

On the Asymmetric Simple Exclusion Process with Multiple Species

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Abstract In previous work the authors, using the Bethe Ansatz, found for the N -particle asymmetric simple exclusion process on the integers a formula—a sum of multiple integrals—for the probability that a system is in a particular configuration at time t given an initial configuration. The present work extends this to the case where particles are of different species, with particles of a higher species having priority over those of a lower species. Here the integrands in the multiple integrals are defined by a system of relations whose consistency requires verifying that the Yang-Baxter equations hold.

Keywords Asymmetric simple exclusion process · Multiple species · Bethe Ansatz · Yang-Baxter equations

1 Introduction

The one-dimensional asymmetric simple exclusion process (ASEP) [10, 14, 15] is one of the simplest interacting particle systems with a single conservation law (density of particles) and as such is a basic model in both probability theory and nonequilibrium statistical physics. Its importance is further enhanced by the fact that weakly asymmetric limits of ASEP distributions can be interpreted as distributions of the height function which solves the Kardar-Parisi-Zhang equation [3, 12, 17].

Recall that in the asymmetric simple exclusion process particles are at sites of the lattice \mathbb{Z} .¹ Each particle waits exponential time, then with probability p it moves one step to the right if the site is unoccupied, otherwise it stays put; and with probability $q = 1 - p$

¹Many authors consider ASEP on the circle or the lattice $[1, L]$ with open boundary conditions.

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it moves one step to the left if the site is unoccupied, otherwise it stays put. Each particle does this independently of the other particles. For a finite number of particles this defines a Markov process; and for infinitely many particles, with further work [15] this too defines a Markov process.

In *multispecies ASEP* particles belong to different species, labeled $1, 2, \dots, M$. Particles of a higher species have priority over those of a lower species.² Thus, if a particle of species s tries to move to a neighboring site occupied by a particle of species s' it is blocked if $s \leq s'$, but if $s > s'$ the particles interchange positions. Second-class particles were introduced by Liggett [13] and subsequently developed and generalized by several authors [1, 4, 5, 7–9, 20].

A configuration in ASEP with N particles is the set of occupied sites

$$X = \{x_1, \dots, x_N\} \quad (x_1 < \dots < x_N).$$

Theorem 2.1 of [18] (with proof corrected in [19]) was a formula for $P_Y(X; t)$, the probability that the system is in configuration X at time t , given that the initial configuration was $Y = \{y_1, \dots, y_N\}$. It is a sum over the permutation group \mathcal{S}_N of multiple integrals. If $p \neq 0$ then

$$P_Y(X; t) = \sum_{\sigma \in \mathcal{S}_N} \frac{1}{(2\pi i)^N} \int_{C_r^N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i (\xi_i^{-y_i-1} e^{\varepsilon(\xi_i)t}) d^N \xi, \tag{1}$$

where C_r is a circle about zero in \mathbb{C} with sufficiently small radius r , where

$$\varepsilon(\xi) = p\xi^{-1} + q\xi - 1,$$

and where $A_\sigma(\xi_1, \dots, \xi_N)$ is given explicitly by (7) below.

In multispecies ASEP a configuration \mathcal{X} is a pair (X, π) where $X = \{x_1, \dots, x_N\}$ as before and π is a function from $[1, N]$ to $[1, M]$. If the system is in configuration \mathcal{X} then the i th particle from the left is at x_i and belongs to species π_i . A special case is that of first- and second-class particles, a first-class particle having priority over a second-class particle. For example, if $\pi = (1 \ 2 \ 2 \ 2)$ the left-most particle is second-class and the other three are first-class.

The purpose of this paper is to establish for multispecies ASEP a formula analogous to (1) for $P_{\mathcal{Y}}(\mathcal{X}; t)$, the probability that the system is in configuration $\mathcal{X} = (X, \pi)$ at time t , given that the initial configuration is $\mathcal{Y} = (Y, \nu)$. We show that there is an entirely analogous formula,

$$P_{\mathcal{Y}}(\mathcal{X}; t) = \sum_{\sigma \in \mathcal{S}_N} \frac{1}{(2\pi i)^N} \int_{C_r^N} A_\sigma^\pi(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i (\xi_i^{-y_i-1} e^{\varepsilon(\xi_i)t}) d^N \xi, \tag{2}$$

but now the factors A_σ^π are not (except in special cases) given explicitly. They are determined by (21) and (22) below. Note that they also depend on ν .

In the next section we present the proof of (1) in some detail. This is partly because for a correct proof one must refer to both [18] and [19], but also because we shall show how to prove (2) by a modification of the proof of (1).

There is another difference here. The factors A_σ^π must satisfy the family of identities determined by (21) and (22) below. For ordinary ASEP there is no issue about the existence of

²This is sometimes called the $M + 1$ species model, empty sites behaving as particles of another species. With our convention, a particle of species M is first-class, having priority over all others.

a solution since it can be written down explicitly. But for multispecies ASEP we must show that the identities define A_{σ}^{π} consistently. If, as in [1], the multispecies ASEP is formulated as a nested Bethe Ansatz problem, one is led to show that the *Yang-Baxter equations* [6, 21] are satisfied. That the model has nontrivial solutions to the Yang-Baxter equations can be traced back to early work of Perk and Schultz on multistate vertex models [16]. It was Alcaraz *et al.* [2] who recognized that these Bethe Ansatz solvable multistate vertex models lead to integrable stochastic models. In our formulation below, these consistency conditions are stated in terms of representations of braid relations.

