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The asymmetric simple exclusion process with an open boundary

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We consider the asymmetric simple exclusion process confined to the nonnegative integers with an open boundary at 0. The point 0 is connected to a reservoir where particles are injected and ejected at prescribed rates subject to the exclusion rule. We derive formulas for the transition probability as a function of time from states where initially there are m particles to states where there are n particles. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4822418]

I. INTRODUCTION

In previous work[10] the authors considered the asymmetric simple exclusion process (ASEP) where particles are confined to the nonnegative integers $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. Each particle waits exponential time, and then with probability p it moves one step to the right if the site is unoccupied, otherwise it does not move; and with probability $q = 1 - p$ a particle not at 0 moves one step to the left if the site is unoccupied, otherwise it does not move. For n-particle ASEP a possible cofiguration is

$$\mathbf{x} = \{x_1, \ldots, x_n\}, \quad (0 \le x_1 < \cdots < x_n).$$

The x_i are the occupied sites. We denote by $\mathcal{X}_n$ the set of possible configurations for n-particle ASEP, and by $\mathfrak{p}_n(\mathbf{x}, \mathbf{y}; t)$ the probability that at time t the system is in configuration $\mathbf{x}$ given that initially it was in configuration $\mathbf{y}$. (We shall drop the subscript "n" when it is understood.)

In Ref. 10 a formula was found for this probability. It was the sum over the Weyl group $\mathbb{B}_n$ of multiple integrals. (For ASEP on $\mathbb{Z}$ it was a sum over the permutation group $\mathbb{S}_n$.[8,9])

Here we consider the ASEP on $\mathbb{Z}^+$ with an open boundary at zero. The stationary measure for ASEP on the finite lattice $[1, L]$ or on the semi-infinite lattice $\mathbb{Z}^+$ with boundaries connected to reservoirs has been the subject of much research starting with Spohn[7] and Derrida *et al.*[1] We refer the reader to the recent work of Sasamoto and Williams[6] for an up-to-date account of these developments. Here we consider the time-dependent properties of ASEP on $\mathbb{Z}^+$ with an open boundary. Specifically, the point 0 is connected to a reservoir where a particle is *injected* into site 0 from the reservoir at a rate α, assuming that the site 0 is empty, and a particle at site 0 is *ejected* into the reservoir at a rate β. Now the number of particles is not conserved and for ASEP with open boundary the configuration $\mathbf{x}$ may lie in $\mathcal{X}_n$ while $\mathbf{y}$ may lie in $\mathcal{X}_m$ with $m \neq n$.

We find an infinite tri-diagonal matrix with operator entries in which the Laplace transforms of the probabilities can be read off from the entries of the inverse matrix. When either $\alpha = 0$ or $\beta = 0$ the matrix is triangular and so the inverse can be computed more explicitly. The result is obtained by solving a system of differential equations for the probabilities. The final formulas involve inverses of operators with kernels the Laplace transforms of certain $\mathfrak{p}(\mathbf{x}, \mathbf{y}; t)$ obtained in Ref. 10.

There are two special cases in which the results are more explicit. For TASEP with $p = 1$, the inverse operator is computable in terms of $\mathfrak{p}(\mathbf{x}, \mathbf{y}; t)$ itself, and the probabilities are given in terms of certain determinants. For SSEP ($p = q$) and general α and β we find formulas analogous to the ones described above for the probability that sites $x_1, \ldots, x_n$ are occupied. This is for infinite systems as well as finite ones.

