

Formulas for joint probabilities for the asymmetric simple exclusion process

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In earlier work, the authors [Tracy, C. A. and Widom, H., “Integral formulas for the asymmetric simple exclusion process,” *Commun. Math. Phys.* **279**, 815 (2008)] obtained integral formulas for probabilities for a single particle in the asymmetric simple exclusion process. Here, formulas are obtained for joint probabilities for several particles. In the case of a single particle, the derivation here is simpler than the one in the earlier work for one of its main results. © 2010 American Institute of Physics. [doi:10.1063/1.3431977]

I. INTRODUCTION

The *asymmetric simple exclusion process* (ASEP) is a continuous time Markov process of interacting particles on the integer lattice \mathbb{Z} . Each particle waits exponential time; then with probability p , it jumps one step to the right if the site is unoccupied and otherwise it stays put; with probability $q=1-p$, it jumps one step to the left if the site is unoccupied and otherwise it stays put. We refer the reader to Refs. 15 and 16 for a precise definition of the model.

When particles can jump only to the right or only to the left, the model is referred to as the *totally asymmetric simple exclusion process* (TASEP). This process is quite special in that it, unlike ASEP, is a *determinantal process*.^{4,12,22,23} For example, in N -particle TASEP the transition probability from an initial state $Y=\{y_1, \dots, y_N\}$ to a state $X=\{x_1, \dots, x_N\}$ in time t is given by an $N \times N$ determinant.²² For TASEP with step initial condition ($y_i=i$), Johansson, in the seminal paper,¹² proved a limit law for the particle current fluctuations. (The limiting distribution is the distribution function F_2 of random matrix theory.)

Subsequently, these particle current limit laws for TASEP were extended to other initial conditions including periodic initial conditions,⁶ stationary initial conditions,¹⁰ and the general two-sided Bernoulli initial condition.^{3,10} Much recent work on TASEP has focused on the *joint distributions* of the particle current.^{2,5,7,8,11,13,17} In particular, for step initial condition, the joint distribution of the associated height function and its KPZ scaling (KPZ refers to Kardar, Parisi, and Zhang¹⁴) to the Airy_2 process^{11,18} is now well developed for TASEP.¹³ We refer the reader to Ref. 9 for a recent review of TASEP and closely related random growth models.

For ASEP the results have been restricted to probabilities for a single particle.^{24–27} In Refs. 25 and 26, the present authors extended Johansson’s limit law for TASEP to ASEP, thus proving a stronger form of KPZ universality. Using results from Ref. 25, Sasamoto and Spohn^{20,21,19} and Amir, Corwin, and Quastel¹ obtained explicit formulas for the exact height distribution for the KPZ equation with narrow wedge initial condition. This height distribution interpolates between a standard Gaussian distribution for small time and the F_2 distribution for large time. To extend the results of Refs. 1, 20, 21, and 19 and show that the Airy_2 process is the large time limit of KPZ with narrow wedge initial condition, we would first try to extend the results of Refs. 24 and 25 to joint distributions of particle positions in ASEP. In this paper, we take a *first step* in this direction

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in that we derive integral formulas for joint probabilities. Whether these formulas can be used to study the KPZ scaling limit remains to be seen.

Here is an outline of this paper. We consider an N -particle system with an initial configuration $Y = \{y_1, \dots, y_N\}$, where $y_1 < \dots < y_N$, and denote by $\{x_1(t), \dots, x_N(t)\}$ the configuration of the system at time t . Our starting point is Theorem 2.1 of Ref. 24, a formula for the probability

$$P_Y(x_i(t) = x_i, i = 1, \dots, N) \quad (1)$$

(the subscript indicating the initial configuration), which is a sum of $N!$ multiple integrals, one for each permutation in the symmetric group S_N . The integrals are over small circles about the origin. In Sec. II we obtain a formula for

$$P_Y(x_i(t) = x_i, i = 1, \dots, m), \quad (2)$$

the joint probability for the first m particles, by summing (1) over all x_i with $i > m$ which satisfy $x_m < x_{m+1} < \dots < x_N < \infty$. We may do this because the integrals are over small circles. Application of combinatorial identity (1.6) of Ref. 24 replaces the sum of $N!$ integrals by a sum of $N!/(N-m+1)!$ integrals. This is Theorem 1.

