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On the Singularities in the Susceptibility Expansion for the Two-Dimensional Ising Model

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Abstract For temperatures below the critical temperature, the magnetic susceptibility for the two-dimensional isotropic Ising model can be expressed in terms of an infinite series of multiple integrals. With respect to a parameter related to temperature and the interaction constant, the integrals may be extended to functions analytic outside the unit circle. In a groundbreaking paper, Nickel (J Phys A 32:3889–3906, 1999) identified a class of singularities of these integrals on the unit circle. In this note we show that there are no other singularities on the unit circle.

Keywords Ising model · Magnetic susceptibility · Singularities

1 Introduction

For the two-dimensional zero-field Ising model on a square lattice, the magnetic susceptibility as a function of temperature is usually studied through its relation with the zero-field spin-spin correlation function:

$$\beta^{-1}\chi = \sum_{M,N\in\mathbb{Z}} \left\{\langle\sigma_{0,0}\,\sigma_{M,N}\rangle - \mathcal{M}^2\right\} \tag{1}$$

where $\beta = (k_B T)^{-1}$, T is temperature, k_B is Boltzmann's constant and $\mathcal{M}$ is the spontaneous magnetization (see, e.g., [8]). Fisher [4] in 1959 initiated the analysis of the analytic structure of χ near the critical temperature T_c by relating it to the long-distance asymptotics of the correlation function at T_c (a result known to Kaufman and Onsager). Subsequently Wu et al. [15] derived the exact *form factor expansion* of χ which has the structure of an infinite series

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whose nth order term is an n-dimensional integral. In later work [12,13,16,17] the structure of the integrands of these n-dimensional integrals was simplified.

The analysis of χ as a function of the *complex variable* T was initiated by Guttmann and Enting [5] where, by the use of high-temperature series expansions, they were led to conjecture that χ, as a function of T, possesses a natural boundary. In two groundbreaking papers, Nickel [10,11] analyzed the n-dimensional integrals appearing in the form factor expansion of χ and identified a class of complex singularities, now called *Nickel singularities*, that lie on a curve and which become ever more dense with increasing n. This work of Nickel provides very strong support for the existence of a natural boundary for χ.[1] For further developments see Chan et al. [3] and the review article [9].

We recall that if T_c denotes the critical temperature, then for the isotropic Ising model, where horizontal and vertical interaction constants have the same value J, the spontaneous magnetization is given for $T < T_c$ by [8,18]

$$\mathcal{M} = (1 - k^2)^{1/8},$$

where $k := (\sinh 2\beta J)^{-2}$; and $\mathcal{M}$ is zero for $T > T_c$. Thus $k = 1$ defines the critical temperature T_c and $0 < k < 1$ is the region $0 < T < T_c$. Boukraa et al. [1] (building on work of Lyberg and McCoy [7]) introduced a simplified model for χ, called the *diagonal susceptibility* χ_d which has the following analogous representation to (1):

$$\beta^{-1}\chi_d = \sum_{N\in\mathbb{Z}} \left\{\langle\sigma_{0,0}\sigma_{N,N}\rangle - \mathcal{M}^2\right\}.$$

By an analysis similar to that of Nickel, they are led to conjecture a natural boundary for χ_d; which in terms of the complex variable k, is the unit circle $|k| = 1$. This conjecture thus says that the low temperature phase $T < T_c$ is separated from the high-temperature phase $T > T_c$ by the natural boundary $|k| = 1$. This conjecture for χ_d is precisely the same as the conjectured natural boundary for χ. In the low-temperature phase, the present authors proved that $|k| = 1$ is a natural boundary for χ_d [14], thus adding additional support for the conjecture for χ.