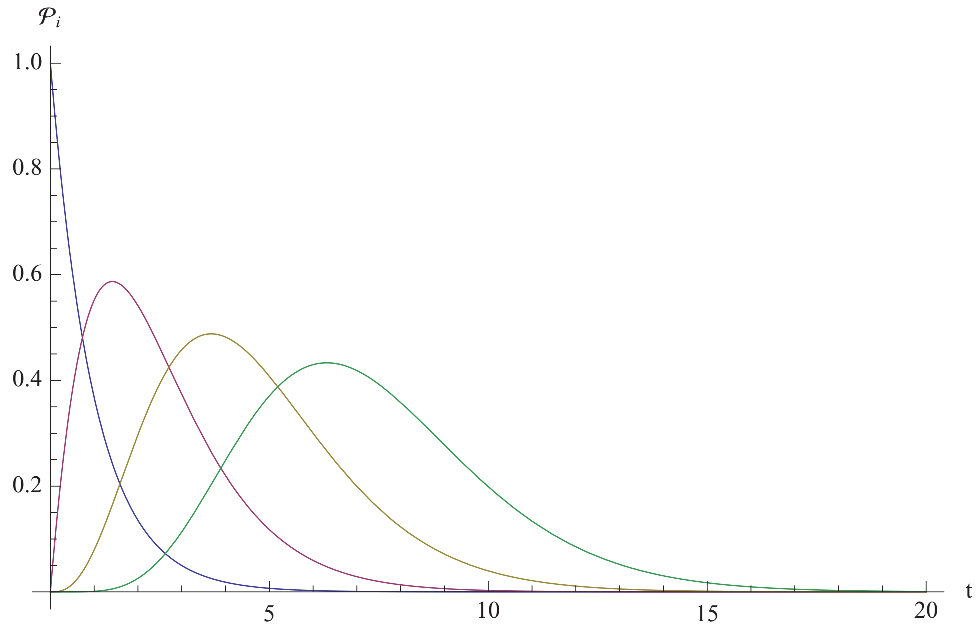

FIG. 1. Plotted are the TASEP probabilities $\mathcal{P}_n(t)$ for $n = 0, 1, 2, 3$ with $\alpha = 1$. Increasing n moves the maximum to the right.

We state the formulas for $\mathfrak{p}(\mathbf{x}, \mathbf{y}; t)$ in the Appendix.

II. STATEMENT OF RESULTS

We denote by $\mathcal{E}_n$ the Lebesgue space $L^1(\mathcal{X}_n)$. From the fact that

$$\sum_{\mathbf{x}\in\mathcal{X}_n} \mathfrak{p}(\mathbf{x}, \mathbf{y}; t) = 1 \tag{1}$$

for each $\mathbf{y}$, it follows that the operator on $\mathcal{E}_n$ with this kernel is bounded with norm one. We denote the Laplace transform of $\mathfrak{p}(\mathbf{x}, \mathbf{y}; t)$ by $\widehat{\mathfrak{p}}(\mathbf{x}, \mathbf{y}; s)$:

$$\widehat{\mathfrak{p}}(\mathbf{x}, \mathbf{y}; s) = \int_0^\infty \mathfrak{p}(\mathbf{x}, \mathbf{y}; t)\, e^{-st}\, dt.$$

The operator with this kernel is bounded on $\mathcal{E}_n$ with norm at most $(\mathrm{Re}\, s)^{-1}$. We denote it by $L_n(s)$. (When $n = 0$ we interpret $\mathfrak{p}(\mathbf{x}, \mathbf{y}; t)$ as 1, and so $L_0(s)$ is multiplication by s^{-1}.) In the results stated below it is tacitly assumed that Re s is sufficiently large.

Now for ASEP with open boundary at zero, we define $P_n(\mathbf{x}; t)$ to be the probability that the system is in configuration $\mathbf{x} \in \mathcal{X}_n$ at time t. We shall usually drop the "$\mathbf{x}$" in the notation, and do not specify an initial configuration. (In the examples, we clearly state the initial conditions.) We denote its Laplace transform by $\widehat{P}_n(s)$.

We define vector functions

$$\widehat{P}(s) = (\widehat{P}_n(s))_{n\geq 0}, \qquad P(0) = (P_n(0))_{n\geq 0},$$

belonging to the direct sum $\sum_{n=0}^\infty \mathcal{E}_n$.

We define operators $A_n : \mathcal{E}_{n-1} \to \mathcal{E}_n$ and $B_n : \mathcal{E}_{n+1} \to \mathcal{E}_n$ by

$$(A_n F)(x_1, \ldots, x_n) = \delta(x_1)\, F(x_2, \ldots, x_n), \tag{2}$$

$$(B_n F)(x_1, \ldots, x_n) = (1 - \delta(x_1))\, F(0, x_1, \ldots, x_n). \tag{3}$$

(When $n = 0$ we interpret $\delta(x_1)$ as zero. In particular $A_0 = 0$ and $B_0 F = F(0)$.)