To obtain from this a formula for arbitrary consecutive particles,

$$P_Y(x_i(t) = x_i, i = n, \dots, m), \quad (3)$$

we have to sum over all x_i with $i < n$, which satisfies $-\infty < x_1 < \dots < x_{n-1} < x_n$, and to do this requires first replacing integrals over small circles by integrals over large ones. This is accomplished with the help of Lemma 3.1 of Ref. 24 and gives Theorem 2. Then summing over the indicated x_i and applying combinatorial identity (1.7) of Ref. 24 give a formula for (3) as a sum of integrals over large circles. This is Theorem 3.

One could obtain a formula for the general joint probability,

$$P_Y(x_{m_1}(t) = x_{m_1}, \dots, x_{m_r}(t) = x_{m_r}),$$

by taking $n = m_1$ and $m = m_r$ in Theorem 3 and summing over those x_i satisfying $x_{m_j} < x_i < x_{m_{j+1}}$. These are finite sums and lead to a rather complicated expression which we shall not write down. There does not seem to be a combinatorial identity that simplifies it.

Finally, we consider the special case $P_Y(x_m(t) = x_m)$, the probability for a single particle. The formula for consecutive particles simplifies by using combinatorial identity (1.9) of Ref. 24 and yields Theorem 4. This is exactly Theorem 5.2 of Ref. 24. The proof here is much simpler than the one in Ref. 24, although they use the same ingredients. (The theorem was the starting point for some of the advances mentioned above, and so it is useful to have this more straightforward proof.) The awkwardness of that proof made it seem unlikely at first that one could find formulas for joint probabilities without great effort.

Although we assumed that the initial configuration Y was finite, Theorems 2–4 extend to configurations that are semi-infinite on the right, in particular, to $Y = \mathbb{Z}^+$ (step initial condition). This is shown in Sec. VI.

II. THE FIRST m PARTICLES: SMALL CONTOURS

Theorem 2.1 of Ref. 24 is the formula that is valid when $p \neq 0$,

$$P_Y(x_i(t) = x_i, i = 1, \dots, N) = \sum_{\sigma \in S_N} \int_{C_r} \dots \int_{C_r} A_\sigma \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \xi_i^{-y_i-1} e^{\sum_i \varepsilon(\xi_i)t} d\xi_1 \dots d\xi_N, \quad (4)$$

where

$$A_\sigma = \text{sgn } \sigma \frac{\prod_{i < j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)})}{\prod_{i < j} f(\xi_i, \xi_j)}, \quad f(\xi, \xi') = p + q\xi\xi' - \xi, \quad \varepsilon(\xi) = p\xi^{-1} + q\xi - 1. \quad (5)$$

(In general, the product symbol Π refers only to the first term after it.) Here, C_r is a circle with center zero and radius r , which is so small that all nonzero poles of the integrand lie outside C_r . (All contour integrals are to be given a factor $1/2\pi i$.)

To obtain $P_Y(x_i(t)=x_i, i=1, \dots, m)$, we sum over all x_{m+1}, \dots, x_N such that $x_m < x_{m+1} < \dots < x_N < \infty$, which we may do when $r < 1$. The result of the summation is that the product $\prod_i \xi_{\sigma(i)}^{x_i}$ in (4) is replaced by

$$\prod_{i < m} \xi_{\sigma(i)}^{x_i} \prod_{i \geq m} \xi_{\sigma(i)}^{x_m} \frac{\xi_{\sigma(m+1)} \xi_{\sigma(m+2)}^2 \cdots \xi_{\sigma(N)}^{N-m}}{(1 - \xi_{\sigma(m+1)} \xi_{\sigma(m+2)}) \cdots (1 - \xi_{\sigma(N)})}$$

We take a fixed ordered $(m-1)$ -tuple $\Lambda = \{\ell_1, \dots, \ell_{m-1}\}$ with distinct $\ell_i \leq N$ and consider first the sum over all permutations σ such that

$$\sigma(1) = \ell_1, \dots, \sigma(m-1) = \ell_{m-1}.$$

[This meaning of Λ as an $(m-1)$ -tuple will be retained throughout.]

The product $\prod_{i < m} \xi_{\sigma(i)}^{x_i} \prod_{i \geq m} \xi_{\sigma(i)}^{x_m}$ becomes

$$\prod_{i < m} \xi_{\ell_i}^{x_i} \prod_{j \notin \Lambda} \xi_j^{x_m}$$

(in the second factor here, and elsewhere below, Λ is thought of as a set) and the numerator in the expression for A_σ may be written as

$$\begin{aligned} & \prod_{j \neq \ell_1} f(\xi_{\ell_1}, \xi_j) \prod_{j \neq \ell_1, \ell_2} f(\xi_{\ell_2}, \xi_j) \cdots \prod_{j \neq \ell_1, \dots, \ell_{m-1}} f(\xi_{\ell_{m-1}}, \xi_j) \prod_{m \leq i < j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)}) \\ &= \prod_{i \in \Lambda, j \notin \Lambda} f(\xi_i, \xi_j) \prod_{i < j} f(\xi_{\ell_i}, \xi_{\ell_j}) \prod_{m \leq i < j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)}). \end{aligned}$$

