J Stat Phys (2011) 143: 60–71
DOI 10.1007/s10955-011-0155-5

Asymptotics for the Covariance of the Airy$_2$ Process

Gregory Shinault · Craig A. Tracy

Received: 3 December 2010 / Accepted: 9 February 2011 / Published online: 1 March 2011

Abstract In this paper we compute some of the higher order terms in the asymptotic behavior of the two point function $\mathbb{P}(\mathcal{A}_2(0) \le s_1, \mathcal{A}_2(t) \le s_2)$, extending the previous work of Adler and van Moerbeke (arXiv:math.PR/0302329; Ann. Probab. 33, 1326–1361, 2005) and Widom (J. Stat. Phys. 115, 1129–1134, 2004). We prove that it is possible to represent any order asymptotic approximation as a polynomial and integrals of the Painlevé II function q and its derivative q'. Further, for up to tenth order we give this asymptotic approximation as a linear combination of the Tracy-Widom GUE density function f_2 and its derivatives. As a corollary to this, the asymptotic covariance is expressed up to tenth order in terms of the moments of the Tracy-Widom GUE distribution.

Keywords Airy process · Asymptotics

1 Introduction

The *Airy*$_2$ *process*, $\mathcal{A}_2(t)$, introduced by Prähofer and Spohn [15] in the context of the polynuclear growth (PNG) model, is a stationary stochastic process whose joint distributions for $t_1 < \cdots < t_m$ are given by

$$\mathbb{P}\left(\mathcal{A}_2(t_1) \le s_1, \ldots, \mathcal{A}_2(t_m) \le s_m\right) = \det(I - \chi K_2 \chi)_{L^2(\{1,\ldots,m\}\times\mathbb{R})} \tag{1}$$

where $K_2 \doteq K_2(x, y)$ is a $m \times m$ matrix kernel, called the *extended Airy kernel*, given by

$$K_{i,j}(x,y) = \begin{cases} \int_0^\infty e^{-z(t_i - t_j)} \operatorname{Ai}(x+z)\operatorname{Ai}(y+z)\,dz & \text{if } i \ge j, \\ -\int_{-\infty}^0 e^{-z(t_i - t_j)} \operatorname{Ai}(x+z)\operatorname{Ai}(y+z)\,dz & \text{if } i < j, \end{cases}$$

G. Shinault (✉) · C.A. Tracy
Department of Mathematics, University of California, Davis, CA 95616, USA
e-mail: gshinault@math.ucdavis.edu

C.A. Tracy
e-mail: tracy@math.ucdavis.edu

and $\chi = \chi(x)$ is the $m \times m$ diagonal matrix whose ith diagonal term is the indicator function $\chi_{s_i}(x) = 1_{x>s_i}$. The one-point function $\mathbb{P}(\mathcal{A}_2(t) \le s)$ is the Tracy-Widom GUE distribution, $F_2(s)$ [20].

For stochastic growth models and their closely related interacting particle systems, the Airy$_2$ process is fundamental since it is expected to describe the limiting process for height fluctuations belonging to the *KPZ Universality Class* with droplet (or step) initial conditions (see [9, 19] for recent reviews). This has been proved for the PNG model [15], the discrete PNG model [13], the boundary of the north polar region of the Aztec diamond [14] and the totally asymmetric simple exclusion process (TASEP) [14]. At the level of the one-point function, this universality has been established for the asymmetric simple exclusion process (ASEP) [23] and for the KPZ equation [3, 8, 17, 18]. Recent work [16] using replica methods have extended this KPZ work to the 2-point function. The Airy$_2$ process also describes the limiting process of the largest eigenvalue in Dyson's Brownian motion model in random matrix theory. For further appearances of the Airy$_2$ process see [6, 7, 10, 12].

We summarize some known properties of the Airy$_2$ process:

1. $\mathcal{A}_2(t)$ has continuous sample paths [13, 15].
2. $\mathcal{A}_2(t)$ locally looks like Brownian motion (see Hägg [11] for a precise statement).
3. The distribution functions (1) satisfy nonlinear differential equations [1, 2, 24].
4. The covariance $\mathrm{cov}_2(t) := \mathrm{cov}(\mathcal{A}_2(t)\mathcal{A}_2(0))$ has the following asymptotic expansions:

$$\mathrm{cov}_2(t) = \begin{cases} \mathrm{var}(F_2) - t + \mathcal{O}(t^2), & t \to 0^+, \\ \sum_{n=1}^{N} \dfrac{C_n}{t^n} + \mathcal{O}(\frac{1}{t^{N+1}}), & t \to \infty. \end{cases} \tag{2}$$

The small-t expansion of cov_2 was given by Prähofer and Spohn [15] and they also found the leading large-t term $C_1 = 0$, $C_2 = 1$. The existence of the higher order terms in the large-t expansion of cov_2 was established by Adler and van Moerbeke [1, 2] and by Widom [25] using different methods. In both [1, 2] and [25] the coefficient C_4 was expressed as a double integral whose integrand was in terms of the Hastings-McLeod solution of Painlevé II appearing in the distribution F_2. One of the main results of this paper is to prove that the coefficients C_{2n}, $2 \le n \le 5$, are expressible in terms of the moments of F_2. (The odd coefficients are all zero.) Precisely, if

$$\mu_n := \int_{-\infty}^{\infty} s^n f_2(s)\, ds, \qquad f_2 := F_2',$$

then

$$C_4 = 2\mu_1, \tag{3}$$

$$C_6 = 2\mu_2 + \frac{10}{3}\mu_1^2, \tag{4}$$

$$C_8 = 2\mu_3 + 14\mu_2\mu_1 + \frac{13}{2}, \tag{5}$$

$$C_{10} = 2\mu_4 + 24\mu_3\mu_1 + \frac{126}{5}\mu_2^2 + 116\mu_1. \tag{6}$$

We conjecture that C_{2n} can be expressed in terms of a polynomial in μ_i for $i \le n-1$. To prove these results we follow the program established by Widom [25] and first prove