In the last section we focus on the explicit determination of some A_{σ}^{π} , with particular attention to the case of one second-class particle.

2 ASEP with One Species

2.1 Bethe Ansatz Solution

The probability $P_Y(X; t)$ satisfies the differential equation (the *master equation*)

$$\frac{\partial u}{\partial t} = \sum_{i=1}^N [pu(x_i - 1)(1 - \delta(x_i - x_{i-1} - 1)) + qu(x_i + 1)(1 - \delta(x_{i+1} - x_i - 1)) - pu(x_i)(1 - \delta(x_{i+1} - x_i - 1)) - qu(x_i)(1 - \delta(x_i - x_{i-1} - 1))]. \tag{3}$$

Here u is $u(x_1, \dots, x_N; t)$, and in the i th summand above the j th variable is x_j when $j \neq i$. We use the convention that any δ term involving x_0 or x_{N+1} is zero.

The probability $P_Y(X; t)$ is the solution of this equation that also satisfies the initial condition

$$u(X; 0) = \delta_Y(X). \tag{4}$$

The particles interact through the exclusion constraint. If they did not interact then $P_Y(X; t)$ would satisfy the differential equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^N [pu(x_i - 1) + qu(x_i + 1) - u(x_i)]. \tag{5}$$

If u satisfies this equation then it would also satisfy (3) if in addition the difference of the right sides were zero. This difference is

$$\sum_{i=1}^{N-1} [pu(x_i, x_{i+1} - 1) + qu(x_i + 1, x_{i+1}) - u(x_i, x_{i+1})] \delta(x_{i+1} - x_i - 1).$$

Here we displayed entries i and $i + 1$ in $u(x)$. For the first and last summands, we changed indices of summation from what they were in (3).

Thus $u(x)$ satisfies (3) if it satisfies both (5) and, for $i = 1, \dots, N - 1$, the *boundary conditions*

$$pu(x_i, x_i) + qu(x_i + 1, x_i + 1) - u(x_i, x_i + 1) = 0. \tag{6}$$

For any nonzero complex numbers ξ_1, \dots, ξ_N , a solution of (5) is $\prod_i (\xi_i^{x_i} e^{\epsilon(\xi_i)t})$. We may permute the ξ_i , take linear combinations, and integrate. In this way we obtain a class of solutions

$$\int \sum_{\sigma \in S_N} F_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i e^{\epsilon(\xi_i)t} d^N \xi,$$

where the functions $F_\sigma(\xi)$ are arbitrary. We look for F_σ such that the integrand satisfies the boundary conditions pointwise. This is the *Bethe Ansatz*.

Substituting the left side of (6) into the part of the integrand that depends on x_i we get $(\xi_{\sigma(i)} \xi_{\sigma(i+1)})^{x_i}$ times

$$(p + q \xi_{\sigma(i)} \xi_{\sigma(i+1)} - \xi_{\sigma(i+1)}) F_\sigma.$$

Define $T_i \sigma$ to be σ with the entries $\sigma(i)$ and $\sigma(i + 1)$ interchanged. If we replace σ by $T_i \sigma$ nothing else in the integrand changes, so a sufficient condition that the integrand is zero is that

$$(p + q \xi_{\sigma(i)} \xi_{\sigma(i+1)} - \xi_{\sigma(i+1)}) F_\sigma + (p + q \xi_{\sigma(i)} \xi_{\sigma(i+1)} - \xi_{\sigma(i)}) F_{T_i \sigma} = 0.$$

If we define

$$S(\xi, \xi') = -\frac{p + q \xi \xi' - \xi}{p + q \xi \xi' - \xi'},$$

then the conditions become

$$\frac{F_{T_i \sigma}}{F_\sigma} = S(\xi_{\sigma(i+1)}, \xi_{\sigma(i)}).$$

We can find the general solution of this system of equations for the F_σ . An *inversion* in σ is a pair (i, j) with $i > j$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$. It is straightforward to check that one solution is

$$A_\sigma(\xi) = \prod_{\text{inversions } (i, j)} S(\xi_i, \xi_j). \tag{7}$$

All F_σ are determined by F_e , where e is the identity permutation. Since this can be an arbitrary function $\varphi(\xi)$, the general solution is $F_\sigma(\xi) = A_\sigma(\xi) \varphi(\xi)$.

2.2 Satisfying the Initial Condition

We choose $\varphi(\xi)$ so that the initial condition is satisfied by the $\sigma = e$ summand. Since

$$\frac{1}{(2\pi i)^N} \int_{\mathcal{C}^N} \prod_i \xi_i^{x_i - y_i - 1} d^N \xi = \delta_Y(X),$$

where \mathcal{C} is a circle about zero, we take $\varphi(\xi) = (2\pi i)^{-N} \prod_i \xi_i^{-y_i - 1}$ and the domain of integration to be \mathcal{C}^N .

When $\sigma \neq e$ it matters which contours of integration we take because of the denominators in the A_σ . If $p \neq 0$ then all the denominators will be nonzero on and inside \mathcal{C}_r if r is small enough. It is such an r that we take in (1).