As for $\text{sgn } \sigma$, we will consider a new guise for σ as its restriction to $\Lambda^c = [1, N] \setminus \Lambda$. (It may be associated in an obvious way with a permutation in S_{N-m+1} , as Λ may be associated in an obvious way with a permutation in S_{m-1} .) What we now call $\text{sgn } \sigma$ will be different from the old $\text{sgn } \sigma$ because the number of inversions in the original σ equals the number of inversions of the new guise σ plus the number of inversions in the original σ which involve some ℓ_i . This number is

$$\sum_{i < m} (\ell_i - i) + \text{No. of inversions in } \Lambda.$$

Thus, the original $\text{sgn } \sigma$ in (5) is to be replaced by

$$(-1)^{\sum_{i < m} (\ell_i - i)} \text{sgn } \Lambda \text{sgn } \sigma,$$

where now σ is in its new guise.

Identity (1.6) of Ref. 24 tells us that, with the new guise σ ,

$$\begin{aligned} & \sum_{\sigma} \text{sgn } \sigma \prod_{m \leq i < j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)}) \frac{\xi_{\sigma(m+1)} \xi_{\sigma(m+2)}^2 \cdots \xi_{\sigma(N)}^{N-m}}{(1 - \xi_{\sigma(m+1)} \xi_{\sigma(m+2)}) \cdots (1 - \xi_{\sigma(N)})} \\ &= p^{(N-m+1)(N-m)/2} \left(1 - \prod_{j \in \Lambda} \xi_j \right) \frac{\prod_{i < j, i, j \notin \Lambda} (\xi_j - \xi_i)}{\prod_{j \notin \Lambda} (1 - \xi_j)}. \end{aligned}$$

Now we sum over all ordered $(m-1)$ -tuples Λ and obtain the following.

Theorem 1: We have when $p \neq 0$,

$$\begin{aligned} P_Y(x_i(t) = x_i, i = 1, \dots, m) &= \sum_{\Lambda} (-1)^{\sum_{i < m} (\ell_i - i)} \operatorname{sgn} \Lambda p^{(N-m+1)(N-m)/2} \\ &\times \int_{C_r} \dots \int_{C_r} \frac{\prod_{i \in \Lambda, j \notin \Lambda} f(\xi_i, \xi_j) \prod_{i < j} f(\xi_i, \xi_j)}{\prod_{i < j} f(\xi_i, \xi_j)} \\ &\times \left(1 - \prod_{j \notin \Lambda} \xi_j \right) \frac{\prod_{i < j, i, j \notin \Lambda} (\xi_j - \xi_i)}{\prod_{j \notin \Lambda} (1 - \xi_j)} \prod_{i < m} \xi_i^{x_i} \prod_{j \notin \Lambda} \xi_j^{x_m} \prod_i \xi_i^{-y_i - 1} e^{\sum_i \varepsilon(\xi_i) t} d\xi_1 \dots d\xi_N. \end{aligned}$$

(From now until Sec. VI, $N = |Y|$, all indices belong to $[1, N]$, and all sets of indices are subsets of $[1, N]$.)

III. THE FIRST m PARTICLES: LARGE CONTOURS

Here, we shall find another representation of the same probability in which all contours of integration are C_R , where R is arbitrarily large. Observe that some factors $f(\xi_i, \xi_j)$ in the denominator in the integrand are canceled by factors in the numerator. We assume at first that $p, q \neq 0$.

Step 1. We show first that we may take all ξ_ℓ -contours with $\ell \in \Lambda$ to be C_R . Let us first expand the ξ_{ℓ_α} -contour, where $\ell_\alpha = \max \Lambda$. There appear to be poles at

$$\xi_{\ell_\alpha} = \frac{p}{1 - q\xi_j}$$

coming from the factor $f(\xi_{\ell_\alpha}, \xi_j)$ in the denominator. Since $\ell_\alpha = \max \Lambda$, we must have $j \notin \Lambda$. But then this factor is cancelled by the same factor in the numerator. So the only poles when we expand the ξ_{ℓ_α} -contour are at

$$\xi_{\ell_\alpha} = \frac{\xi_i - p}{q\xi_i},$$

which comes from the factor $f(\xi_i, \xi_{\ell_\alpha})$ in the denominator when $i < \ell_\alpha$.