$$\mathbb{P}\left(\mathcal{A}_2(t) \le s_2, \mathcal{A}_2(0) \le s_1\right) = \sum_{n=0}^{N} \frac{c_n(s_1, s_2)}{t^n} + \mathcal{O}\left(\frac{1}{t^{N+1}}\right) \tag{7}$$

as $t \to \infty$. It was previously shown [1, 2, 25] that for $n \le 4$ each c_n could be written as polynomials and integrals of the Painlevé II function, its derivative, and the variables s_1 and s_2. A feature of our analysis is that we show through order t^{-10} that each c_{2n} can be expressed in terms of f_2, its derivatives, and polynomials in s_k; see (18)–(20), (24).

Bornemann [4, 5] has given a high precision numerical evaluation of $\mathrm{cov}_2(t)$, $0 \le t \le 100$. His method involves a numerical evaluation of the Fredholm determinant appearing in (1) for $m = 2$ followed by numerical integrations to give $\mathrm{cov}_2(t)$. In Appendix 2 we compare the large-t asymptotics with these numerical results.

In the present paper we begin by showing how to obtain an asymptotic expression for the extended Airy kernel following Widom [25]. The large-t asymptotics of the two-point distribution is then given in terms of f_2 and its derivatives. This in turn allows the easy computation of the large-t expansion of the covariance, which ends the main body of the paper. The appendices contain some of the higher order terms, and comparison to high precision numerical results.

2 Asymptotics for $\chi K_2 \chi$

The first step in our asymptotic analysis is a large t expression for the extended Airy kernel $\chi K_2 \chi$. In the $m = 2$ case the $\chi K_2 \chi$ operator has a matrix kernel of the form

$$\begin{bmatrix} \chi_{s_1}(x)\int_0^\infty \mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz\,\chi_{s_1}(y) & \chi_{s_1}(x)\int_0^\infty e^{-zt}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz\,\chi_{s_2}(y) \\ -\chi_{s_2}(x)\int_{-\infty}^0 e^{zt}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz\,\chi_{s_1}(y) & \chi_{s_2}(x)\int_0^\infty \mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz\,\chi_{s_2}(y) \end{bmatrix}.$$

We must compute the Fredholm determinant of this operator for large-t. To this end we will split the operator into two manageable components so that $\chi K_2 \chi = K + L$:

$$K(x,y) := \begin{bmatrix} \chi_{s_1}(x)\int_0^\infty \mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz\,\chi_{s_1}(y) & 0 \\ 0 & \chi_{s_2}(x)\int_0^\infty \mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz\,\chi_{s_2}(y) \end{bmatrix},$$

$$L(x,y) := \begin{bmatrix} 0 & \chi_{s_1}(x)\int_0^\infty e^{-zt}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz\,\chi_{s_2}(y) \\ -\chi_{s_2}(x)\int_{-\infty}^0 e^{zt}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz\,\chi_{s_1}(y) & 0 \end{bmatrix}.$$

The determinant computation is simplified via

$$\begin{aligned} \det(I - \chi K_2 \chi) &= \det(I - K - L) = \det\left[(I-K)(I - (I-K)^{-1}L)\right] \\ &= \det(I-K)\det(I - (I-K)^{-1}L). \end{aligned}$$

The determinant of $I - K$ is $F_2(s_1)F_2(s_2)$, and has no dependence on t. So we need only look at L to determine the asymptotics. For that determinant, we make an expansion of the terms in $L(x,y)$. By repeatedly applying integration by parts, the upper-right corner is

$$\begin{aligned} L_{12} &= \chi_{s_1}(x)\int_0^\infty dz\,e^{-zt}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,\chi_{s_2}(y) \\ &= \sum_{n=0}^{N} \frac{1}{t^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \chi_{s_1}\,\mathrm{Ai}^{(k)} \otimes \mathrm{Ai}^{(n-k)}\,\chi_{s_2} + \mathcal{O}\left(t^{-(N+1)}\right) \end{aligned}$$

and the lower-left corner

$$
\begin{aligned}
L_{21} &= \chi_{s_2}(x) \int_{-\infty}^{0} dz\, e^{zt}\, \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z) \chi_{s_1}(y) \\
&= \sum_{n=0}^{N} \frac{(-1)^{n+1}}{t^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \chi_{s_2}\, \mathrm{Ai}^{(k)} \otimes \mathrm{Ai}^{(n-k)}\, \chi_{s_1} + \mathcal{O}\left(t^{-(N+1)}\right).
\end{aligned}
$$

We take this approximation up to $N = 10$, which is valid in the trace norm, because this is the highest order term we aim to calculate.