Then we define matrices $\delta, L(s), A, B$, with operator entries, acting on $\sum_{n=0}^{\infty}\mathcal{E}_n$. The first is diagonal with (n, n)-entry multiplication by $\delta(x_1)$, the second is diagonal with n, n-entry $L_n(s)$, the third is subdiagonal (one diagonal below the main diagonal) with $n, n-1$-entry A_n, and the last is superdiagonal (one diagonal above the main diagonal) with $n, n+1$-entry B_n.

Theorem 1. With this notation we have

$$\widehat{P}(s-\alpha) = \Big(I - L(s)\,((\alpha-\beta)\,\delta + \alpha\,A + \beta\,B)\Big)^{-1} L(s)\,P(0). \tag{4}$$

There are expressions for the entries of the inverse operator as infinite series of products. But when either $\beta = 0$ or $\alpha = 0$ the operator has only one subdiagonal or one superdiagonal and each entry of the inverse is a single product. We state the results as recursion formulas. Define

$$M_n(s) = (I - (\alpha-\beta)\,L_n(s)\,\delta)^{-1}. \tag{5}$$

Corollary 1.1. Suppose $\beta = 0$ and that initially there are k particles at $\mathbf{y} \in \mathcal{X}_k$. Then

$$\widehat{P}_k(s-\alpha) = M_k(s)\,L_k(s)\,\delta_{\mathbf{y}},$$

(interpreted as s^{-1} when $k = 0$) and when $n > k$

$$\widehat{P}_n(s-\alpha) = \alpha\,M_n(s)\,L_n(s)\,A_n\,\widehat{P}_{n-1}(s-\alpha).$$

Corollary 1.2. Suppose $\alpha = 0$ and that initially there are k particles at $\mathbf{y} \in \mathcal{X}_k$. Then

$$\widehat{P}_k(s) = M_k(s)\,L_k(s)\,\delta_{\mathbf{y}},$$

and when $n < k$

$$\widehat{P}_n(s) = \beta\,M_n(s)\,L_n(s)\,B_n\,\widehat{P}_{n+1}(s).$$

In connection with the corollaries we show the following.

Remark 1.1. The operators appearing in the inverses can be replaced by lower-dimensional ones. This will be useful for computation. Define

$$\mathcal{X}_n^+ = \{\{x_1, \ldots, x_n\} \in \mathcal{X}_n : x_1 > 0\}, \quad \mathcal{E}_n^+ = L^1(\mathcal{X}_n^+),$$

and then operators:
$L_{n-1}^0(s) : \mathcal{E}_{n-1}^+ \to \mathcal{E}_{n-1}^+$ with kernel $\widehat{\mathfrak{p}}((0, \mathbf{x}), (0, \mathbf{y}); s)$,
$L_{n,n-1}^0(s) : \mathcal{E}_{n-1}^+ \to \mathcal{E}_n$ with kernel $\widehat{\mathfrak{p}}(\mathbf{x}, (0, \mathbf{y}); s)$,
$L_{n-1,n}^0(s) : \mathcal{E}_n \to \mathcal{E}_{n-1}^+$ with kernel $\widehat{\mathfrak{p}}((0, \mathbf{x}), \mathbf{y}; s)$.
(a) The operator $M_n(s)\,L_n(s)\,A_n : \mathcal{E}_{n-1} \to \mathcal{E}_n$ in Corollary 1.1 is equal to

$$L_{n,n-1}^0(s)\,(I - (\alpha-\beta)\,L_{n-1}^0(s))^{-1}\,R_{n-1},$$

(The I here is the identity operator on $\mathcal{E}_{n-1}^+$ while the I in (5) is the identity operator on $\mathcal{E}_n$ where $R_{n-1} : \mathcal{E}_{n-1} \to \mathcal{E}_{n-1}^+$ is the restriction operator).
(b) The operator $M_k(s)\,L_k(s) : \mathcal{E}_k \to \mathcal{E}_k$ in Corollaries 1.1 and 1.2 is equal to

$$L_k(s) + (\alpha-\beta)\,L_{k,k-1}^0(s)\,(I - (\alpha-\beta)\,L_{k-1}^0(s))^{-1}\,L_{k-1,k}^0(s).$$

Remark 1.2. In the special case of TASEP when $p = 1$ we have the simplification $(I - \alpha\,L_n^0(s))^{-1} = I + \alpha\,L_n^0(s-\alpha)$.