We show that the integral with respect to ξ_i of the residue at this pole equals zero. When we make the substitution $\xi_{\ell_\alpha} \rightarrow (\xi_i - p)/q\xi_i$, we find that $\varepsilon(\xi_i) + \varepsilon(\xi_{\ell_\alpha})$ becomes analytic at $\xi_i = 0$. As for the first quotient in the integrand, each factor f having ξ_{ℓ_α} as one of its variables is on the order of ξ_i^{-1} as $\xi_i \rightarrow 0$. There are $N - 1$ such factors in the numerator and $N - 2$ in the denominator [since we do not include the factor $f(\xi_i, \xi_{\ell_\alpha})$ that gave rise to the pole]. Thus, this quotient is $O(\xi_i^{-1})$. The residue of $1/f(\xi_i, \xi_{\ell_\alpha})$ at the pole is $1/q\xi_i$, so these combine to give the power ξ_i^{-2} . As for the product of powers of the variables at the end of the integrand, when $i \notin \Lambda$, the product of those involving ξ_i is

$$\left(\frac{\xi_i - p}{q\xi_i} \right)^{x_\alpha} \xi_i^{x_m} \left(\frac{\xi_i - p}{q\xi_i} \right)^{-y_{\ell_\alpha}} \xi_i^{-y_i - 1} = O(\xi_i^{-x_\alpha + x_m + y_{\ell_\alpha} - y_i}) = O(\xi_i^2)$$

since $x_\alpha < x_m$ and $i < \ell_\alpha$. Thus, the integrand is analytic inside C_r and so its integral is zero. If $i \in \Lambda$, then $i = \ell_\beta$ with $\beta > \alpha$. [Otherwise, the factor $f(\xi_{\ell_\beta}, \xi_{\ell_\alpha})$ in the denominator is cancelled by the same factor in the numerator.] Then the exponent x_m above is replaced by x_β and $x_\beta > x_\alpha$.

We have shown that we may expand the ξ_{ℓ_α} -contour to C_R . Next we expand the ξ_{ℓ_β} -contour where ℓ_β is the second-largest element of Λ . The only difference from what went before is that there could be a pole at

$$\xi_{\ell_\beta} = \frac{P}{1 - q\xi_{\ell_\alpha}}$$

coming from the factor $f(\xi_{\ell_\beta}, \xi_{\ell_\alpha})$ in the denominator. However, $\xi_{\ell_\alpha} \in C_R$, and if R is large enough, the pole would lie inside C_r and so would not be passed in the expansion of the ξ_{ℓ_β} -contour. Thus, we may expand the ξ_{ℓ_β} -contour to C_R

Continuing in this way we see that all the ξ_ℓ -contours with $\ell \in \Lambda$ may be taken to be C_R .

Step 2. Next we expand the ξ_i -contours with $i \notin \Lambda$, and when we do this we encounter poles at $\xi_i=1$ as well as those coming from the first quotient in the integrand. As above, the latter poles will not contribute, but those from $\xi_i=1$ will. An application of Lemma 3.1 of Ref. 24 will tell us the result.

We restate the lemma to make it compatible with the present notation. Let T be a finite set of indices and let

$$g = g(\xi_i)_{i \in T}$$

be a function that is analytic for all $\xi_i \neq 0$. Assume that for $i > k$

$$g|_{\xi_i \rightarrow (\xi_k - p)/q\xi_k} = O(\xi_k) \tag{6}$$

as $\xi_k \rightarrow 0$, uniformly when all the ξ_j with $j \neq i, k$ are bounded and bounded away from zero. For $S \subset T$ denote by g_S the function obtained from g by setting all ξ_i with $i \notin S$ equal to 1. (In particular, $g = g_T$.) Define

$$I_S(\xi) = \prod_{\substack{i < j \\ i, j \in S}} \frac{\xi_j - \xi_i}{f(\xi_i, \xi_j)} \frac{g_S(\xi)}{\prod_{i \in S} (1 - \xi_i)}.$$