Now we analyze $T := (I - K)^{-1} L$. At this point we introduce the notation

$$
Q_{n,k} = (I - \chi_{s_k} K_{\mathrm{Ai}} \chi_{s_k})^{-1} \chi_{s_k}\, \mathrm{Ai}^{(n)}.
$$

The kernel K_{Ai} is the classical Airy kernel of random matrix theory, while the purpose of the χ's are to ensure the kernels are integrated over the appropriate domain. Using this notation in the asymptotic expansion of $L(x, y)$, the upper right term in the matrix kernel of T is written

$$
T_{12} = \sum_{n=0}^{N} \frac{1}{t^{n+1}} \sum_{k=0}^{n} \binom{n}{k} Q_{k,1} \otimes \mathrm{Ai}^{(n-k)}\, \chi_{s_2} + \mathcal{O}(t^{-(N+1)})
$$

and the lower left term is

$$
T_{21} = \sum_{n=0}^{N} \frac{(-1)^{n+1}}{t^{n+1}} \sum_{k=0}^{n} \binom{n}{k} Q_{k,2} \otimes \mathrm{Ai}^{(n-k)}\, \chi_{s_1} + \mathcal{O}(t^{-(N+1)}).
$$

The determinant we are now interested in is best analyzed in terms of the following trace formula:

$$
\begin{aligned}
\det(I - T) &= \det(I - T_{12} T_{21}) = \exp\left[\operatorname{tr} \log(I - \mathcal{T})\right] \\
&= 1 - \operatorname{tr} \mathcal{T} + \frac{1}{2}\left((\operatorname{tr} \mathcal{T})^2 - \operatorname{tr} \mathcal{T}^2\right) \\
&\quad - \frac{1}{6}\left((\operatorname{tr} \mathcal{T})^3 - 3 \operatorname{tr} \mathcal{T}^2 \operatorname{tr} \mathcal{T} + 2 \operatorname{tr} \mathcal{T}^3\right) + \cdots. \qquad (8)
\end{aligned}
$$

Here we have introduced $\mathcal{T} = T_{12} T_{21}$. To evaluate an expression for $\mathcal{T}$ we will repeatedly use the fact that $(f \otimes g)(e \otimes h) = (g, e)_{L^2} f \otimes h$. By introducing the notation

$$
u_{n,j}(s_k) = (Q_{n,k}, \chi_{s_k}\, \mathrm{Ai}^{(j)})_{L^2} = (Q_{j,k}, \chi_{s_k}\, \mathrm{Ai}^{(n)})_{L^2} = u_{j,n}(s_k)
$$

we are able to write

$$
\begin{aligned}
\mathcal{T} = &\sum_{n_1+n_2 \le N-2} \frac{(-1)^{n_2+1}}{t^{n_1+n_2+2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1} \binom{n_2}{k_2} u_{n_1-k_1,k_2}(s_2) Q_{k_1,1} \otimes \mathrm{Ai}^{(n_2-k_2)}\, \chi_{s_1} \\
&+ \mathcal{O}\left(\frac{1}{t^{N+1}}\right). \qquad (9)
\end{aligned}
$$

From (9) above, we know that we will also need the formula for $(T_{12}T_{21})^2$:

$$\mathcal{T}^2 = \sum_{\substack{n_1+n_2\\+m_1+m_2\le N-4}} \frac{(-1)^{n_2+m_2}}{t^{n_1+n_2+m_1+m_2+4}} \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\sum_{\ell_1=0}^{m_1}\sum_{\ell_2=0}^{m_2} \binom{n_1}{k_1}\binom{n_2}{k_2}\binom{m_1}{\ell_1}\binom{m_2}{\ell_2}$$
$$\times\, u_{n_1-k_1,k_2}(s_2)u_{m_1-\ell_1,\ell_2}(s_2)u_{n_2-k_2,\ell_1}(s_1)Q_{k_1,1}\otimes\chi_{s_1}\,\mathrm{Ai}^{(m_2-\ell_2)} + \mathcal{O}\left(\frac{1}{t^{N+1}}\right). \qquad (10)$$

We continue multiplying the sums $\mathcal{T}^n$ in this manner as necessary for the order of our desired approximation.

3 The Two-Point Distribution

From (8) we conclude $\operatorname{tr}\mathcal{T}$ is order t^{-2}, while (9) and (10) tell us $(\operatorname{tr}\mathcal{T})^2 - \operatorname{tr}\mathcal{T}^2$ is order t^{-4}. Terms can be added as necessary to get any order approximation one wishes for. In other words,

$$\mathbb{P}\left(\mathcal{A}_2(t)\le s_2, \mathcal{A}_2(0)\le s_1\right) = \sum_{n=0}^{N}\frac{c_n(s_1,s_2)}{t^n} + \mathcal{O}\left(\frac{1}{t^{N+1}}\right)$$

for $t\to\infty$.

It has already been established that $c_1 = c_3 = 0$, and c_2 and c_4 can be written in terms of the Hastings-McLeod solution, which we denote by $q(s)$, to the Painlevé II equation [1, 2, 25]. This result can be extended to any order. It is clear from (9) and (10) that all of our coefficients $c_n(s_1,s_2)$ are polynomials in the variables $u_{k,j}$ multiplied by $F_2(s_1)F_2(s_2)$. In our notation the first two terms are

$$c_2(s_1,s_2) = F_2(s_1)u_{0,0}(s_1)F_2(s_2)u_{0,0}(s_2), \qquad (11)$$
$$c_4(s_1,s_2) = F_2(s_1)F_2(s_2)\left[u_{1,0}(s_1)u_{1,0}(s_2) + u_{0,0}(s_1)\left(2u_{2,0}(s_2) - u_{1,1}(s_2)\right)\right] + \text{reversed}, \qquad (12)$$

where "reversed" denotes the interchange of s_1 and s_2 in the previous terms. The formula for F_2 in terms of q,

$$F_2(s) = \exp\left[-\int_s^\infty (x-s)q(x)^2\,dx\right],$$

is well known so we need only focus on the $u_{k,j}$'s. There are three facts that will enable us to show each $u_{k,j}$ can be expressed in terms of q and q' and integrals of q and q':

1. $q_0 := (I - K_{\mathrm{Ai}})^{-1}\chi_s\,\mathrm{Ai}(s)$ is a solution to Painlevé II ($q_0 = q$),
2. $u'_{k,j} = -q_k q_j$, where $q_j := (I-K_{\mathrm{Ai}})^{-1}\chi_s\,\mathrm{Ai}^{(j)}(s)$,
3. $q_n = (n-2)q_{n-3} + sq_{n-2} - u_{n-2,1}\,q + u_{n-2,0}\,q_1$ for $n\ge 3$.