We now state the results of this paper. We set

$$s = 1/\sqrt{k} = \sinh 2\beta J,$$

so that the low-temperature phase corresponds to $s > 1$. If we define

$$D(x, y; s) = s + s^{-1} - (x + x^{-1})/2 - (y + y^{-1})/2, \tag{2}$$

then we have the expansion

$$\beta^{-1}\chi = \mathcal{M}^2 \sum_{n=1}^{\infty} \chi^{(2n)}, \tag{3}$$

where

$$\chi^{(n)} = \frac{1}{n!}\,\frac{1}{(2\pi i)^{2n}} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \frac{(1+\prod_j x_j^{-1})\,(1+\prod_j y_j^{-1})}{(1-\prod_j x_j)\,(1-\prod_j y_j)} \prod_{j<k} \frac{x_j - x_k}{x_j x_k - 1}\,\frac{y_j - y_k}{y_j y_k - 1}$$
$$\times \prod_j \frac{dx_j\, dy_j}{D(x_j, y_j; s)}.$$

[1] As Nickel noted, for a rigorous proof one must show that there are no cancellations of the singularities in the infinite sum.

Here $\mathcal{C}_r$ denotes the circle with center zero and radius $r < 1$ and r sufficiently close to 1 (depending on s). All indices in the integrand run from 1 to n. A derivation of this representation will be given in Appendix 1.[2]

We extend $\chi^{(n)}$ to a function of the complex variable s with $|s| > 1$. A *Nickel singularity* is a point s^0 on the unit circle $\mathbb{T}$ such that the real part of s^0 is the average of the real parts of two nth roots of unity.

We shall show that for n even these are the only singularities of $\chi^{(n)}$. More precisely, $\chi^{(n)}$ extends from the exterior of $\mathbb{T}$ to a C^∞ function on $\mathbb{T}$ except for the Nickel singularities.[3]

Here we use the term "singularity" to denote a point in no neighborhood of which a function is C^∞. In the physics litearture it usually means a point beyond which a function cannot be continued analytically. It appears that $\chi^{(n)}$ satisfies a linear differential equation with only regular singular points (although the authors admit not having seen a derivation of this that they understand).[4] At a regular singular point the function has a series expansion whose leading term is a fractional or negative power, or a power times a power of the logarithm. Such a function cannot extend from outside $\mathbb{T}$ to be C^∞ in a neighborhood of that point. Therefore we get the stronger result that for n even $\chi^{(n)}$ extends analytically across the unit circle except at the Nickel singularities.

2 Outline of the Proof

With the notations

$$F(x) = \frac{1}{1-\prod_j x_j}, \quad F(y) = \frac{1}{1-\prod_j y_j}, \quad F_{jk}(x) = \frac{1}{1-x_j\,x_k}, \quad F_{jk}(y) = \frac{1}{1-y_j\,y_k},$$

$$G_j(x,y;s) = \frac{1}{D(x_j,y_j;s)} \quad \Delta(x,y) = \left(1+\prod_j x_j^{-1}\right)\left(1+\prod_j y_j^{-1}\right)\prod_{j<k}(x_j-x_k)\,(y_j-y_k),$$

all thought of as functions on $\mathbb{R}^n \times \mathbb{R}^n$, the integral becomes

$$\int_{\mathcal{C}_r^n}\int_{\mathcal{C}_r^n} F(x)\,F(y)\prod_{j<k}F_{jk}(x)\prod_{j<k}F_{jk}(y)\prod_j G_j(x,y;s)\,\Delta(x,y)\,dx\,dy.$$

This equals r^{2n} times

$$\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} F(rx)\,F(ry)\prod_{j<k}F_{jk}(rx)\prod_{j<k}F_{jk}(ry)\prod_j G_j(rx,ry;s)\,\Delta(rx,ry)\,dx\,dy. \quad (4)$$

A partition of unity allows us to localize. At any given point $(x^0, y^0) \in \mathbb{T}^n \times \mathbb{T}^n$ some of the F-factors may become singular as $r \to 1$, and after letting $r \to 1$ some of the G-factors may become singular as $s \to s^0 \in \mathbb{T}$. We represent each of these potentially singular factors as an exponential integral over $\mathbb{R}^+$. The gradient of the exponent in the resulting integrand is approximately a linear combination with positive coefficients of certain vectors, one from each factor. Unless s^0 is a Nickel singularity, the convex hull of these vectors does not contain

[2] Our $\chi^{(2n)}$ is equal to the $\hat{\chi}^{(2n)}$ of [10,11].