Denote by $I(\sigma)$ the σ -summand in (1) with $t = 0$. To prove (1) we must show that

$$\sum_{\sigma \neq e} I(\sigma) = 0.$$

This will be proved by induction on N . For $N = 1$ there is nothing to show. Assuming the result for $N - 1$, the sum over all permutations in $\mathcal{S}_N \setminus \{e\}$ such that $\sigma(N) = N$ will be zero, by the induction hypothesis. (No S -factor in A_σ will involve ξ_N , and there is an obvious correspondence between $\sigma \in \mathcal{S}_{N-1}$ and those $\sigma \in \mathcal{S}_N$ satisfying $\sigma(N) = N$.) So it suffices to show that

$$\sum_{\sigma(N) \neq N} I(\sigma) = 0. \tag{8}$$

For each nonempty subset B of $[1, N - 1]$ define

$$\mathcal{S}_N(B) = \{ \sigma \in \mathcal{S}_N : \text{the inversions in } \sigma \text{ involving } N \text{ are the } (N, i) \text{ with } i \in B \}.$$

We shall show that for each B we have

$$\sum_{\sigma \in \mathcal{S}_N(B)} I(\sigma) = 0. \tag{9}$$

Once we have this, (8) will follow since the set of σ with $\sigma(N) \neq N$ is the disjoint union of the various $\mathcal{S}_N(B)$.

Start of the proof The integrands in $I(\sigma)$ with $\sigma \in \mathcal{S}_N(B)$ may be written

$$\prod_{i \in B} S(\xi_N, \xi_i) \times \prod_{i \leq N} \xi_i^{x_{\sigma^{-1}(i)} - y_i - 1} \times \prod \{ S(\xi_k, \xi_\ell) : N > k > \ell, \sigma^{-1}(k) < \sigma^{-1}(\ell) \}.$$

In these integrals we make the substitution

$$\xi_N \rightarrow \frac{\eta}{\prod_{i < N} \xi_i}, \tag{10}$$

so that η runs over a circle of radius r^N . The integrand becomes

$$(-1)^{|B|} \prod_{i \in B} \frac{p + q\eta \prod_{\ell \neq i, N} \xi_\ell^{-1} - \eta \prod_{\ell \neq N} \xi_\ell^{-1}}{p + q\eta \prod_{\ell \neq i, N} \xi_\ell^{-1} - \xi_i} \tag{11}$$

$$\times \eta^{x_{\sigma^{-1}(N)} - y_N - 1} \prod_{i < N} \xi_i^{x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(N)} + y_N - y_i - 1} \tag{12}$$

$$\times \prod \{ S(\xi_k, \xi_\ell) : N > k > \ell, \sigma^{-1}(k) < \sigma^{-1}(\ell) \}. \tag{13}$$

The reason we still have -1 in the exponents in (12) is that $d\xi_N = \prod_{i < N} \xi_i^{-1} d\eta$.

Lemma 1 *When $|B| = 1$ we have $I(\sigma) = 0$ for all $\sigma \in \mathcal{S}_N(B)$.*

Proof There is a single $i \in B$ and (11) is analytic inside the ξ_i -contour except for a simple pole at $\xi_i = 0$. The exponent of ξ_i in (12) is positive since $N > i$, and so $y_N > y_i$, and $\sigma^{-1}(i) > \sigma^{-1}(N)$, and so $x_{\sigma^{-1}(i)} > x_{\sigma^{-1}(N)}$. Therefore the integrand is analytic inside the ξ_i -contour, so the integral is zero. \square

Lemma 2 When $|B| > 1$ we have for all $\sigma \in \mathcal{S}_N(B)$,

$$I(\sigma) = \sum_{(i,j)} I_{B,(i,j)}(\sigma),$$

where in the sum (i, j) runs over all unordered pairs with $i, j \in B$ and $i \neq j$, where each $I_{B,(i,j)}(\sigma)$ is a lower-order integral in which (11) is replaced by a factor depending only on B and (i, j) (the other factors remaining the same), and where $\xi_i = \xi_j$ in the domain of integration.

Proof We may assume that $q \neq 0$. This case follows by a limiting argument. We are going to shrink some of the ξ_i -contours with $i \in B$. Due to the defining property of r , the only poles we pass will come from the product (11). In fact, to avoid double poles later we take $\xi_i \in \mathcal{C}_{r_i}$ with the r_i all slightly different.

Take $j = \max B$ and shrink the ξ_j -contour. The product (11) has a simple pole at $\xi_j = 0$ (the j -factor has the pole and the i -factors with $i \neq j$ are analytic there) and the power of ξ_j in (12) is positive as before, so the integrand is analytic at $\xi_j = 0$. For each $k \in B$ with $k \neq j$ we pass the pole at

$$\xi_j = \frac{q\eta \prod_{\ell \neq j,k,N} \xi_\ell^{-1}}{\xi_k - p} \tag{14}$$

coming from the k -factor in (11). (Our assumption on the r_i assures that there are no double poles.) For the residue we replace the k -factor by

$$-\frac{p + q\eta \prod_{\ell \neq k,N} \xi_\ell^{-1} - \eta \prod_{\ell \neq N} \xi_\ell^{-1}}{q\eta \xi_j^{-2} \prod_{\ell \neq j,k,N} \xi_\ell^{-1}}, \tag{15}$$

where in this and the j -factor we replace ξ_j by the right side of (14). When $i \neq j, k$ the i -factor becomes

$$\frac{p + q\eta \prod_{\ell \neq i,N} \xi_\ell^{-1} - \eta \prod_{\ell \neq N} \xi_\ell^{-1}}{p(1 - \xi_i \xi_k^{-1})},$$

and we replace ξ_j in the numerator by the right side of (14).