The first statement is established explicitly in [20, 22], where a template is also developed for the proof of the second statement. We begin by proving the second statement. Henceforth we will use the notation

$$Q_k(x;s) := (I-K_{\mathrm{Ai}})^{-1}\chi_s\,\mathrm{Ai}^{(k)}.$$

Proof of Statement 2 We proceed by first differentiating $u_{k,j}$.

$$\frac{du_{k,j}}{ds} = \frac{d}{ds}\left[\int_s^\infty Q_k(x;s)\,\mathrm{Ai}^{(j)}(x)\,dx\right] = -q_k(s)\,\mathrm{Ai}^{(j)}(s) + \int_s^\infty \frac{\partial Q_k}{\partial s}\,\mathrm{Ai}^{(j)}(x)\,dx. \quad (13)$$

We now need a formula for $\partial_s Q_k$:

$$\frac{\partial Q_k}{\partial s} = \frac{\partial}{\partial s}(I - K_{\mathrm{Ai}})^{-1}\chi_s\,\mathrm{Ai}^{(k)} = (I - K_{\mathrm{Ai}})^{-1}\frac{\partial K_{\mathrm{Ai}}}{\partial s}(I - K_{\mathrm{Ai}})^{-1}\chi_s\,\mathrm{Ai}^{(k)}.$$

This necessitates a formula $\partial_s K_{\mathrm{Ai}}$. Direct computation shows that this has kernel

$$\frac{\partial K_{\mathrm{Ai}}}{\partial s} = -K_{\mathrm{Ai}}(x,s)\delta(y-s).$$

By introducing the operator $R = (I - K_{\mathrm{Ai}})^{-1}K_{\mathrm{Ai}}$ we can write

$$\frac{\partial Q_k}{\partial s}(x;s) = -R(x,s)q_k(s).$$

Returning to (13), we now have

$$\begin{aligned}\frac{du_{k,j}}{ds} &= -q_k(s)\,\mathrm{Ai}^{(j)}(s) - \int_s^\infty R(s,x)q_k(s)\,\mathrm{Ai}^{(j)}(x)\,dx\\ &= -q_j(s)\left(\int_s^\infty (\delta(s-x) + R(s,x))\,\mathrm{Ai}^{(k)}(x)\,dx\right)\\ &= -q_j(s)(I-K)^{-1}\chi_s\,\mathrm{Ai}^{(k)}(s) = -q_j(s)q_k(s).\end{aligned}$$

Note that we pass from the second line to the third line because the kernel of $(I - K_{\mathrm{Ai}})^{-1}$ is $\delta(x-y) + R(x,y)$. □

Proof of Statement 3 First, we develop an explicit formula for q_n. We start with

$$Q_n(x;s) = (I - K_{\mathrm{Ai}})^{-1}\chi_s\,\mathrm{Ai}^{(n)}$$

and the formula

$$\mathrm{Ai}^{(n)}(x) = (n-2)\,\mathrm{Ai}^{(n-3)}(x) + x\,\mathrm{Ai}^{(n-2)}(x)$$

for $n \geq 3$. This gives

$$Q_n(x;s) = (n-2)Q_{n-3}(x;s) + P_{n-2}(x;s)$$

where $P_n(x;s) := (I - K_{\mathrm{Ai}})^{-1}x\,\mathrm{Ai}^{(n)}(x)\chi_s$. Now we apply the commutator relations

$$[M, (I - K_{\mathrm{Ai}})^{-1}]\,\mathrm{Ai}^{(n)} = x\,Q_n(x;s) - P_n(x;s)$$

and

$$[M, (I - K_{\mathrm{Ai}})^{-1}] \doteq Q_0(x;s)Q_1(y;s) - Q_1(x;s)Q_0(y;s)$$

to arrive at

$$P_n(x;s) = xQ_n(x;s) - u_{n,1}(s)Q_0(x;s) + u_{n,0}(s)Q_1(x;s).$$

Taking $x = s$ we get the recursive relation

$$q_n = (n-2)q_{n-3} + sq_{n-2} - u_{n-2,1}q + u_{n-2,0}q_1$$

for $n \geq 3$. From [21, 22, 25] we know that $q_1 = q' + u_{00}\,q$ and $q_2 = sq_1 - u_{11}\,q + u_{00}\,q_1$, so we have formulas for all q_n. □

From here, we can complete the proof inductively. If $u_{k,j}$ is expressible as a polynomial and integrals of the q, q' and s for $j \leq k$ then so is q_{k+1} due to the identity above. The differential equations for each $u_{k,j}$ give us $u_{k+1,j} = \int_s^\infty q_{k+1}(x)q_j(x)dx$ for $j \leq k+1$. This completes the proof, so long as enough base cases are satisfied.

While it is interesting that we can write these u_{jk} formulas as a polynomial and integrals in terms of s, q, and q', there are explicit polynomial formulas in the aforementioned terms for each $u_{j,k}$ with $j + k \leq 8$. For example,

$$u_{0,0} = (q')^2 - sq^2 - q^4, \tag{14}$$

$$u_{1,0} = \frac{1}{2}u_{0,0}^2 - \frac{1}{2}q^2, \tag{15}$$

$$u_{1,1} = \frac{1}{3}u_{0,0}^3 - \left(q^2 + \frac{s}{3}\right)u_{0,0} - \frac{2}{3}qq', \tag{16}$$

$$u_{2,0} = \frac{1}{2}u_{1,1}^2 + \frac{1}{2}su_{0,0}. \tag{17}$$

We conjecture this is true for all $u_{j,k}$. The conclusion is the two-point distribution has an asymptotic formula up to order t^{-10} in terms of q.