In the case of SSEP ($p = q$), even for infinitly many particles, there are formulas for correlations that are no more complicated when both α and β are nonzero. For $\mathbf{x} = \{x_1, \ldots, x_n\} \in \mathcal{X}_n$ we define $\Psi_n(\mathbf{x};\, t)$ to be the probability that sites $x_1, \ldots, x_n$ are occupied at time t. We denote its Laplace transform by $\widehat{\Psi}_n(s) = \widehat{\Psi}_n(\mathbf{x}; s)$ and introduce the vector functions

$$\widehat{\Psi}(s) = (\widehat{\Psi}_n(s))_{n\geq 0}, \quad \Psi(0) = (\Psi_n(0))_{n\geq 0}.$$

(We define $\Psi_0(t) = 1$, and so $\widehat{\Psi}_0(s) = s^{-1}$.)

Let the operators $L_n(s)$ and A_n, and the matrices $\delta, L(s), A$ with operator entries be the same as before. Now we define $\mathcal{E}_n = L^\infty(\mathcal{X}_n)$, and observe that by (1) and the symmetry of the kernel we have $\sum_{\mathbf{y}\in\mathcal{X}_n} \mathfrak{p}(\mathbf{x},\mathbf{y};t) = 1$ for each $\mathbf{x}\in\mathcal{X}_n$. It follows that $L_n(s)$ is a bounded operator on this $\mathcal{E}_n$ with norm at most $(\mathrm{Re}\, s)^{-1}$.

Theorem 2. We have

$$\widehat{\Psi}(s) = \Big(I + L(s)\,((\alpha+\beta)\,\delta - \alpha\,A)\Big)^{-1} L(s)\,\Psi(0).$$

We now set

$$M_n(s) = (I + (\alpha+\beta)\,L_n(s)\,\delta)^{-1}.$$

Corollary 2.1. For $n > 0$,

$$\widehat{\Psi}_n(s) = \alpha\, M_n(s) L_n(s) A_n\, \widehat{\Psi}_{n-1}(s) + M_n(s) L_n(s)\, \Psi_n(0).$$

Corollary 2.2. In the case of Bernoulli initial condition with density ρ,

$$\widehat{\Psi}_n(s) = \alpha\, M_n(s) L_n(s) A_n\, \widehat{\Psi}_{n-1}(s) + s^{-1}\,\rho^n\, M_n(s)\,1,$$

where "1" is the constant function on $\mathcal{X}_n$.

Corollary 2.3. When initially no sites are occupied

$$\widehat{\Psi}_n(s) = \alpha\, M_n(s) L_n(s) A_n\, \widehat{\Psi}_{n-1}(s).$$

The analogue of Remark 1.1 holds here.

From the abstract formulas, Theorems 1 and 2 and their corollaries, we derive some concrete results.

Suppose, in ASEP, that at time zero there is a single particle at y. From Corollaries 1.1 and 1.2 we show:

$\alpha = \mathbf{0}$: When $p > q$, with probability

$$1 - \frac{\beta\,(q^{-1}-1)^{-y}}{p-q+\beta},$$

the particle is never ejected. When $p \le q$, with probability one the particle will eventually be ejected. The expected value of the time at which this occurs is infinite when $p = q$, and when $p < q$ it is

$$\frac{y + q/\beta}{q-p}.$$

$\beta = \mathbf{0}$: With probability one a second particle will eventually be injected. The expected value of the time at which this occurs is

$$\frac{1}{\alpha} + \frac{\xi_+(\alpha)^{-y}}{q\,(\xi_+(\alpha)-1) - \alpha},$$

where

$$\xi_+(\alpha) = \frac{1}{2q}\,(\alpha + 1 + \sqrt{(\alpha+1)^2 - 4pq}\,).$$

Combining Remark 1.2 with the determinant formula (A2) for $\mathfrak{p}(\mathbf{x},\mathbf{y};t)$ in TASEP it is practical to compute some exact results. Denote by $\mathcal{P}_n(t)$ the probability that starting with no particles at time 0 there are exactly n particles at time t. Then

