac{1-v}{2}\right)^2 - s2^{-4/3}(1-v^2)^{2/3}t^{-2/3} + o(t^{-2/3}).$$

Since this is exactly the statement that $m=\sigma t$ satisfies (6), we see that the Theorem is established.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Grant Nos. DMS–0553379 (C.A.T.) and DMS–0552388 (H.W.).

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