The formulas for q_n and $u'_{j,k}$ also enable us to show that $F_2 u_{j,k}$ is multilinear in $f_2 = F_2'$, its derivatives, and s for j and k needed through c_{10}. A table of these formulas is included in Appendix 1. The verification of each equation simply requires the differentiation of both sides and looking at $s \to \infty$ to verify the constants of integration are zero.

Using this table, (9), and (10), a direct computation yields the formulas

$$c_2(s_1,s_2) = f_2(s_1)f_2(s_2), \tag{18}$$

$$c_4(s_1,s_2) = (s_1+s_2)f_2(s_1)f_2(s_2) + \frac{1}{2}f_2'(s_1)f_2'(s_2), \tag{19}$$

$$\begin{aligned} c_6(s_1,s_2) = {} & \frac{1}{3}(3s_1+s_2)(3s_2+s_1)f_2(s_1)f_2(s_2) + 3\left(f_2'(s_1)f_2(s_2) + f_2(s_1)f_2'(s_2)\right) \\ & + (s_1+s_2)f_2'(s_1)f_2'(s_2) + \frac{1}{6}f_2''(s_1)f_2''(s_2). \end{aligned} \tag{20}$$

The odd terms are not listed here because they are zero. The coefficient formula for c_8 is given below in (24).

From a logical standpoint the work we have done is sufficient. However, it is instructional to see an alternate derivation of (18)–(19) from (11)–(12). First note that we have already established $u_{0,0}(s) = \int_s^\infty q(x)^2\,dx$. Using this and the integral formula for F_2, it is easy to check that $f_2 = F_2' = F_2 u_{0,0}$. This gives us (18). For the next term, we use the identities (17), (15), and $f_2 = F_2 u_{0,0}$ in the formula (12) to get

$$c_4(s_1,s_2) = \frac{1}{2} F_2(s_1)(u_{0,0}(s_1)^2 - q(s_1)^2) F_2(s_2)(u_{0,0}(s_2)^2 - q(s_2)^2) + (s_1+s_2) f_2(s_1) f_2(s_2).$$

Finally, observe that $f_2' = [F_2 u_{0,0}]' = F_2 u_{0,0}^2 - F_2 q^2$ so we arrive at (19).

4 The Covariance

For the covariance of the Airy$_2$ process we denote the Nth order asymptotic approximation by

$$\mathrm{cov}_{2,N}(t) = \sum_{n=1}^{N} \frac{C_n}{t^n}.$$

It was established by Adler and van Moerbeke [1, 2] and Widom [25] that the coefficients up to $N = 4$ are $C_1 = 0$, $C_2 = 1$, $C_3 = 0$, and

$$C_4 = \int\int_{\mathbb{R}^2} c_4(u,v)\, du\, dv.$$

Through an application of Fubini's theorem and (19), we get

$$C_4 = 2 \int_{\mathbb{R}} u f_2(u) du \int_{\mathbb{R}} f_2(v)\, dv + \frac{1}{2} \left(\int_{\mathbb{R}} f_2'(u) du \right)^2 = 2\mu_1. \tag{21}$$

Following this same procedure for C_6 we have

$$\begin{aligned} C_6 &= \int_{\mathbb{R}^2} c_6(s_1,s_2)\, ds_1\, ds_2 \\ &= \int_{\mathbb{R}^2} \frac{1}{3} (3s_1 + s_2)(3s_2 + s_1) f_2(s_1) f_2(s_2)\, ds_1\, ds_2 \\ &= 2\mu_2 + \frac{10}{3}\mu_1^2 \end{aligned} \tag{22}$$

since all the other integrals appearing in (20) integrate to zero. The analogous formulas for C_8 and C_{10} appear in (5) and (6) with some details of their computation in Appendix 1.

Acknowledgements The authors acknowledge helpful conversations with Harold Widom. We would also like to thank Folkmar Bornemann for providing us with his table of values for the Airy$_2$ covariance. This work was supported by the National Science Foundation through grant DMS-0906387 and VIGRE DMS0636297.

Appendix 1: Identities

In this appendix we give the formulas necessary to attain (18)–(20). The table below gives $u_{j,k} F_2$ in terms of f_2 and its derivatives.