We now shrink the ξ_k -contour. There is a pole of order 2 at $\xi_k = 0$ coming from (15) and the j -factor in (11). Since $k < j = \max B < N$, we have $y_N - y_k \geq 2$, so the exponent of ξ_k in (12) is at least 2. Therefore the integrand is analytic at $\xi_k = 0$. The factor (15) has no other poles inside \mathcal{C}_{r_k} . An i -factor with $i \neq j, k$ will have a pole at $\xi_k = \xi_i$ if $r_i < r_k$. There is also the pole at

$$\xi_k = \frac{q\eta \prod_{\ell \neq j,k,N} \xi_\ell^{-1}}{\xi_j - p}$$

coming from the j -factor. This relation and (14) imply $\xi_j = \xi_k$.

Thus when we shrink the ξ_j -contour and the ξ_k -contours with $k \neq j$ we obtain $(N - 2)$ -dimensional integrals in each of which two of the ξ -variables corresponding to indices in B are equal. This proves the lemma. □

Lemma 3 In the notation of Lemma 2, for each (i, j) there is a partition of $\mathcal{S}_N(B)$ into pairs σ, σ' such that $I_{B,(i,j)}(\sigma) + I_{B,(i,j)}(\sigma') = 0$ for each pair.

Proof We pair σ and σ' if they agree except for the positions of i and j , which are interchanged. The factor (12) is clearly the same for both when $\xi_i = \xi_j$, and we shall show that the σ - and σ' -factors in (13) are negatives of each other when $\xi_i = \xi_j$.

Assume for definiteness that

$$i < j \quad \text{and} \quad \sigma^{-1}(i) < \sigma^{-1}(j). \tag{16}$$

(Otherwise we reverse the roles of σ and σ' .) Then the factor $S(\xi_j, \xi_i)$ does not appear for σ in (13) but it does appear for σ' . This factor equals -1 when $\xi_i = \xi_j$.

To complete the proof it is enough to show that for any $k \neq i, j$ the product of S -factors involving k and either i or j is the same for σ and σ' when $\xi_i = \xi_j$. If $\sigma^{-1}(k)$ is outside the interval $(\sigma^{-1}(i), \sigma^{-1}(j))$ the S -factors in question are the same for σ and σ' , so we assume $\sigma^{-1}(k)$ is inside the interval. There are three cases, with the results displayed in the table below. The first column gives the position of k relative to i and j , the second column gives the product of S -factors involving k and either i or j for σ , and the third column gives the corresponding product for σ' ,

$i < k < j,$	1,	$S(\xi_j, \xi_k)S(\xi_k, \xi_i),$
$k < i,$	$S(\xi_i, \xi_k),$	$S(\xi_j, \xi_k),$
$k > j,$	$S(\xi_k, \xi_j),$	$S(\xi_k, \xi_i).$

In all cases but the second the S -factors are exactly the same for σ and σ' when $\xi_i = \xi_j$. For the second we use $S(\xi, \xi_k)S(\xi_k, \xi) = 1$. □

Clearly (9) follows from Lemmas 1–3, and this completes the proof of (1). □

3 ASEP with Multiple Species

3.1 Bethe Ansatz Solution

Observe that an interchange of particles at positions x_i and x_{i+1} has the same effect as leaving the particles as they were but interchanging π_i and π_{i+1} . Thus X remains the same but π is replaced by $T_i\pi$. (This is the same T_i as before, but applied to π rather than σ .)

Write $u^\pi(X; t)$ for $P_{\mathcal{Y}}(\mathcal{X}; t)$. The master equation for u^π differs from Eq. (3) for u since particles are not blocked as much, so there are other terms on the right side.

We define

$$\alpha_i(\pi) = \begin{cases} 0 & \text{if } \pi_i = \pi_{i+1}, \\ p & \text{if } \pi_i < \pi_{i+1}, \\ q & \text{if } \pi_i > \pi_{i+1}, \end{cases}$$

and define $\beta_i(\pi)$ as above but with p and q interchanged. (Thus $\beta_i(\pi) = \alpha_i(T_i\pi)$.) We compute that what must be added to the right side of (3) is

$$\sum_{i=1}^{N-1} [\alpha_i(\pi)u^{T_i\pi}(x_i, x_{i+1}) - \beta_i(\pi)u^\pi(x_i, x_{i+1})] \delta(x_{i+1} - x_i - 1).$$

(As before, we displayed entries i and $i + 1$.) The terms here may be incorporated into the boundary conditions, as the δ -terms in (3) were. We conclude that if $u^\pi(X; t)$ satisfies the

equation

$$\frac{\partial u^\pi}{\partial t} = \sum_{i=1}^N [pu^\pi(x_i - 1) + qu^\pi(x_i + 1) - u^\pi(x_i)], \tag{17}$$

the boundary conditions for $i = 1, \dots, N - 1$

$$pu^\pi(x_i, x_i) + qu^\pi(x_i + 1, x_i + 1) - u^\pi(x_i, x_i + 1) - \alpha_i(\pi)u^{T_i\pi}(x_i, x_i + 1) + \beta_i(\pi)u^\pi(x_i, x_i + 1) = 0, \tag{18}$$

and the initial condition

$$u^\pi(X; t) = \delta_Y(X)\delta_v(\pi),$$

then $P_Y(\mathcal{X}; t) = u^\pi(X; t)$.