Then when $p, q \neq 0$,

$$\int_{C_r^{|T|}} I_T(\xi) \prod_{i \in T} d\xi_i = \sum_{S \subset T} \tau^{\sigma(S, T) - |S|} \frac{q^{|S|(|S|-1)/2}}{p^{|T|(|T|-1)/2}} \int_{C_R^{|S|}} I_S(\xi) \prod_{i \in S} d\xi_i, \tag{7}$$

where R is so large that all the zeros of the denominators lie inside C_R . [When S is empty, the integral on the right side is interpreted as $g(1, \dots, 1)$.] Here, $\tau = p/q$ and

$$\sigma(S, T) = \#\{(i, j) : i \in S, j \in T, i \geq j\}.$$

We apply this with $T = \Lambda^c$ and

$$g(\xi_i)_{i \in \Lambda^c} = \left(1 - \prod_{i \in \Lambda^c} \xi_i\right) \prod_{i, j \in \Lambda^c} \xi_i^{x_m - y_i - 1} e^{\sum_{i \in \Lambda^c} \varepsilon(\xi_i)t}$$

$$\times \int_{C_R^{|\Lambda|}} \frac{\prod_{i \in \Lambda, j \notin \Lambda} f(\xi_i, \xi_j) \prod_{i < j} f(\xi_i, \xi_j)}{\prod_{i < j, i \text{ or } j \in \Lambda} f(\xi_i, \xi_j)} \prod_{i < m} \xi_i^{x_i} \prod_{i \in \Lambda} \xi_i^{-y_i - 1} e^{\sum_{i \in \Lambda} \varepsilon(\xi_i)t} \prod_{i \in \Lambda} d\xi_i.$$

The integral on the left side of (7) is then the integral in Theorem 1. That g is analytic for $\xi_i \neq 0$ despite the denominator in the integrand follows from the fact that R may be arbitrarily large. The argument leading to (6) is like the analogous argument in step 1. There are $m-1$ factors on the order of ξ_k^{-1} in both the numerator and denominator of the quotient in the integrand, and the product $\xi_i^{-y_i - 1} \xi_k^{-y_k - 1}$ becomes

$$\left(\frac{\xi_k - p}{q\xi_k}\right)^{-y_i - 1} \xi_k^{-y_k - 1} = O(\xi_k)$$

since $i > k$.

So the lemma applies. If $|S|=k$, then the coefficient on the right side of (7) is

$$p^{-(N-m+1)(N-m)/2} \tau^{\sigma(S, \Lambda^c) - k} q^{k(k-1)/2}. \tag{8}$$

However, there is another factor arising from

$$\frac{\prod_{i \in \Lambda, j \notin \Lambda} f(\xi_i, \xi_j)}{\prod_{i < j, i \text{ or } j \in \Lambda} f(\xi_i, \xi_j)}$$

when any ξ_k with $k \in \Lambda^c \setminus S$ is set equal to 1. This factor is

$$\frac{\prod_{i \in \Lambda} f(\xi_i, 1)}{\prod_{k < j, j \in \Lambda} f(1, \xi_j) \cdot \prod_{i < k, i \in \Lambda} f(\xi_i, 1)} = \prod_{\substack{j > k \\ j \in \Lambda}} \frac{f(\xi_j, 1)}{f(1, \xi_j)} = (-\tau)^{\#\{j \in \Lambda: j > k\}}.$$

Hence, the total factor when ξ_k with $k \in \Lambda^c \setminus S$ is set equal to 1 is

$$(-\tau)^{\sigma(\Lambda, S^c \cap \Lambda^c)}. \tag{9}$$

If we use the bilinearity of $\sigma(U, V)$ in its two variables, and the easy fact that

$$\sigma(U, V) + \sigma(V, U) = |U||V| + |U \cap V|,$$

we see that

$$\sigma(\Lambda, S^c \cap \Lambda^c) = \sigma(\Lambda, \Lambda^c) - \sigma(\Lambda, S) = \sum \ell_i - m(m-1)/2 + \sigma(S, \Lambda) - (m-1)k.$$

Thus (8) times (9) equals

$$p^{-(N-m+1)(N-m)/2} (-1)^{\sum_{i < m} (\ell_i - i) + \sigma(S, \Lambda) - (m-1)k} \tau^{\sum_{i < m} (\ell_i - i) + \sigma(S) - mk} q^{k(k-1)/2},$$

where we have written $\sigma(S)$ for $\sigma(S, \Lambda) + \sigma(S, \Lambda^c) = \sigma(S, [1, N])$.

We replace each integral in the formula of Theorem 1 by the sum over $S \subset \Lambda^c$ we obtain using (7). The result is as follows.