j,k	f_2	f_2'	f_2''	$f_2^{(3)}$	$f_2^{(4)}$	$f_2^{(5)}$	$f_2^{(6)}$	$f_2^{(7)}$	$f_2^{(8)}$
0, 0	1	0	0	0	0	0	0	0	0
1, 0	0	$\frac{1}{2}$	0	0	0	0	0	0	0
1, 1	$-\frac{s}{3}$	0	$\frac{1}{3}$	0	0	0	0	0	0
2, 0	$\frac{s}{3}$	0	$\frac{1}{6}$	0	0	0	0	0	0
2, 1	$-\frac{1}{4}$	0	0	$\frac{1}{8}$	0	0	0	0	0
3, 0	$\frac{7}{12}$	$\frac{s}{3}$	0	$\frac{1}{24}$	0	0	0	0	0
2, 2	$\frac{s^2}{5}$	$-\frac{3}{10}$	0	0	$\frac{1}{20}$	0	0	0	0
3, 1	$-\frac{s^2}{5}$	$\frac{2}{15}$	$\frac{s}{6}$	0	$\frac{1}{30}$	0	0	0	0
4, 0	$\frac{s^2}{5}$	$\frac{47}{60}$	$\frac{s}{6}$	0	$\frac{1}{120}$	0	0	0	0
3, 2	$\frac{2s}{9}$	$\frac{s^2}{18}$	$-\frac{1}{12}$	$\frac{s}{18}$	0	$\frac{1}{72}$	0	0	0
4, 1	$-\frac{13s}{18}$	$-\frac{s^2}{18}$	$\frac{11}{24}$	$\frac{s}{9}$	0	$\frac{1}{144}$	0	0	0
5, 0	$\frac{101s}{90}$	$\frac{23s^2}{90}$	$\frac{59}{120}$	$\frac{s}{18}$	0	$\frac{1}{720}$	0	0	0
3, 3	$-\frac{s^3}{7}+\frac{34}{63}$	$\frac{17s}{42}$	$\frac{s^2}{9}$	$-\frac{1}{18}$	$\frac{s}{36}$	0	$\frac{1}{252}$	0	0
4, 2	$\frac{s^3}{7}-\frac{11}{42}$	$-\frac{13s}{84}$	0	$\frac{1}{8}$	$\frac{s}{24}$	0	$\frac{1}{336}$	0	0
5, 1	$-\frac{s^3}{7}-\frac{74}{105}$	$-\frac{47s}{420}$	$\frac{s^2}{10}$	$\frac{7}{20}$	$\frac{s}{24}$	0	$\frac{1}{840}$	0	0
6, 0	$\frac{s^3}{7}+\frac{1151}{630}$	$\frac{733s}{420}$	$\frac{7s^2}{45}$	$\frac{71}{360}$	$\frac{s}{72}$	0	$\frac{1}{5040}$	0	0
4, 3	$-\frac{s^2}{4}$	$\frac{127}{288}$	$\frac{5s}{18}$	$\frac{s^2}{18}$	$\frac{5}{288}$	$\frac{s}{72}$	0	$\frac{1}{1152}$	0
5, 2	$\frac{43s^2}{60}$	$\frac{s^3}{15}-\frac{123}{160}$	$-\frac{3s}{20}$	$\frac{s^2}{30}$	$\frac{61}{480}$	$\frac{s}{60}$	0	$\frac{1}{1920}$	0
6, 1	$-\frac{73s^2}{60}$	$-\frac{s^3}{15}+\frac{271}{1440}$	$\frac{29s}{36}$	$\frac{17s^2}{180}$	$\frac{221}{1440}$	$\frac{s}{90}$	0	$\frac{1}{5760}$	0
7, 0	$\frac{691s^2}{420}$	$\frac{22s^3}{105}+\frac{4873}{1440}$	$\frac{394s}{315}$	$\frac{11s^2}{180}$	$\frac{83}{1440}$	$\frac{s}{360}$	0	$\frac{1}{40320}$	0
4, 4	$\frac{s^4}{9}-\frac{118s}{81}$	$-\frac{17s^2}{54}$	$-\frac{s^3}{81}+\frac{119}{144}$	$\frac{31s}{108}$	$\frac{s^2}{27}$	$\frac{1}{144}$	$\frac{s}{216}$	0	$\frac{1}{5184}$
5, 3	$-\frac{s^4}{9}+\frac{667s}{810}$	$\frac{197s^2}{540}$	$\frac{32s^3}{405}-\frac{4}{45}$	$\frac{119s}{1080}$	$\frac{29s^2}{1080}$	$\frac{11}{360}$	$\frac{11s}{2160}$	0	$\frac{1}{6480}$
6, 2	$\frac{s^4}{9}+\frac{322s}{405}$	$\frac{103s^2}{540}$	$\frac{s^3}{162}-\frac{35}{72}$	$\frac{113s}{540}$	$\frac{37s^2}{1080}$	$\frac{11}{180}$	$\frac{s}{216}$	0	$\frac{1}{12960}$
7, 1	$-\frac{s^4}{9}-\frac{21167s}{5670}$	$-\frac{1999s^2}{3780}$	$\frac{37s^3}{567}+\frac{115}{63}$	$\frac{6107s}{7560}$	$\frac{47s^2}{1080}$	$\frac{17}{360}$	$\frac{s}{432}$	0	$\frac{1}{45360}$
8, 0	$\frac{s^4}{9}+\frac{19912s}{2835}$	$\frac{5297s^2}{1890}$	$\frac{409s^3}{2835}+\frac{28319}{10080}$	$\frac{4273s}{7560}$	$\frac{19s^2}{1080}$	$\frac{19}{1440}$	$\frac{s}{2160}$	0	$\frac{1}{362880}$

Thus, for example, $u_{2,1}F_2 = -\frac{1}{4}f_2 + \frac{1}{8}f_2^{(3)}$. To verify these identities, first divide both sides by F_2, differentiate both sides, then compare the behavior at infinity to check that the constant of integration is the same. To illustrate this, we check the identity for $u_{1,0}F_2$. After differentiation we must verify that

$$-q_1q_0 = \frac{(f_2''/2)F_2 - (f_2'/2)f_2}{F_2^2},$$

which is equivalent to

$$-(q' + u_{0,0}q)qF_2 = \frac{1}{2}(-2qq'F_2 - 2q^2 f_2 + u_{0,0}f_2') - \frac{1}{2}f_2' u_{0,0}.$$

Expanding each side, we see they are both $-qq'F_2 - q^2 f_2$. The remainder of the table is proved in this manner.