$$\mathcal{P}_0(t) = e^{-\alpha t},$$

$$\mathcal{P}_1(t) = \frac{\alpha}{1-\alpha}\left(t - \frac{\alpha}{1-\alpha}\right)e^{-\alpha t} + \frac{\alpha^2}{(1-\alpha)^2}e^{-t},$$

$$\mathcal{P}_2(t) = \left(\frac{\alpha^2}{2(1-\alpha)^2}t^2 - \frac{\alpha^2}{(1-\alpha)^3}t + \frac{\alpha^2}{(1-\alpha)^2}\right)e^{-\alpha t}$$

$$+\left(\frac{\alpha^2}{2(1-\alpha)^2}t^2 - \frac{\alpha(1-3\alpha+\alpha^2)}{(1-\alpha)^3} + \frac{1-2\alpha}{(1-\alpha)^2}\right)e^{-t} - e^{-(1+\alpha)t}.$$

These are for $\alpha \neq 1$. When $\alpha = 1$ there is no singularity; we take a limit and the formulas simplify (see Fig. 1.).

For SSEP we obtain the following consequence of Corollary 2.2. Define

$$\Delta N(t) = \sum_{x\geq 0} (\eta_x(t) - \eta_x(0)),$$

the net number of particles that have entered the system at time t. (Which may be negative.) With Bernoulli initial condition when $\langle \eta_x(0)\rangle = \rho$, we show for the expected value that

$$\langle \Delta N(t)\rangle \sim \sqrt{\frac{2}{\pi}}\,\frac{\alpha-(\alpha+\beta)\rho}{\alpha+\beta}\,t^{1/2} \quad \text{as } t\to\infty.$$

(This result was obtained in Ref. 3 in the case $\beta = 0$ and $\rho = 0$.) By a laborious computation (not included) we can show that the second moment of $\Delta N(t)$ is finite.

In the special case where initially there are no occupied sites, we use Corollary 2.3 and present a not completely rigorous (to say the least) argument that

$$\langle \Delta N(t)^2\rangle \sim \frac{2}{\pi}\frac{\alpha^2}{(\alpha+\beta)^2}\,t \quad \text{as } t\to\infty.$$

Combining this with the first moment asymptotics when $\rho = 0$ we conclude that the variance of $\Delta N(t)/t^{1/2}$ tends to zero as $t \to \infty$. (As predicted in Ref. 3 in the case $\beta = 0$.) The derivation is quite long, but the result with the precise constants came out so nicely in the end that we could not resist including it.

III. PROOFS OF THE RESULTS

A. ASEP

Proof of Theorem 1: The probability $\mathfrak{p}(\mathbf{x}, \mathbf{y};\, t)$ for n-particle ASEP on $\mathbb{Z}^+$ is the solution of the differential equation

$$\frac{d}{dt}\,u(X;t) = \sum_{i=1}^{N}\Big[p\,u(x_i-1)\,(1-\delta(x_i-x_{i-1}-1)) + q\,u(x_i+1)\,(1-\delta(x_{i+1}-x_i-1))$$

$$-p\,u(x_i)\,(1-\delta(x_{i+1}-x_i-1)) - q\,u(x_i)\,(1-\delta(x_i-x_{i-1}-1))\Big] + [q\,u(x_1) - p\,u(x_1-1)]\,\delta(x_1)$$

that satisfies the initial condition

$$u(\mathbf{x};\,0) = \delta_{\mathbf{y}}(\mathbf{x}).$$

(In the ith summand entry i is displayed and entry j is x_j when $j \neq i$.)

If we denote by $\mathcal{Q}_n$ the operator given by the right side of the equation, then $\mathfrak{p}(\mathbf{x}, \mathbf{y};\, t)$ is the kernel of $e^{t\mathcal{Q}_n}$. Thus the equation is

$$\frac{d\mathfrak{p}_n}{dt} = \mathcal{Q}_n\,\mathfrak{p}_n(\mathbf{x};\,t),$$

where we have written $\mathfrak{p}_n(\mathbf{x};\, t)$ for $\mathfrak{p}(\mathbf{x}, \mathbf{y};\, t)$.