Theorem 2: We have when $q \neq 0$,

$$\begin{aligned} P_Y(x_i(t) = x_i, i = 1, \dots, m) &= \sum_{\substack{\Lambda \\ S \subset \Lambda^c}} (-1)^{\sigma(S, \Lambda) - (m-1)k} \text{sgn } \Lambda \tau^{\sum_{i < m} (\ell_i - i) + \sigma(S) - mk} q^{k(k-1)/2} \\ &\times \int_{C_R} \dots \int_{C_R} \frac{\prod_{i \in \Lambda, j \in S} f(\xi_i, \xi_j) \prod_{i < j} f(\xi_i, \xi_j)}{\prod_{i < j} f(\xi_i, \xi_j)} \\ &\times \left(1 - \prod_{j \in S} \xi_j \right) \frac{\prod_{i < j, i, j \in S} (\xi_j - \xi_i)}{\prod_{j \in S} (1 - \xi_j)} \prod_{i < m} \xi_i^{x_i} \prod_{j \in S} \xi_j^{x_m} \prod_i \xi_i^{-y_i - 1} e^{\sum_{i \in S} (\xi_i) t} \prod_i d\xi_i. \end{aligned}$$

The summation is over all sets S and $(m-1)$ -tuples Λ disjoint from S . Here, $k = |S|$ and in the integrand indices that are not specified belong to $S \cup \Lambda$.

We assumed at first that $p, q \neq 0$ since that was required by Lemma 3.1 of Ref. 25. To obtain the relation when $p=0$ by passing to the $p \rightarrow 0$ limit, we need to only observe that the power of τ in the coefficient is non-negative and that the integrand is continuous at $p=0$ when $R > 1$.

IV. CONSECUTIVE PARTICLES

Here we find a formula for $P_Y(x_i(t) = x_i, i = n, \dots, m)$. If we take the preceding formula and sum over all x_1, \dots, x_{n-1} with $-\infty < x_1 < \dots < x_{n-1} < x_n$, which we may do when $R > 1$, the partial product $\prod_{i < n} \xi_i^{x_i}$ is replaced by

$$\frac{(\xi_{\ell_1} \cdots \xi_{\ell_{n-1}})^{x_n}}{(\xi_{\ell_1} - 1)(\xi_{\ell_1}\xi_{\ell_2} - 1) \cdots (\xi_{\ell_1} \cdots \xi_{\ell_{n-1}} - 1)}. \tag{10}$$

Our $(m-1)$ -tuple Λ may be written in the obvious way as $\{\Lambda_1, \Lambda_2\}$, where

$$\Lambda_1 = \{\ell_1, \dots, \ell_{n-1}\}, \quad \Lambda_2 = \{\ell_n, \dots, \ell_{m-1}\}. \tag{11}$$

Until now we did not differentiate in the notation between an ordered set and the corresponding unordered set. Which was meant was always clear from the context. Now we introduce a notation when we want to distinguish the two: for an ordered set Λ , we denote by $\chi(\Lambda)$ the corresponding unordered set.

In the integrand in Theorem 2 the only factor that depends on Λ and not just on the set $\chi(\Lambda)$ is

$$\prod_{i < j} f(\xi_{\ell_i}, \xi_{\ell_j}) = \prod_{i < j < n} f(\xi_{\ell_i}, \xi_{\ell_j}) \prod_{i < n, j \geq n} f(\xi_{\ell_i}, \xi_{\ell_j}) \prod_{n \leq i < j} f(\xi_{\ell_i}, \xi_{\ell_j}).$$

The first factor depends on Λ_1 , the second depends only on $\chi(\Lambda_1)$ and $\chi(\Lambda_2)$, while the third depends on Λ_2 .

In Theorem 2 we are to sum over all sets S and all $(m-1)$ -tuples Λ disjoint from S . Let us fix S first and sum over all Λ disjoint from S . This sum over Λ is the sum over all Λ_1 and Λ_2 as in (11). We do this sum by first taking a fixed set S_1 disjoint from S with $|S_1|=n-1$ and summing over all Λ_1 with $\chi(\Lambda_1)=S_1$. Then we must sum over S_1 and, of course, Λ_2 (which is disjoint from S_1 and S) and S . To recapitulate: the summation in the statement of Theorem 2 may be replaced by

$$\sum_{S, S_1, \Lambda_1, \Lambda_2},$$

where S and S_1 run over all disjoint sets with $|S_1|=n-1$, and Λ_1 and Λ_2 run over all $(n-1)$ - and $(m-n)$ -tuples, respectively, satisfying

$$\chi(\Lambda_1) = S_1, \quad \chi(\Lambda_2) \subset (S \cup S_1)^c.$$

With this notation the coefficient in Theorem 2 may be written as

$$(-1)^{\sigma(S \cup S_1, \Lambda_2) - (m-1)k} (-1)^{\sigma(S_1, \Lambda_2)} \text{sgn } \Lambda_1 \text{sgn } \Lambda_2 \tau^{\sigma(S \cup S_1 \cup \Lambda_2) - m(m-1)/2 - mk} q^{k(k-1)/2}.$$