For $n = 2, 4, 6$, the identities presented in this table are sufficient to write c_n in terms of f_2 and its derivatives. At c_8 we have a potential problem. There is a nonzero eighth order term appearing in $\frac{1}{2}[(\operatorname{tr}\mathcal{T})^2 - \operatorname{tr}\mathcal{T}^2]$:

$$\frac{1}{t^8}\left[u_{1,0}(s_1)^2 - u_{0,0}(s_1)u_{1,1}(s_1)\right]\left[u_{1,0}(s_2)^2 - u_{0,0}(s_2)u_{1,1}(s_2)\right].$$

This term must be multiplied $F_2(s_1)F_2(s_2)$, and the table does not provide a way to reduce this to an f_2-type expression. However, a direct computation will verify the formula

$$\left[u_{1,0}^2 - u_{0,0}u_{1,1}\right]F_2 = -\frac{1}{6}f_2 + \frac{1}{3}sf_2' - \frac{1}{12}f_2^{(iii)}. \tag{23}$$

Using this formula, the table of identities, and (9) and (10) we arrive at the expression[1]

$$\begin{aligned}
c_8(s_1,s_2) &= \left[\frac{149}{6} + s_1^3 + 7s_1s_2^2 + 7s_1^2s_2 + s_2^3\right] f_2(s_1)f_2(s_2) \\
&\quad + \left[15s_1 + \frac{34}{3}s_2\right] f_2(s_1)f_2'(s_2) + \left[15s_2 + \frac{34}{3}s_1\right] f_2(s_2)f_2'(s_1) \\
&\quad + \left[\frac{3}{2}s_1^2 + \frac{13}{3}s_1s_2 + \frac{3}{2}s_2^2\right] f_2'(s_1)f_2'(s_2) + 3[f_2''(s_1)f_2'(s_2) + f_2''(s_2)f_2'(s_1)] \\
&\quad + \frac{1}{2}(s_1+s_2)f_2''(s_1)f_2''(s_2) + \frac{1}{24}f_2^{(iii)}(s_1)f_2^{(iii)}(s_2).
\end{aligned} \tag{24}$$

Performing the same sort of integration as in (21), (22) results in (5).

We can find t^{-10} order terms in trace formulas of the form $\operatorname{tr}\mathcal{T}^j(\operatorname{tr}\mathcal{T})^k$ for $j+k \le 5$ in the expansion (8). Remarkably, when the computation of tenth order terms in (8) is carried out it is found that all terms cancel except those contained in

$$-\operatorname{tr}\mathcal{T} + \frac{1}{2}\left[(\operatorname{tr}\mathcal{T})^2 - \operatorname{tr}\mathcal{T}^2\right].$$

The terms needed from $-\operatorname{tr}\mathcal{T}$ can be dealt with using the table, as before. The same is not true of $\frac{1}{2}[(\operatorname{tr}\mathcal{T})^2 - \operatorname{tr}\mathcal{T}^2]$. There is a great amount of cancellation within this term, so much so that by introducing the notation

$$\begin{aligned}
A(s) &= u_{1,0}(s)^2 - u_{0,0}(s)u_{1,1}(s), \\
B(s) &= -2u_{2,0}(s)^2 + u_{1,1}(s)u_{2,0}(s) - u_{1,0}(s)u_{2,1}(s)2u_{0,0}(s)u_{2,2}(s) \\
&\qquad + 3u_{1,0}(s)u_{3,0}(s) - 3u_{0,0}(s)u_{3,1}(s), \\
C(s) &= u_{1,0}(s)u_{2,0}(s) - u_{0,0}(s)u_{2,1}(s)
\end{aligned}$$

we find the tenth order term in $\frac{1}{2}[(\operatorname{tr}\mathcal{T})^2 - \operatorname{tr}\mathcal{T}^2]$ is given by

$$2A(s_1)F_2(s_1)B(s_2)F_2(s_2) + 2A(s_2)F_2(s_2)B(s_1)F_2(s_1) + 2C(s_1)F_2(s_1)C(s_2)F_2(s_2).$$

[1] We do not explicitly provide the formula for c_{10}, as the result is rather cumbersome.

Analogous to the above table and (23), we may verify the following identities in the same fashion:

$$A(s)F_2(s) = -\frac{1}{6}f_2(s) + \frac{1}{3}sf_2'(s) - \frac{1}{12}f_2^{(3)}(s),$$

$$B(s)F_2(s) = -\frac{1}{3}sf_2(s) + \frac{2}{3}s^2 f_2'(s) - \frac{1}{6}f_2^{(3)}(s),$$

$$C(s)F_2(s) = \frac{1}{12}f_2'(s) + \frac{1}{6}sf_2''(s) - \frac{1}{24}f_2^{(4)}(s).$$

Using these formulas to integrate terms in $\frac{1}{2}[(\operatorname{tr}\mathcal{T})^2 - \operatorname{tr}\mathcal{T}^2]$, and the table of identities to do the same for terms in $\operatorname{tr}\mathcal{T}$, we find (6) for C_{10}.

Appendix 2: Numerical Comparisons

Folkmar Bornemann has generously provided us with a table of values for the Airy$_2$ covariance obtained by directly computing appropriate Fredholm determinants. See [4, 5] for details on the numerical methods used. We denote these values $\operatorname{cov}_B(t)$ and consider the information for $5 \le t \le 25$. Using the following high precision computations courtesy of Prähofer,

$$\mu_1 = -1.771\,086\,807\,411\,601\,626\ldots$$

$$\mu_2 = 3.949\,943\,272\,220\,377\,513\ldots$$

$$\mu_3 = -9.711\,844\,753\,027\,647\,354\ldots$$

$$\mu_4 = 26.025\,435\,426\,839\,994\,565\ldots$$

together with (3)–(6), we obtain the expressions

$$C_4 = 2\mu_1 = -3.542\,173\,614\,823\,203\,252\ldots$$

$$C_6 = 2\mu_2 + \frac{10}{3}\mu_1^2 = 18.355\,714\,809\,065\,487\ldots$$

$$C_8 = 2\mu_3 + 14\mu_2\mu_1 + \frac{13}{2} = -110.863\,383\,378\,407\,421\ldots$$

$$C_{10} = 2\mu_4 + 24\mu_3\mu_1 + \frac{126}{5}\mu_2^2 + 116\mu_1 = 652.588\,990\,733\,866\,004\ldots.$$