For open ASEP we write the corresponding probability as $P_n(\mathbf{x};t)$ (and do not specify any initial condition). The equation for P_n is

$$\frac{dP_n}{dt}(x_1,\ldots,x_n\,:\,t)=\mathcal{Q}_n\,P_n(x_1,\ldots,x_n\,:\,t)$$

$$+\alpha\,\delta(x_1)\,P_{n-1}(x_2,\ldots,x_n\,:\,t)-\alpha\,(1-\delta(x_1))\,P_n(x_1,\ldots,x_n\,:\,t)$$

$$-\beta\,\delta(x_1)\,P_n(x_1,\ldots,x_n\,:\,t)+\beta(1-\delta(x_1))\,P_{n+1}(0,x_1,\ldots,x_n\,:\,t).$$

Define $\Phi_n(\mathbf{x};t)=e^{\alpha t}P_n(\mathbf{x};t)$. In terms of the operators A_n and B_n defined in (2) and (3), and $\delta = \delta(x_1)$, the equation for Φ_n becomes

$$\frac{d\Phi_n(t)}{dt}=\mathcal{Q}_n\,\Phi_n+(\alpha-\beta)\,\delta\,\Phi_n+\alpha\,A_n\,\Phi_{n-1}+\beta\,B_n\,\Phi_{n+1}. \tag{6}$$

The equation and initial condition are satisfied if

$$\Phi_n(t)=\int_0^t e^{(t-u)\mathcal{Q}_n}\Big((\alpha-\beta)\,\delta\,\Phi_n(u)+\alpha\,A_n\,\Phi_{n-1}(u)+\beta\,B_n\,\Phi_{n+1}(u)\Big)\,du+e^{t\mathcal{Q}_n}\,\Phi_n(0).$$

We use the fact that the Laplace transform of a convolution is the product of the Laplace transforms. Recall that the kernel of $L_n(s)$ is $\widehat{\mathfrak{p}}(\mathbf{x},\mathbf{y};s)$, which is the Laplace transform of the kernel of $e^{t\mathcal{Q}_n}$. In other words, the Laplace transform of the operator $e^{t\mathcal{Q}_n}$ is $L_n(s)$. So taking Laplace transforms in the last equation gives

$$\widehat{\Phi}_n(s)=L_n(s)\Big((\alpha-\beta)\,\delta\,\widehat{\Phi}_n(s)+\alpha\,A_n\,\widehat{\Phi}_{n-1}(s)+\beta\,B_n\,\widehat{\Phi}_{n+1}(s)\Big)+L_n(s)\,\Phi_n(0).$$

If we now introduce the vector functions

$$\widehat{\Phi}(s)=(\widehat{\Phi}_n(s))_{n\ge 0},\quad \Phi(0)=(\Phi_n(0))_{n\ge 0},$$

and the operator matrices $\delta, L(s), A, B$ defined earlier we see that the system may be written as

$$\widehat{\Phi}(s)=L(s)\Big((\alpha-\beta)\,\delta+\alpha\,A+\beta\,B\Big)\,\widehat{\Phi}(s)+L(s)\,\Phi(0).$$

Since $\widehat{\Phi}(s)=\widehat{P}(s-\alpha)$, this gives the statement of Theorem 1.

Corollaries 1.1 and 1.2: We write the operator inverse in (4) as

$$\Big(I-(M(s)\,L(s)\,(\alpha\,A+\beta\,B))\Big)^{-1}\,M(s)\,L(s),$$

where $M(s)$ is the diagonal matrix with operator entries $M_n(s)$. When $\beta = 0$ this equals

$$(I-\alpha\,M(s)\,L(s)\,A)^{-1}\,M(s)\,L(s). \tag{7}$$

The operator matrix $M(s)\,L(s)\,A$ consists of one subdiagonal, with m, $m-1$-entry $M_m(s)\,L_m(s)\,A_m$. Therefore, the n, n'-entry of the inverse ($n'\le n$) is

$$\alpha^{n-n'}\,M_n(s)L_n(s)A_n\cdots M_{n'+1}(s)L_{n'+1}(s)A_{n'+1},$$

where for $n = n'$ this equals I. So the n, n'-entry of (7) is

$$\alpha^{n-n'}\,M_n(s)L_n(s)A_n\cdots M_{n'+1}(s)L_{n'+1}(s)A_{n'+1}M_{n'}(s)L_{n'}(s).$$