[3] For n odd our argument leaves open the possibility of other singularities. See footnote 6.

[4] The equations for $n \le 6$ have been found [2], and all their singularities are regular.

0, a fact that allows us to find a lower bound for the length of the gradient. (This is the crucial point in the proof.[5]) Then several applications of the divergence theorem give the bound $O(1)$ for the integral, uniformly in s and r. The same is true after differentiating with respect to s any number of times. This will imply that $\chi^{(n)}$ extends to a C^∞ function on $\mathbb{T}$ excluding these points.

3 The Proof

For a given point $(x^0, y^0) = ((x_j^0), (y_j^0)) \in \mathbb{T}^n \times \mathbb{T}^n$ some of the factors in (4) become singular as $r \to 1$ and $s \to s^0$, as described above. For example $F(rx)$ becomes singular when $\prod_j x_j^0 = 1$ and $G_j(rx, ry; s)$ becomes singular when

$$\mathcal{R}e\, x_j^0 + \mathcal{R}e\, y_j^0 = 2\,\mathcal{R}e\, s^0.$$

There is a neighborhood of (x^0, y^0) in which no other factors become singular, so that outside this neighborhood the rest of the integrand is a smooth function of x and y and bounded for s in a neighborhood of s^0, together with each of its derivatives with respect to s. Let $\psi(x, y)$ be a C^∞ function with support in this neighborhood. (Eventually the support will be taken even smaller.) We shall show that the integral (4), with the function $\psi(x, y)$ inserted in the integrand, is uniformly bounded for s in a neighborhood of s^0, together with each derivative with respect to s, when r is taken close enough (depending on s) to 1.

In our neighborhood we make the variable changes

$$x_j = x_j^0\, e^{i\theta_j}, \quad y_j = y_j^0\, e^{i\varphi_j}.$$

Below we give the behavior of the reciprocals of the F-factors, in terms of the θ_j, φ_j, if the factors become singular at (x^0, y^0).

$$1/F(rx) = -i \sum_j \theta_j + O\Big((1-r) + \sum_j \theta_j^2\Big),$$

$$1/F(ry) = -i \sum_j \varphi_j + O\Big((1-r) + \sum_j \varphi_j^2\Big),$$

$$1/F_{jk}(rx) = -i\,(\theta_j + \theta_k) + O\Big((1-r) + \theta_j^2 + \theta_k^2\Big),$$

$$1/F_{jk}(ry) = -i\,(\varphi_j + \varphi_k) + O\Big((1-r) + \varphi_j^2 + \varphi_k^2\Big).$$

We note that the real parts of these reciprocals are at least $1 - r$, and so are all positive.

For any G-factor that becomes singular at $(x^0, y^0; s^0)$ we have

$$i/G_j(rx, ry; s) = -\,i\,(\alpha_j\,\theta_j + \beta_j\,\varphi_j) - i\Big[s + s^{-1} - \Big(s^0 + {s^0}^{-1}\Big)\Big] + O\Big((1-r) + \theta_j^2 + \varphi_j^2\Big).$$

[5] Each of the singular limiting factors $F(x)$, $F(y)$, $F_{jk}(x)$, $F_{jk}(y)$, $G_j(x, y; s^0)$ may be interpreted as a distribution on $\mathbb{T}^n \times \mathbb{T}^n$. That 0 is not in the convex hull of the vectors is precisely the condition that allows one to define the product of these distributions as a distribution [6]. This is what led us to the present proof.

where

$$\alpha_j = \mathcal{I}m\, x_j^0, \quad \beta_j = \mathcal{I}m\, y_j^0.$$

The reason we put the factor i on the left is that now the real part of the right side, which is equal to the imaginary part of the expression in brackets, is positive when $\mathcal{I}m\, s > 0$ and r is sufficiently close to 1 (depending on s). This we assume. (Otherwise we replace the factor i by $-i$ and change signs in the definitions of α_j and β_j.)