The following table collects values of $\operatorname{cov}_B(t)$, the approximations $\operatorname{cov}_{2,n}(t)$, and the error between the two values measured by $\operatorname{cov}_B(t) - \operatorname{cov}_{2,n}(t)$.

t	$\operatorname{cov}_B(t)$	$\operatorname{cov}_{2,6}(t)$	Error	$\operatorname{cov}_{2,8}(t)$	Error	$\operatorname{cov}_{2,10}(t)$	Error
5	.03527955721	.03550728796	-2×10^{-4}	.03522347770	6×10^{-5}	.03529030281	-1×10^{-5}
10	.00966309240	.00966413835	-1×10^{-6}	.00966302972	6×10^{-8}	.00966309498	-3×10^{-9}
15	.004376044913	.00437608706	-4×10^{-8}	.00437604380	1×10^{-9}	.00437604493	-2×10^{-11}
20	.002478143955	.00247814822	-4×10^{-9}	.00247814389	6×10^{-11}	.00247814396	-1×10^{-12}
25	.001591006500	.00159100722	-7×10^{-10}	.00159100649	6×10^{-12}	.00159100650	-8×10^{-13}

References

1. Adler, M., van Moerbeke, P.: A PDE for the joint distributions of the Airy process. Preprint. arXiv:math.PR/0302329
2. Adler, M., van Moerbeke, P.: PDEs for the joint distributions of the Dyson, Airy and sine processes. Ann. Probab. **33**, 1326–1361 (2005)
3. Amir, G., Corwin, I., Quastel, J.: Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions. Commun. Pure Appl. Math. **64**, 466–537 (2011)
4. Bornemann, F.: On the numerical evaluation of Fredholm determinants. Math. Comput. **79**, 871–915 (2010)
5. Bornemann, F.: On the numerical evaluation of distributions in random matrix theory: A review. Markov Process. Relat. Fields **16**, 803–866 (2010)
6. Borodin, A., Ferrari, P.L., Sasamoto, T.: Transition between the Airy_1 and Airy_2 processes and TASEP fluctuations. Commun. Pure Appl. Math. **61**, 1603–1629 (2008)
7. Borodin, A., Ferrari, P.L.: Large time asymptotics of growth models on space-like paths. Electron. J. Probab. **13**, 1380–1418 (2008)
8. Dotsenko, V.: Replica Bethe ansatz derivation of the Tracy-Widom distribution of the free energy fluctions in one-dimensional directed polymers. J. Stat. Mech., P07010 (2010)
9. Dotsenko, V.: Universal randomness. Preprint. arXiv:1009.3116
10. Ferrari, P.L., Spohn, H.: Step fluctuations for a facet crystal. J. Stat. Phys. **113**, 1–46 (2003)
11. Hägg, J.: Local Gaussian fluctuations in the Airy and discrete PNG processes. Ann. Probab. **36**, 1059–1092 (2008)
12. Imamura, T., Sasamoto, T.: Dynamics of a tagged particle in the asymmetric exclusion process with the step initial condition. J. Stat. Phys. **128**, 799–846 (2007)
13. Johansson, K.: Discrete polynuclear growth and determinantal processes. Commun. Math. Phys. **242**, 277–329 (2003)
14. Johansson, K.: The arctic circle boundary and the Airy process. Ann. Probab. **33**, 1–30 (2005)
15. Prähofer, M., Spohn, H.: Scale invariance of the PNG droplet and the Airy process. J. Stat. Phys. **108**, 1071–1106 (2002)
16. Prolhac, S., Spohn, H.: Two-point generating function of the free energy for a directed polymer in a random medium. Preprint. arXiv:1011.4014
17. Sasamoto, T., Spohn, H.: Exact height distributions for the KPZ equation with narrow wedge initial condition. Nucl. Phys. B **834**, 523–542 (2010)
18. Sasamoto, T., Spohn, H.: The one-dimensional KPZ equation: an exact solution and its universality. Phys. Rev. Lett. **104**, 230602 (2010)
19. Sasamoto, T., Spohn, H.: The $1+1$ dimensional Kardar-Parisi-Zhang equation and its universality class. J. Stat. Mech., P11013 (2010)
20. Tracy, C.A., Widom, H.: Level-spacing distributions and the Airy kernel. Commun. Math. Phys. **159**, 151–174 (1994)
21. Tracy, C.A., Widom, H.: Fredholm determinants, differential equations and matrix models. Commun. Math. Phys. **163**, 38–72 (1994)
22. Tracy, C.A., Widom, H.: Airy kernel and Painlevé II. In: Its, A., Harnad, J. (eds.) Isomonodromic Deformations and Applications in Physics. CRM Proceedings & Lecture Notes, vol. 31, pp. 85–98. Amer. Math. Soc., Providence (2002)
23. Tracy, C.A., Widom, H.: Asymptotics in ASEP with step initial condition. Commun. Math. Phys. **290**, 129–154 (2009)
24. Tracy, C.A., Widom, H.: Differential equations for Dyson processes. Commun. Math. Phys. **252**, 7–41 (2004)
25. Widom, H.: On asymptotics for the Airy process. J. Stat. Phys. **115**, 1129–1134 (2004)