Thus,

$$\widehat{P}_n(s-\alpha)=\sum_{n'\le n}\alpha^{n-n'}\,M_n(s)L_n(s)A_n\cdots M_{n'+1}(s)L_{n'+1}(s)A_{n'+1}\,M_{n'}(s)\,L_{n'}(s)\,P_{n'}(0), \tag{8}$$

When $\alpha = 0$ the matrix $M(s)\,L(s)\,B$ consists of one superdiagonal, and we obtain similarly

$$\widehat{P}_n(s)=\sum_{n'\ge n}\beta^{n'-n}\,M_n(s)L_n(s)B_n\cdots M_{n'-1}(s)L_{n'-1}(s)B_{n'-1}\,M_{n'}(s)\,L_{n'}(s)\,P_{n'}(0).$$

If initially there are k particles at $\mathbf{y} \in \mathcal{X}_k$ then in both cases $P_{n'}(0)$ is nonzero only for $n' = k$. The formulas become

$$\widehat{P}_n(s-\alpha) = \alpha^{n-k}\, M_n(s) L_n(s) A_n \cdots M_{k+1}(s) L_{k+1}(s) A_{k+1}\, M_k(s)\, L_k(s)\, \delta_y,$$

$$\widehat{P}_n(s) = \beta^{k-n}\, M_n(s) L_n(s) B_n \cdots M_{k-1}(s) L_{k-1}(s) B_{k-1}\, M_k(s)\, L_k(s)\, \delta_y,$$

and the corollaries follow.

Remark 1.1: For (a) we use the fact that because A_n has the factor δ,

$$M_n(s) L_n(s) A_n = (I - (\alpha-\beta)\, L_n(s)\, \delta)^{-1} L_n(s)\, \delta\, A_n = L_n(s)\, \delta\, (I - (\alpha-\beta)\, \delta\, L_n(s)\, \delta)^{-1}\, A_n.$$

Since $(A_n f)(x_1, \mathbf{x}) = \delta(x_1)\, (R_{n-1} f)(\mathbf{x})$ and $\mathbf{x} \in \mathcal{E}^+_{n-1}$, we obtain statement (a) in different notation. For (b) we use

$$\begin{aligned} M_k(s) &= (I - (\alpha-\beta)\, L_k(s)\delta)^{-1} = I + (\alpha-\beta)\, L_k(s)\delta\, (I - (\alpha-\beta)\, L_k(s)\delta)^{-1} \\ &= I + (\alpha-\beta)\, L_k(s)\delta\, (I - (\alpha-\beta)\, \delta L_k(s)\delta)^{-1}. \end{aligned} \tag{9}$$

Thus,

$$M_k(s) L_k(s) = L_k(s) + (\alpha-\beta)\, L_k(s)\delta\, (I - (\alpha-\beta)\, \delta L_k(s)\delta)^{-1} \delta L_k(s),$$

and statement (b) follows.

Initially a single particle: Let $\mathcal{P}_1(y;t) = \sum_{x\ge 0} P_1(x, y; t)$ denote the probability that, starting with one particle at y, we still have one particle at time t. Denote its Laplace transform by $\widehat{\mathcal{P}}_1(y;s)$.

We begin with the case $\beta = 0$, so $\mathcal{P}_1(y;t)$ is the probability that no new particle has been injected by time t.

By Corollary 1.1 the Laplace transform $\widehat{P}_1(x, y; s-\alpha)$ is equal to $(M_1(s)L_1(s)\delta_y)(x)$. By Remark 1.1(b) this equals

$$\widehat{\mathfrak{p}}(x, y; s) + \alpha\, \widehat{\mathfrak{p}}(x, 0; s)\, (1 - \alpha\, \widehat{\mathfrak{p}}(0, 0; s))^{-1}\, \widehat{\mathfrak{p}}(0, y; s).$$

Then using $\sum_{x\ge 0} \widehat{\mathfrak{p}}(x, y; s) = s^{-1}$ we obtain

$$\widehat{\mathcal{P}}_1(y; s-\alpha) = \frac{1}{s}\Big[1 + \frac{\alpha\, \widehat{\mathfrak{p}}(0, y; s)}{1 - \alpha\, \widehat{\mathfrak{p}}(0, 0; s)}\Big].$$