All estimates are consistent with differentiation. For example, the result of differentiating $1/F(rx)$ with respect to θ_k is $-i + O((1-r) + \sum_j |\theta_j|)$.

In what follows we exclude $s^0 = \pm 1, \pm i$, which are Nickel singularities for even n. Thus we assume $(\alpha_j, \beta_j) \neq (0, 0)$.

Because all real parts of the reciprocals are positive they may be represented as integrals over $\mathbb{R}^+$. Thus, we have for any potentially singular factor,

$$F(rx) = \int_{\mathbb{R}^+} e^{i\xi\,(\sum_j \theta_j + \text{correction})}\, d\xi,$$

$$F(ry) = \int_{\mathbb{R}^+} e^{i\eta\,(\sum_j \varphi_j + \text{correction})}\, d\eta,$$

$$F_{jk}(rx) = \int_{\mathbb{R}^+} e^{i\xi_{jk}\,(\theta_j + \theta_k + \text{correction})}\, d\xi_{jk},$$

$$F_{jk}(ry) = \int_{\mathbb{R}^+} e^{i\eta_{jk}\,(\varphi_j + \varphi_k + \text{correction})}\, d\eta_{jk},$$

$$G_j(rx, ry; s) = i \int_{\mathbb{R}^+} e^{i\zeta_j\,(\alpha_j \theta_j + \beta_j \varphi_j + s + s^{-1} - s^0 - s^{0^{-1}} + \text{correction})}\, d\zeta_j.$$

In all of these, “correction” denotes i times the O terms above.

Thus, the integral (4) is replaced by one in which the cut-off function $\psi(x, y)$ is inserted into the integrand and each potentially singular factor is replaced by an integral over $\mathbb{R}^+$. Denote the number of these factors (and so the number of (ξ, η, ζ)-integrations) by N. We change the order of integration and integrate first with respect to the θ_j, φ_j. We want to apply the divergence theorem so that we eventually get a bound $O(R^{-N-1})$, where R is the radial variable in the N-dimensional (ξ, η, ζ)-space. To do this we have to find a lower bound for the length of the gradient of the sum of the exponents coming from the (ξ, η, ζ)-integrations.

We define the following vectors in $\mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{aligned} X &= (1\,1\,\cdots\,1\,0\,0\,\cdots\,0) \\ Y &= (0\,0\,\cdots\,0\,1\,1\,\cdots\,1) \\ X_{jk} &= (0\cdots1\cdots1\cdots\,0\,0\,\cdots) \\ Y_{jk} &= (\cdots\,0\,0\,\cdots0\,1\cdots1\cdots) \\ Z_j &= (0\,\cdots0\,\alpha_j\,0\cdots0\,\beta_j\cdots0). \end{aligned}$$

Let us explain. The first n components are the θ_j components, the last n the φ_j components. For X the ones are the first n components and the zeros are the rest, and for Y these are reversed. For X_{jk} the ones are components j and k and the others are zero, and for Y_{jk} the ones are components $n+j$ and $n+k$ and the others are zero. For Z_j component j is α_j and component $n+j$ is β_j, and the others are zero.

Aside from the factor i and the correction term from each summand, the gradient of the sum of the exponents is the subsum of

$$\xi\, X + \eta\, Y + \sum_{j<k} \xi_{jk}\, X_{jk} + \sum_{j<k} \eta_{jk}\, Y_{jk} + \sum_{j} \zeta_j\, Z_j \tag{5}$$

containing the N $(\xi,\ \eta,\ \zeta)$-variables that actually appear.