We assume a solution of the form

$$u^\pi(X; t) = \sum_{\sigma \in S_N} \frac{1}{(2\pi i)^N} \int_{C_r^N} A_\sigma^\pi(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i (\xi_i^{-y_i-1} e^{\varepsilon(\xi_i)t}) d^N \xi,$$

where $A_e^\pi = \delta_v(\pi)$ so that the initial condition is satisfied by the $\sigma = e$ summand.

Substituting the left side of (18) into the part of the integrand that depends on x_i we now get $(\xi_{\sigma(i)}\xi_{\sigma(i+1)})^{x_i}$ times

$$[f(b, a) + \beta_i(\pi)b]A_\sigma^\pi - \alpha_i(\pi)bA_{T_i\sigma}^{T_i\pi},$$

where

$$f(\xi, \xi') = p + q\xi\xi' - \xi, \quad a = \xi_{\sigma(i)}, \quad b = \xi_{\sigma(i+1)}. \tag{19}$$

If we replace σ by $T_i\sigma$ nothing else in the integrand changes, so a sufficient condition that the integral be zero is that

$$[f(b, a) + \beta_i(\pi)b]A_\sigma^\pi + [f(a, b) + \beta_i(\pi)a]A_{T_i\sigma}^\pi - \alpha_i(\pi)bA_{T_i\sigma}^{T_i\pi} - \alpha_i(\pi)aA_{T_i\sigma}^{T_i\pi} = 0.$$

If this is to hold for all π it must hold for π replaced by $T_i\pi$, so

$$[f(b, a) + \alpha_i(\pi)b]A_\sigma^{T_i\pi} + [f(a, b) + \alpha_i(\pi)a]A_{T_i\sigma}^{T_i\pi} - \beta_i(\pi)bA_\sigma^\pi - \beta_i(\pi)aA_{T_i\sigma}^\pi = 0.$$

If $\pi_{i+1} = \pi_i$ then we obtain simply

$$f(b, a)A_\sigma^\pi + f(a, b)A_{T_i\sigma}^\pi = 0. \tag{20}$$

If $\pi_{i+1} \neq \pi_i$ we use $\beta_i(\pi) = 1 - \alpha_i(\pi)$ to help with the computation and find by eliminating $A_{T_i\sigma}^{T_i\pi}$ from the two equations above that

$$A_{T_i\sigma}^\pi = \alpha_i(\pi) \frac{b - a}{f(a, b)} A_\sigma^{T_i\pi} - \left[\alpha_i(\pi) \frac{a - b}{f(a, b)} + 1 \right] A_\sigma^\pi.$$

(Here (19) was also used.) If we replace the second $\alpha_i(\pi)$ by $1 - \beta_i(\pi)$ we get

$$A_{T_i\sigma}^\pi = \alpha_i(\pi) \frac{b-a}{f(a,b)} A_\sigma^{T_i\pi} - \frac{f(b,a) - \beta_i(\pi)(a-b)}{f(a,b)} A_\sigma^\pi,$$

which now agrees with (20) when $\pi_{i+1} = \pi_i$.

The relation is nicer for the quantity h_σ^π defined by

$$A_\sigma^\pi = h_\sigma^\pi A_\sigma, \tag{21}$$

where A_σ is defined by (7) as before. The condition $A_e^\pi = \delta_v(\pi)$ becomes $h_e^\pi = \delta_v(\pi)$.

If we use $a-b = f(b,a) - f(a,b)$, divide by $A_{T_i\sigma} = S(b,a)A_\sigma$, and recall (19), our formula becomes

$$h_{T_i\sigma}^\pi = h_\sigma^\pi + (1 + S(\xi_{\sigma(i)}, \xi_{\sigma(i+1)})) [\alpha_i(\pi) h_\sigma^{T_i\pi} - \beta_i(\pi) h_\sigma^\pi]. \tag{22}$$

We have to show that these formulas together with $h_e^\pi = \delta_v(\pi)$ define h_σ^π consistently. This means that if we obtain σ from e by a sequence of operations T_i on S_N then what we obtain from this same sequence acting on the h_σ^π is independent of the particular sequence chosen. This follows from the relations

- (i) $T_i T_i = I$;
- (ii) $T_i T_j = T_j T_i$ when $|i - j| > 1$;
- (iii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

We should be more precise as to what exactly these T_i are. Let \mathcal{H}_0 be the set of all functions

$$h : \pi \rightarrow \text{function of } \xi,$$

and $\mathcal{H} = S_N \times \mathcal{H}_0$. Define $T_i^0 : \mathcal{H} \rightarrow \mathcal{H}_0$ by

$$T_i^0(\sigma, h) = h + (1 + S(\xi_{\sigma(i)}, \xi_{\sigma(i+1)})) [\alpha_i \cdot (h \circ T_i) - \beta_i \cdot h]. \tag{23}$$

Then define $T_i : \mathcal{H} \rightarrow \mathcal{H}$ by

$$T_i(\sigma, h) = (T_i\sigma, T_i^0(\sigma, h)). \tag{24}$$

(The same symbol T_i is used to denote these operators and operators on S_N . The context should make it clear which is meant.) It is the T_i defined by (24) that satisfy relations (i)–(iii). Once we have these relations it follows from the Coxeter presentation of S_N [11, Sect. 1.9] that there is a homomorphism from S_N to the group of mappings from \mathcal{H} to itself, such that each T_i acting on S_N goes to the corresponding T_i acting on \mathcal{H} . Because this is a homomorphism, if two products of transpositions in S_N are equal then so are the corresponding products of the T_i on \mathcal{H} , which is what was claimed.