Notice that Λ_1 appears here only in the factor $\text{sgn } \Lambda_1$. The sum of those terms involving Λ_1 as distinguished from S_1 is

$$\sum_{\Lambda_1} \text{sgn } \Lambda_1 \prod_{i < j < n} f(\xi_{\ell_i}, \xi_{\ell_j}) \frac{(\xi_{\ell_1} \cdots \xi_{\ell_{n-1}})^{x_n}}{(\xi_{\ell_1} - 1)(\xi_{\ell_1}\xi_{\ell_2} - 1) \cdots (\xi_{\ell_1} \cdots \xi_{\ell_{n-1}} - 1)}.$$

[Recall the factor (10).] Identity (1.7) of Ref. 24 tells us that this sum equals $(\xi_{\ell_1} \cdots \xi_{\ell_{n-1}})^{x_n}$ times

$$q^{(n-1)(n-2)/2} \frac{\prod_{i < j, i, j \in S_1} (\xi_j - \xi_i)}{\prod_{j \in S_1} (\xi_j - 1)} = (-1)^{n-1} q^{(n-1)(n-2)/2} \frac{\prod_{i < j, i, j \in S_1} (\xi_j - \xi_i)}{\prod_{j \in S_1} (1 - \xi_j)}.$$

We write the last product here times the analogous product in the integrand in Theorem 2, where S appears instead of S_1 , as

$$\frac{\prod_{i < j, i, j \in S \cup S_1} (\xi_j - \xi_i)}{\prod_{j \in S \cup S_1} (1 - \xi_j)} \frac{(-1)^{\sigma(S_1, S)}}{\prod_{i \in S_1, j \in S} (\xi_j - \xi_i)}.$$

Putting all these things together and using $\sigma(S, S_1) + \sigma(S_1, S) = (m-1)k$, we obtain the following.

Theorem 3: We have when $q \neq 0$,

$$\begin{aligned} P_Y(x_i(t) = x_i, i = n, \dots, m) &= \sum_{S, S_1, \Lambda_2} (-1)^{\sigma(S \cup S_1, \Lambda_2) + n - 1} \\ &\times \text{sgn } \Lambda_2 \tau^{\sigma(S \cup S_1 \cup \Lambda_2) - m(m-1)/2 - mk} q^{k(k-1)/2 + (n-1)(n-2)/2} \\ &\times \int_{C_R} \dots \int_{C_R} \frac{\prod_{i \in S_1, j \in S \cup \Lambda_2} f(\xi_i, \xi_j) \prod_{i \in \Lambda_2, j \in S} f(\xi_i, \xi_j) \prod_{n \leq i < j} f(\xi_i, \xi_j)}{\prod_{i < j} f(\xi_i, \xi_j)} \\ &\times \left(1 - \prod_{j \in S} \xi_j \right) \frac{\prod_{i < j, i, j \in S \cup S_1} (\xi_j - \xi_i)}{\prod_{j \in S \cup S_1} (1 - \xi_j)} \frac{1}{\prod_{i \in S_1, j \in S} (\xi_j - \xi_i)} \\ &\times \prod_{i \in S_1} \xi_i^{x_n} \prod_{n \leq i < m} \xi_i^{x_i} \prod_{j \in S} \xi_j^{x_m} \prod_i \xi_i^{-y_i - 1} e^{\sum_i \varepsilon(\xi_i) t} \prod_i d\xi_i. \end{aligned}$$

The summation is over all disjoint sets S and S_1 with $|S_1| = n - 1$ and $(m - n)$ -tuples Λ_2 disjoint from $S \cup S_1$. Here, $k = |S|$ and in the integrand indices that are not specified belong to $S \cup S_1 \cup \Lambda_2$.

V. ONE PARTICLE

In this case $n = m$ and so Λ_2 disappears, the first line of the right side becomes

$$\sum_{S, S_1} (-1)^{m-1} \tau^{\sigma(S \cup S_1) - m(m-1)/2 - mk} q^{k(k-1)/2 + (m-1)(m-2)/2},$$

and the factor

$$\prod_{i \in S_1} \xi_i^{x_n} \prod_{n \leq i < m} \xi_i^{x_i} \prod_{j \in S} \xi_j^{x_m}$$

becomes

$$\prod_{i \in S \cup S_1} \xi_i^{x_m}.$$

Now we take a fixed set S_3 with $|S_3| = m + k - 1$ and sum over all partitions $S_3 = \{S, S_1\}$ with $|S| = k, |S_1| = m - 1$. The only part of the sum that depends on S and S_1 individually is

$$\sum_{S, S_1} \frac{\prod_{i \in S_1, j \in S} f(\xi_i, \xi_j)}{\prod_{i \in S_1, j \in S} (\xi_j - \xi_i)} \left(1 - \prod_{j \in S} \xi_j \right).$$