Lemma 1 *Suppose that n is even and that s^0 is not a Nickel singularity. Then 0 is not in the convex hull of those of the vectors $X,\ Y,\ X_{jk},\ Y_{jk},\ Z_j$ that appear in the subsum of (5).*

Proof We show that if a linear combination of these vectors with nonnegative coefficients is zero, but not all the coefficients are zero, then s^0 is a Nickel singularity. We say that a vector "appears" in the linear combination if its coefficient is nonzero. Some Z_j must appear since all the others have nonnegative components and at least one positive component. (Recall that Z_j appears when $\mathcal{R}e\, x_j^0 + \mathcal{R}e\, y_j^0 = 2\,\mathcal{R}e\, s^0$).

If X_{jk} appears then then so must Z_j and Z_k and $\alpha_j,\ \alpha_k < 0$, to cancel the nonzero components of X_{jk}. But X_{jk} appears only when $x_j^0\, x_k^0 = 1$, so $\alpha_j + \alpha_k = 0$, which is a contradiction. Thus no X_{jk} appears. Similarly no Y_{jk} appears.

Since some Z_j appears either X or Y must. Suppose that X appears. (In particular $\prod x_j^0 = 1$.) Then all $\alpha_j < 0$, and if the coefficient of X is c_X the coefficient of Z_j must be $-c_X/\alpha_j$.

There are two subcases:

(i) Y appears: (In particular $\prod y_j^0 = 1$.) In analogy with the above, if the coefficient of Y is c_Y then the coefficient of Z_j is $-c_Y/\beta_j$. Thus $\alpha_j/\beta_j = c_X/c_Y$ for all j. We claim that this implies that all x_j^0 are equal and all y_j^0 are equal. Consider pairs (x, y) with both in the lower half-plane, and $\mathcal{R}e\, x + \mathcal{R}e\, y = 2\,\mathcal{R}e\, s^0$. Set $x = e^{i\theta},\ y = e^{i\varphi}$. It is an exercise in calculus to show that as θ increases while $\cos\,\theta + \cos\,\varphi$ remains constant the ratio $\mathcal{I}m\, x/\mathcal{I}m\, y = \sin\theta/\sin\,\varphi$ strictly decreases if $\mathcal{R}e\, s^0 > 0$ and strictly increases if $\mathcal{R}e\, s^0 < 0$. Therefore this ratio determines θ, and so x. Similarly the ratio determines y. So all x_j^0 are equal and all y_j^0 are equal, as claimed. They must both be nth roots of unity, so s^0 is a Nickel singularity.

(ii) Y does not appear: Since all Z_j appear, we must have all $\beta_j = 0$ in this case. So all $y_j^0 = \pm 1$. If some $y_j^0 = 1$ then $\mathcal{R}e\, s^0 > 0$, because if $\mathcal{R}e\, s^0$ were negative it could not be the average of 1 and some $\mathcal{R}e\, x_j^0$. Then all $y_j^0 = 1$, for the same reason. Hence each $\mathcal{R}e\, x_j^0 = 2\,\mathcal{R}e\, s - 1$, and since all $\alpha_j < 0$ this implies that all x_j^0 are equal, and equal to some nth root of unity. Thus s^0 is a Nickel singularity. If some $y_j^0 = -1$, and therefore all $y_j^0 = -1$, this again implies that all x_j^0 equal some nth root of unity. Since n is even s^0 is again a Nickel singularity.[6]

□

If 0 is not in the convex hull of vectors then there is a lower bound for linear combinations of them with nonnegative coefficients, even when the vectors are perturbed.

[6] Since -1 is not an nth root of unity when n is odd, these s^0 are not Nickel singularities.