Relations (i)–(iii) are the braid relations and are the consistency equations referred to in Sect. 1. Relations (i) and (ii) can be checked by hand. Since (iii) involves only three consecutive indices, it is enough to check the case $N = 3$. This was verified by a computer computation.

3.2 Satisfying the Initial Condition

We have to show that

$$\sum_{\sigma \in \mathcal{S}_N(B)} I^\pi(\sigma) = 0, \tag{25}$$

where $I^\pi(\sigma)$ is the σ -summand in (2) with $t = 0$.

In Sect. 2.2 we made a variable change, shrank contours, and found that all $I(\sigma)$ with $\sigma \in \mathcal{S}_N(B)$ are sums of lower-dimensional integrals $I_{B,(i,j)}(\sigma)$, one for each unordered pair (i, j) with $i, j \in B$, such that $\xi_i = \xi_j$ in the domain of integration.

To extend this to the $I^\pi(\sigma)$ we have to know first that no new poles arise from the factors h_σ^π . These would come from S -factors involving ξ_N , because it is only these that might introduce poles after the variable change (10). Using (22) to see which such factors could arise in the expressions for h_σ^π we start with $\sigma = e$ and do the following to get to our $\sigma \in \mathcal{S}_N(B)$: first, bring to the front and rearrange the $i \notin B \cup \{N\}$. This gives some factors $1 + S(\xi_i, \xi_j)$ with $i, j \neq N$. (Which exact factors occur depends on π as well as σ .) Then move N through the $i \in B$. Since at each step N was to the right of where it moves to, we get some factors $1 + S(\xi_i, \xi_N)$ with $i \in B$. Then rearrange the elements of B to reach σ , which introduces some factors $1 + S(\xi_i, \xi_j)$ with $i, j \in B$. Thus the only factors involving ξ_N are some $1 + S(\xi_i, \xi_N)$ with $i \in B$.³ If we multiply this factor by the $S(\xi_N, \xi_i)$ from the product in (7) for A_σ we get $S(\xi_N, \xi_i) + 1$. Thus, no new poles arise from h_σ^π with $\sigma \in \mathcal{S}_N(B)$.

We can now proceed as in Sect. 2.2, making the variable change and shrinking contours.

We say that σ, σ' are (i, j) -paired if they agree except for the positions of i and j , which are interchanged. We saw in the proof of Lemma 3 that if σ and σ' are (i, j) -paired then the integrands in $I_{B,(i,j)}(\sigma)$ and $I_{B,(i,j)}(\sigma')$ are negatives of each other. Therefore to prove (25) it suffices to show that if σ and σ' are (i, j) -paired then $h_\sigma^\pi = h_{\sigma'}^\pi$, when $\xi_i = \xi_j$. We show this by induction on $|\sigma^{-1}(i) - \sigma^{-1}(j)|$, and we may assume $\sigma^{-1}(j) > \sigma^{-1}(i)$.

If $\sigma^{-1}(j) = \sigma^{-1}(i) + 1$ we replace i by $\sigma^{-1}(i)$ in (22). Since $S(\xi_i, \xi_j) = -1$ when $\xi_i = \xi_j$, the statement holds then.

Suppose $m > 1$, that the statement holds when $\sigma^{-1}(j) = \sigma^{-1}(i) + m - 1$, and that $\sigma^{-1}(j) = \sigma^{-1}(i) + m$. For convenience of notation we assume that

$$\sigma = (1, 2, \dots, m, m + 1 \dots), \quad \sigma' = (m + 1, 2, \dots, m, 1 \dots).$$

(Thus σ and σ' are $(1, m + 1)$ -paired. Neither the actual labels nor the exact positions for the indices is relevant; only their relative positions is.) The dots represent other entries that are equal for σ and σ' .

We want to show that $h_\sigma^\pi = h_{\sigma'}^\pi$, when $\xi_1 = \xi_{m+1}$. We have

$$T_1\sigma = (2, 1, \dots, m, m + 1, \dots), \quad T_1\sigma' = (2, m + 1, \dots, m, 1, \dots).$$

Observe that $T_1\sigma$ and $T_1\sigma'$ are also $(1, m + 1)$ -paired but $(T_1\sigma)^{-1}(1) = 2$ while $(T_1\sigma')^{-1}(m + 1) = m + 1$, so the induction hypothesis holds. Thus $h_{T_1\sigma}^\pi = h_{T_1\sigma'}^\pi$ for all π when $\xi_1 = \xi_{m+1}$.

³For example, suppose $\sigma = (3\ 2\ 5\ 4\ 1)$, for which $B = \{4, 1\}$. Then the steps might be

$$(1\ 2\ 3\ 4\ 5) \rightarrow (1\ 3\ 2\ 4\ 5) \rightarrow (3\ 1\ 2\ 4\ 5) \rightarrow (3\ 2\ 1\ 4\ 5) \rightarrow (3\ 2\ 1\ 5\ 4) \rightarrow (3\ 2\ 5\ 1\ 4) \rightarrow (3\ 2\ 5\ 4\ 1).$$

The only S -factors involving ξ_5 come from steps four and five, and are $S(\xi_4, \xi_5)$ and $S(\xi_1, \xi_5)$.