If we observe that S is the complement of S_1 in S_3 , we see that we can apply identity (1.9) of Ref. 24 (with S there S_1 here, with m there $m - 1$ here, and with N there $m + k - 1$ here), which tells us that the sum equals

$$q^{(m-1)k} \begin{bmatrix} m + k - 2 \\ m - 1 \end{bmatrix}_\tau \left(1 - \prod_{i \in S_3} \xi_i \right),$$

where the τ -binomial coefficient $\begin{bmatrix} N \\ n \end{bmatrix}_\tau$ is defined by

$$\begin{bmatrix} N \\ n \end{bmatrix}_\tau = \frac{(1 - \tau^N)(1 - \tau^{N-1}) \dots (1 - \tau^{N-n+1})}{(1 - \tau)(1 - \tau^2) \dots (1 - \tau^n)}.$$

Hence, after some algebra, our formula becomes

Theorem 4: We have when $q \neq 0$,

$$\begin{aligned} P_Y(x_m(t) = x_m) &= (-1)^{m-1} \tau^{m(m-1)/2} \sum_{|S_3| \geq m} \tau^{\sigma(S_3)-m|S_3|} q^{|S_3|(|S_3|-1)/2} \left[\begin{matrix} |S_3| - 1 \\ m - 1 \end{matrix} \right]_{\tau} \\ &\times \int_{C_R} \dots \int_{C_R} \prod_{i < j} \frac{\xi_j - \xi_i}{f(\xi_i, \xi_j)} \frac{1 - \prod_i \xi_i}{\prod_i (1 - \xi_i)} \prod_i \xi_i^{x_m - y_i - 1} e^{\sum_i \varepsilon(\xi_i) t} \prod_i d\xi_i, \end{aligned}$$

where all indices in the integrand run over S_3 .

This is exactly Theorem 5.2 of Ref. 24.

VI. SEMI-INFINITE CONFIGURATIONS

Here we show that Theorem 2 and, consequently, also Theorems 3 and 4, hold when Y is semi-infinite on the right, if we take the sums over finite sets $S \subset \Lambda^c$. It follows from the fact that ASEP is a Feller process¹⁵ that the probability for the semi-infinite initial configuration Y equals the $N \rightarrow \infty$ limit of the probability for the initial configuration $\{y_1, \dots, y_N\}$. Thus, we sum over only those $\Lambda \subset [1, N]$ and those $S \subset [1, N]$ in Theorem 2 and then pass to the limit. The limit will be the sum over all Λ and (finite) S if the resulting series is absolutely convergent. This we now show using the fact that R may be taken arbitrarily large.

Consider the integrand first. Each f factor is on the order of R^2 for large R , while each ξ_i is on the order of R . Combining the estimates for all factors in the integrand other than $\prod_i \xi_i^{y_i-1}$ gives, after a little algebra, $O(R^{-k^2/2+O(k)})$. (Recall that m is fixed.) Since each $y_i \geq y_1 + i - 1$, we have $\prod_i \xi_i^{y_i-1} = O(R^{-\sigma(\Lambda \cup S)+O(k)})$. Thus, the integrand is

$$O(R^{-k^2/2-\sigma(\Lambda \cup S)+O(k)}).$$

Another factor R^k comes from the domain of integration, but this does not change the bound.

The coefficient on the right side of Theorem 2 has absolute value at most $\tau^{\sigma(\Lambda \cup S)+O(k)}$. If we take $R > \tau^2$, as we may, then this combined with the preceding bound gives

$$O(R^{-k^2/2-\sigma(\Lambda \cup S)/2+O(k)}). \quad (12)$$

This a bound for the summand in Theorem 2. Now we show that the sum of these over all Λ and S is finite. For any s , we have

$$\#\{(\Lambda, S): \sigma(\Lambda \cup S) = s\} \leq 2^s s^{m-1}.$$

The reason is that if $\sigma(\Lambda \cup S) = s$, then the largest element in $\Lambda \cup S$ is at most s . So there are at most 2^s choices for S , and having chosen S , there are at most s^{m-1} choices for Λ . It follows that the sum of (12) over all (Λ, S) is at most a constant times

$$\sum_{k,s \geq 0} 2^s s^{m-1} R^{-k^2/2-s/2+O(k)},$$

which is finite when $R > 4$.

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