Lemma 2 *Assume 0 is not in the convex hull of the vectors* $V_1, \ldots, V_N$. *Then for sufficiently small* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that, for vectors* U_j *with* $|U_j - V_j| < \varepsilon$ *and coefficients* $c_j \geq 0$, *we have*

$$\Big|\sum_j c_j\, U_j\Big| \geq \delta \sum_j c_j. \tag{6}$$

Proof Suppose the result is not true. Then there is a sequence $\varepsilon_k \to 0$, vectors $U_{j,k}$ with $|U_{j,k} - V_j| \leq \varepsilon_k$, and coefficients $c_{j,k} \geq 0$ such that for each k,

$$\Big|\sum_j c_{j,k}\, U_j\Big| < \frac{1}{k} \sum_j c_{j,k}.$$

By homogeneity we may assume that each $\sum_j c_{j,k} = 1$. Then, by taking subsequences, we may assume that each $c_{j,k}$ converges as $k \to \infty$ to some c_j. Then $\sum_j c_j = 1$, and each $U_{j,k} \to V_j$, so $\sum_j c_j\, V_j = 0$. This is a contradiction. □

Lemma 3 *Assume n is even and* s^0 *is not a Nickel singularity. There is a neighborhood of* (x^0, y^0) *such that if* $\psi(x, y)$ *is a* C^∞ *function with support in that neighborhood then the integral (4), with* ψ *inserted in the integrand and r sufficiently close to 1 (depending on s), is bounded in a neighborhood of* $s = s^0$; *and the same is true for each derivative with respect to s.*

Proof We combine Lemmas 1 and 2 to deduce that if r is close enough to 1 and the support of ψ is small enough, then in the support of ψ the length of the gradient of the exponent in the integral is at least a constant times the sum of the coefficients in the subsum of (5) that arises. Therefore $N + 1$ applications of the divergence theorem shows that the integral over the θ_j, φ_j has absolute value at most a constant times $1/R^{N+1}$, where R is the radial variable in the N-dimensional (ξ, η, ζ)-space.[7] Therefore the integral (4) with $\psi(x, y)$ inserted in the integrand, which results after integration over the (ξ, η, ζ), is $O(1)$ uniformly for s in a neighborhood of s^0. (The integral over $R < 1$ is clearly bounded.) Differentiating with respect to s any number of times just brings down powers of the ζ_j, and so only requires more applications of the divergence theorem. □

Theorem *When n is even* $\chi^{(n)}$ *extends to a* C^∞ *function on* $\mathbb{T}$ *except at the Nickel singularities.*

Proof Assume s^0 is not a Nickel singularity. Each (x^0, y^0) has a neighborhood given by Lemma 3. Finitely many of these neighborhoods cover $\mathbb{T}^n \times \mathbb{T}^n$. We can find a C^∞ partition of unity $\{\psi_i(x, y)\}$ such that the support of each ψ_i is contained in one of these neighborhoods. Each integral (4) with $\psi_i(x, y)$ inserted in the integrand and r sufficiently close to 1, together with each derivative with respect to s, is bounded in a neighborhood of $s = s^0$. Therefore the same is true of (4) itself, and therefore for r^n times (4), which is independent of r, and therefore for $\chi^{(n)}$. This implies[8] that $\chi^{(n)}$ extends to a C^∞ function on $\mathbb{T}$ in a neighborhood of s^0. □

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[7] We explain this in Appendix 2.

[8] We explain this in Appendix 3.

Appendix 1

For $T < T_c$ and $N \geq 0$ we have the following Fredholm determinant representation of the spin-spin correlation function (see [13, p. 375] or [12, p. 142]):

$$\langle \sigma_{00}\, \sigma_{MN} \rangle = \mathcal{M}^2 \det(I + g_{MN}).$$

The operator has kernel

$$g_{MN}(\theta_1, \theta_2) = e^{iM\theta_1 - N\gamma(e^{i\theta_1})}\, h(\theta_1, \theta_2),$$

where

$$h(\theta_1, \theta_2) = \frac{\sinh \frac{1}{2}(\gamma(e^{i\theta_1}) - \gamma(e^{i\theta_2}))}{\sin \frac{1}{2}(\theta_1 + \theta_2)},$$