Applying (22) with $i = 1$ and σ replaced by $T_1\sigma$ and by $T_1\sigma'$ gives the relations

$$\begin{aligned}
 h_\sigma^\pi &= h_{T_1\sigma}^\pi + (1 + S(\xi_2, \xi_1))[\alpha_1(\pi)h_{T_1\sigma}^{T_1\pi} - \beta_1(\pi)h_{T_1\sigma}^\pi], \\
 h_{\sigma'}^\pi &= h_{T_1\sigma'}^\pi + (1 + S(\xi_2, \xi_{m+1}))[\alpha_1(\pi)h_{T_1\sigma'}^{T_1\pi} - \beta_1(\pi)h_{T_1\sigma'}^\pi].
 \end{aligned}$$

When $\xi_1 = \xi_{m+1}$ the right sides are equal and therefore so are the left sides.

This completes the proof.

3.3 Formulas for h_σ^π

3.3.1 A General Formula

For $\sigma \in S_N$ we define $h_\sigma \in \mathcal{H}_0$ to be the function $\pi \rightarrow h_\sigma^\pi$. With this notation formula (22) may be written $h_{T_i\sigma} = T_i^0(\sigma, h_\sigma)$, where T_i^0 is given by (23). From this we get

$$h_{T_i T_j \sigma} = T_i^0(T_j \sigma, h_{T_j \sigma}) = T_i^0(T_j \sigma, T_j^0(\sigma, h_\sigma)) = T_i^0 T_j(\sigma, h_\sigma).$$

And in general,

$$h_{T_{j_1} T_{j_2} \dots T_{j_m} \sigma} = T_{j_1}^0 T_{j_2} \dots T_{j_m}(\sigma, h_\sigma).$$

If we first set $\sigma = e$ above and then choose the various T_i such that

$$T_{j_1} T_{j_2} \dots T_{j_m} e = \sigma, \tag{26}$$

we get, since $h_e^\pi = \delta_v(\pi)$,

$$h_\sigma^\pi = T_{j_1}^0 T_{j_2} \dots T_{j_m}(e, \delta_v). \tag{27}$$

For $\rho \in S_N$, and α and β functions of π , we define

$$\begin{aligned}
 U_i(\rho, \beta) &= 1 - [1 + S(\xi_{\rho(i)}, \xi_{\rho(i+1)})] \cdot \beta, \\
 V_i(\rho, \alpha) &= [1 + S(\xi_{\rho(i)}, \xi_{\rho(i+1)})] \cdot \alpha.
 \end{aligned}$$

These U_i and V_i come from the coefficients of h and $h \circ T_i$, respectively, in (23), and products of them are the coefficients of the δ -summands which result from applying the various T_j in (27). When we apply (23) consecutively to (27) we get a sum of products of the form $W_1 W_2 \dots W_m$, where each W is a U or V .

As an example, we find that $T_j^0 T_i(e, \delta_v)$ is a linear combination of $\delta_v, \delta_{v \circ T_i}, \delta_{v \circ T_j}$, and $\delta_{v \circ T_i \circ T_j}$. The coefficient of δ_v is $U_j(T_i e, \beta_j) U_i(e, \beta_i)$, the coefficient of $\delta_{v \circ T_i}$ is $U_j(T_i e, \beta_j) V_i(e, \alpha_i)$, the coefficient of $\delta_{v \circ T_j}$ is $V_j(T_i e, \alpha_j) U_i(e, \beta_i \circ T_j)$, and the coefficient of $\delta_{v \circ T_i \circ T_j}$ is $V_j(T_i e, \alpha_j) V_i(e, \alpha_i \circ T_j)$. These are the four W -products.

Passing to the general case, we observe that in every factor W the original permutation e has been composed with all earlier (rightward) T_j in (27). Every factor V_i comes from the corresponding $h \circ T_i$ summand in (23). This changes the α or β in each earlier factor W by composing it with these T_i . Otherwise said, each α or β in a factor W is a composition with the later T_i . The W -product is determined by these (ordered) i , and we denote the sequence of them by $\mathcal{I} = \{i_1, \dots, i_n\}$. The resulting δ -term is $\delta_{v \circ T_{i_1} \circ \dots \circ T_{i_n}}$.

In general we get a linear combination of δ -terms of the form $\delta_{v \circ T_{i_1} \circ \dots \circ T_{i_n}}$, where $\mathcal{I} = \{i_1, \dots, i_n\} \subset \mathcal{I}_\sigma$ and $\mathcal{I}_\sigma = \{j_1, \dots, j_m\}$ is the sequence in (26). Each i_k is a j_ℓ while the

other j_ℓ appear between some consecutive i_{k-1} and i_k . (For this we set $i_0 = 0, i_{n+1} = \infty$.) From the above remarks we see that the coefficient of $\delta_{v \circ T_{i_1} \circ \dots \circ T_{i_n}}$ is

$$W_{\mathcal{I}} = \prod_{\ell=1}^m W_{\mathcal{I}, \ell} \tag{28}$$

where

$$W_{\mathcal{I}, \ell} = \begin{cases} V_{i_k}(T_{j_{\ell-1}} \cdots T_{j_m} e, \alpha_{i_k} \circ T_{i_{k-1}} \circ \dots \circ T_{i_1}) & \text{if } j_\ell = i_k, \\ U_{j_\ell}(T_{j_{\ell-1}} \cdots T_{j_m} e, \beta_{j_\ell} \circ T_{i_{k-1}} \circ \dots \circ T_{i_1}) & \text{if } j_\ell \in (i_{k-1}, i_k). \end{cases} \tag{29}$$

This gives

$$h_\sigma^\pi = \sum_{\mathcal{I} \subset \mathcal{I}_\sigma} W_{\mathcal{I}}(\pi) \delta_{v \circ T_{i_1} \circ \dots \circ T_{i_n}}(\pi), \tag{30}$$

where $\mathcal{I} = \{i_1, \dots, i_n\}$.