and $\gamma(z)$ is defined by

$$\cosh \gamma(z) = s + s^{-1} - (z + z^{-1})/2,$$

with the condition that $\gamma(z)$ is real and positive for $|z| = 1$. The operator acts on $L^2(-\pi, \pi)$ with weight function

$$\frac{1}{2\pi\ \sinh \gamma(e^{i\theta})}.$$

Using the identity (see [13, (5.5)] or [12, (2.69)])

$$\det \big(h(\theta_j, \theta_k)\big) = \prod_{j<k} [h(\theta_j, \theta_k)]^2,$$

and the Fredholm expansion we obtain that $\langle \sigma_{00}\, \sigma_{MN} \rangle$ equals

$$\mathcal{M}^2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{1}{(2\pi)^{2n}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{j<k} [h(\theta_j, \theta_k)]^2 \prod_j e^{iM\theta_j - N\gamma(e^{i\theta_j})} \frac{d\theta_j}{\sinh \gamma(e^{i\theta_j})}. \tag{7}$$

Here all indices run from 1 to $2n$. We used the fact that since the matrix $(h(\theta_j, \theta_k))$ is antisymmetric its odd-order determinants vanish.

We have the identity, observed in [11],

$$\frac{\sinh(\frac{1}{2}(\gamma(e^{i\theta_1}) - \gamma(e^{i\theta_2}))}{\sin(\frac{1}{2}(\theta_1 + \theta_2))} = \frac{\sin(\frac{1}{2}(\theta_1 - \theta_2))}{\sinh(\frac{1}{2}(\gamma(e^{i\theta_1}) + \gamma(e^{i\theta_2}))}$$

Therefore, with $x_j = e^{i\theta_j}$,

$$[h(\theta_1, \theta_2)]^2 = \frac{e^{-\gamma(x_1)} - e^{-\gamma(x_2)}}{1 - e^{-\gamma(x_1) - \gamma(x_2)}} \frac{x_1 - x_2}{1 - x_1 x_2}.$$

With $D(x, y; s)$ defined by (2) a short calculation shows that

$$y\, D(x, y; s) = -\frac{1}{2}(y - e^{-\gamma(x)})\, (y - e^{\gamma(x)}).$$

Thus inside the unit circle $1/(y\, D(x, y; s))$ has a pole at $y = e^{-\gamma(x)}$ with residue $1/\sinh\gamma(x)$. It follows that for r sufficiently close to 1,

$$\frac{1}{(2\pi i)^{2n}} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \prod_{j<k} \frac{y_j - y_k}{1 - y_j y_k} \prod_j \frac{y_j^{N-1}\, dy_j}{D(x_j, y_j; s)} = \prod_j \frac{e^{-N\gamma(x_j)}}{\sinh\gamma(x_j)} \prod_{j<k} \frac{e^{-\gamma(x_j)} - e^{-\gamma(x_k)}}{1 - e^{-\gamma(x_j)-\gamma(x_k)}}.$$

We deduce that the integral in (7) equals

$$\frac{1}{(2\pi)^{2n}} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \prod_{j<k} \frac{y_j - y_k}{1 - y_j y_k} \frac{x_j - x_k}{1 - x_j x_k} \prod_j \frac{x_j^M\, y_j^N}{D(x_j, y_j; s)} \prod_j \frac{dx_j}{x_j} \frac{dy_j}{y_j}. \tag{8}$$

It remains to compute

$$\sum_{M,N\in\mathbb{Z}} \left\{\langle\sigma_{0,0}\, \sigma_{M,N}\rangle - \mathcal{M}^2\right\}.$$

Subtracting $\mathcal{M}^2$ in the summand is the same as taking the sum in (7) only over $n > 0$.

To compute the sum over $M, N \in \mathbb{Z}$ we use the fact that $\langle\sigma_{0,0}\, \sigma_{M,N}\rangle$ is even in M and in N, so

$$\sum_{M,N} = 4 \sum_{M,N\geq 0} -2 \sum_{M=0,\, N\geq 0} -2 \sum_{N=0,\, M\geq 0} + \text{the } (0, 0) \text{ term},$$

and find that after summing, the factor $\prod_j x_j^M\, y_j^N$ in the integrand in (8) gets replaced by

$$\frac{(1 + \prod_j x_j)\, (1 + \prod_j y_j)}{(1 - \prod_j x_j)\, (1 - \prod_j y_j)}.$$

This gives (3).

Appendix 2

Suppose f and g are C^∞ functions on $\mathbb{R}^d$, with f having compact support, and we have an integral

$$\int f(\theta)\, e^{g(\theta)}\, d\theta.$$

We write it as

$$\int f(\theta) \frac{\nabla g(\theta)}{|\nabla g(\theta)|^2} \cdot \nabla e^{g(\theta)}\, d\theta.$$

If define the operator L by

$$(Lf)(\theta) = -\nabla \cdot f(\theta) \frac{\nabla g(\theta)}{|\nabla g(\theta)|^2},$$

then q applications of the divergence theorem show that the integral equals

$$\int (L^q f)(\theta)\, e^{g(\theta)}\, d\theta.$$

Now we have

(a) $L^q f$ is a linear combination of (partial) derivatives of f with coefficients that are homogeneous polynomials of degree q in derivatives of the components of $\nabla g/|\nabla g|^2$;
(b) each pth derivative of each component of $\nabla g/|\nabla g|^2$ equals $1/|\nabla g|^{2p+2}$ times a homogeneous polynomial of degree $2p+1$ in derivatives of g.
Assume that we also have
(c) $|\nabla g(\theta)| \geq \mu$ and each derivative of $g(\theta)$ is $O(\mu)$;
(d) each derivative of $f(\theta)$ is $O(1)$.

Then assuming that $\mathcal{R}e\, g$ is uniformly bounded above, we can conclude that

$$\int\limits_{\mathbb{R}^d} f(\theta)\, e^{g(\theta)}\, d\theta = O(\mu^{-q}) \text{ for all } q.$$

In the application in Lemma 3 we have $d = 2n$, g is the sum of the exponents in the integrals, f is the product of other integrands, and μ can be taken to be a small constant times the sum of the coefficients in the subsum of (5).

Appendix 3

Suppose $\mathcal{U}$ is an open set in $\mathbb{T}$, that f is analytic in the region

$$\Omega = \{Rs : s \in \mathcal{U},\quad 1 < R < 1 + \delta\},$$

and that f and each of its derivatives is bounded in Ω. We show that f extends to a C^∞ function on $\Omega \cup \mathcal{U}$.

Pick any $s_0 \in \Omega$. We have for each $k \geq 0$ and $s' \in \Omega$,

$$f^{(k)}(s') = f^{(k)}(s_0) + \int\limits_{s_0}^{s'} f^{(k+1)}(t)\, dt,$$

with the path of integration in Ω. Since $f^{(k+1)}$ is bounded, this shows that that $f^{(k)}$ extends continuously to $\Omega \cup \mathcal{U}$. Denote by $f_k(s)$ this extension. In paticular f_0 is the continuous extension of f. We show that it belongs to C^∞.

We show by induction that $f_0 \in C^k$. We know this for $k = 0$. Assuming this for k, we see that for $s \in \mathcal{U}$,

$$\frac{d^k}{ds^k} f_0(s) = \lim_{s' \to s} \frac{d^k}{ds'^k} f(s') = f^{(k)}(s_0) + \int\limits_{s_0}^{s} f_{k+1}(t)\, dt.$$

It follows that f_0 is $k+1$ times differentiable and

$$\frac{d^{k+1}}{ds^{k+1}} f_0(s) = f_{k+1}(s) = \lim_{s' \to s} \frac{d^{k+1}}{ds'^{k+1}} f(s').$$

This gives the assertion.

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