The sequence \mathcal{I}_σ in (26) is not unique, so we do not yet have an explicit formula for h_σ^π . To find a particular sequence let us see how to get from σ to e by a sequence of T_i . We may do this by first bringing 1 to slot 1 by a sequence of transpositions, then bringing 2 to slot 2 by a sequence of transpositions, etc. We first bring 1 to slot 1 by the product

$$T_1 T_2 \cdots T_{\sigma^{-1}(1)-1}.$$

Suppose we have brought $1, \dots, k-1$ to their slots. Then k itself has been moved to the right by

$$\iota(k) := \#\{j : j < k, \sigma^{-1}(j) > \sigma^{-1}(k)\} = \text{the number of inversions of the form } (k, j),$$

because bringing each such j to its slot has moved k one slot to the right. We then bring i to its slot by the product

$$T_k \cdots T_{\sigma^{-1}(k)+\iota(k)-1}.$$

Thus

$$\prod_{k=1}^{N-1} T_i \cdots T_{\sigma^{-1}(k)+\iota(k)-1} \sigma = e,$$

the factor with the larger k being to the left. This gives

$$\sigma = \prod_{k=1}^{N-1} T_{\sigma^{-1}(k)+\iota(k)-1} \cdots T_k e, \tag{31}$$

the factor with the larger k being to the right. With this representation we have

$$\mathcal{I}_\sigma = \bigcup_{k=1}^{N-1} (\ell_k, \dots, k), \tag{32}$$

where $\ell_k = \sigma^{-1}(k) + \iota(k) - 1$, the interval with the larger k being to the right.

Together with (30) this gives an explicit, albeit complicated, formula for h_σ^π .

3.3.2 One Second-Class Particle

Because all the π now will have 1 in a single position and 2 in the others, most of the α and β terms in (29) will be zero when applied to π . So most of the $W_{\mathcal{I}_\ell}$ will be 0 (when a V_j) or 1 (when a U_j). It is possible systematically to determine which $W_{\mathcal{I}}$ will be nonzero when \mathcal{I}_σ is given, and that makes it practical write down formulas for the h_σ^π in specific cases. We shall not describe the procedure but state two results that use (32), which comes from representation (31).

For $\nu^{-1}(1) = 1$ (the second-class particle initially in position 1) and $\pi^{-1}(1) = j$ (the second-class particle ending in position j) $h_\sigma^\pi = 0$ when $\sigma^{-1}(1) < j$ and

$$h_\sigma^\pi = (p - qS(\xi_1, \xi_{\sigma(j)}))q(1 + S(\xi_1, \xi_{\sigma(j-1)})) \cdots q(1 + S(\xi_1, \xi_{\sigma(1)})),$$

when $\sigma^{-1}(1) \geq j$. (When $\sigma^{-1}(1) = j$ the factor on the left equals 1.)

For $\nu^{-1}(1) = 2$ and $\pi^{-1}(1) = j$ the formula is more complicated. When $\sigma^{-1}(1) \geq j$ and $\sigma^{-1}(2) + \iota(2) \geq j$ it is

$$h_\sigma^\pi = \sum_{i=1}^{j-1} \left[\prod_{k=1}^{i-1} q(1 + S(\xi_2, \xi_{\sigma(k)})) \cdot (p - qS(\xi_2, \xi_{\sigma(i)})) \right. \\ \left. \times (q - pS(\xi_1, \xi_{\sigma(i)})) \cdot \prod_{k=i+1}^{j-1} q(1 + S(\xi_1, \xi_{\sigma(k)})) \cdot (p - qS(\xi_1, \xi_{\sigma(j)})) \right] \\ + \prod_{k=1}^{j-1} q(1 + S(\xi_2, \xi_{\sigma(k)})) \cdot (p - qS(\xi_2, \xi_{\sigma(j)})) \cdot p(1 + S(\xi_1, \xi_{\sigma(j)})).$$

We should point out that in any given case (31) may not be the best representation for computation. For example, take $\nu^{-1}(1) = \pi^{-1}(1) = 3$ and $\sigma = (4\ 3\ 2\ 1)$. Using the representation

$$\sigma = T_3T_2T_1T_3T_2T_3e \tag{33}$$

from (31), there are five nonzero summands in (30). Therefore h_σ^π is given as a sum of five products. But if instead we use the representation

$$\sigma = T_1T_2T_1T_3T_2T_1e, \tag{34}$$

there are only two nonzero summands and we get the relatively simple formula

$$h_\sigma^\pi = q(1 + S(\xi_2, \xi_4))(q - pS(\xi_2, \xi_3))q(1 + S(\xi_1, \xi_3)) \\ + (q - pS(\xi_2, \xi_4))(p - qS(\xi_1, \xi_4))(q - pS(\xi_1, \xi_3)).$$

For another example take $\sigma = (4\ 3\ 2\ 1)$ as before but $\nu^{-1}(1) = 3, \pi^{-1}(1) = 4$. Using (33) we again get five nonzero summands in (30). But using (34) we get only one, and the much simpler answer

$$h_\sigma^\pi = q(1 + S(\xi_1, \xi_4))(q - pS(\xi_1, \xi_3)).$$

So although the representation (31) leads to a general procedure for computing h_σ^π , in any particular case there may well be a better representation for computation.

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