From formula (A1) for $\mathfrak{p}(x, y; t)$ in the case $n = 1$ we compute that

$$\widehat{\mathfrak{p}}(0, y; s) = \frac{1}{q}\, \frac{\xi_+(s)^{-y}}{\xi_+(s) - 1}, \tag{10}$$

where

$$\xi_+(s) = \frac{1}{2q}\, (s + 1 + \sqrt{(s+1)^2 - 4pq});$$

this is the solution of $\varepsilon(\xi) = s$ with positive square root when $s > 0$. Thus

$$\widehat{\mathcal{P}}_1(y; s-\alpha) = \frac{1}{s}\left[1 + \frac{\alpha\, \xi_+(s)^{-y}}{q(\xi_+(s) - 1) - \alpha}\right]. \tag{11}$$

The denominator in the brackets is nonzero for $s = \alpha$ (and positive for $s > \alpha$), from which we conclude that

$$\int_0^\infty \mathcal{P}_1(y;t)\, dt = \widehat{\mathcal{P}}_1(y; 0) = \frac{1}{\alpha} + \frac{\xi_+(\alpha)^{-y}}{q(\xi_+(\alpha) - 1) - \alpha}.$$

It follows that with probability one a second particle will eventually be injected, since $\mathcal{P}(y;t) \to 0$ as $t \to \infty$, and the integral is the expected time at which it occurs. (If $T(y)$ denotes the time when a second particle is injected, then $\text{Prob}\,(T(y) > t) = \mathcal{P}_1(y;t)$, from which the statement follows.)

This was for $\beta = 0$. For $\alpha = 0$, $\mathcal{P}_1(y;t)$ is the probability that the particle has not been ejected by time t. We use Corollary 1.2 (and Remark 1.1), and formula (11) is replaced by

$$\widehat{\mathcal{P}}_1(y;s) = \frac{1}{s}\left[1 - \frac{\beta\,\xi_+(s)^{-y}}{q(\xi_+(s)-1)+\beta}\right].$$

We compute that as $s \to 0$,

$$\widehat{\mathcal{P}}_1(y;s) \sim \frac{1}{s}\left[1 - \frac{\beta\,(q^{-1}-1)^{-y}}{p-q+\beta}\right] \quad \text{if } p > q,$$

$$\widehat{\mathcal{P}}_1(y;s) \sim \frac{1}{\sqrt{2s}}\,(2y + 1/\beta) \ \text{ if } p = q,$$

$$\widehat{\mathcal{P}}_1(y;s) \to \frac{y + q/\beta}{q-p} \ \text{ if } p < q.$$

Applying the Tauberian theorem we deduce

$$\lim_{t\to\infty} \mathcal{P}_1(y;t) = 1 - \frac{\beta\,(q^{-1}-1)^{-y}}{p-q+\beta} \quad \text{if } p > q,$$

$$\mathcal{P}_1(y;t) \sim \frac{1}{\sqrt{2\pi}}\left[2y + \frac{1}{\beta}\right]t^{-1/2} \ \text{ as } t \to \infty \text{ if } p = q,$$

$$\int_0^\infty \mathcal{P}_1(y;t)\,dt = \frac{y + q/\beta}{q-p} \ \text{ if } p < q.$$

When $p > q$ the limit on the first line is the probability that the particle is never ejected. If $p \le q$, then with probability 1 the particle will eventually be ejected. The expected value of the time at which it is ejected is infinite when $p = q$, by the second line, and is given by the integral on the last line when $p < q$.

B. TASEP

Remark 1.2: To compute $(I - \alpha\, L_n^0(s))^{-1}$ we use the fact that for $k \ge 1$ the kernel of $L_n^0(s)^k$ is

$$\sum_{\mathbf{z}_1,\ldots,\mathbf{z}_{k-1}\in\mathcal{X}_n^+} \widehat{\mathfrak{p}}((0,\mathbf{x}),(0,\mathbf{z}_1);s)\,\widehat{\mathfrak{p}}((0,\mathbf{z}_1),(0,\mathbf{z}_2);s)\cdots\widehat{\mathfrak{p}}((0,\mathbf{z}_{k-1}),(0,\mathbf{y});s).$$

The summand is the Laplace transform of the $(k-1)$-fold convolution

$$\int_{u_1+\cdots+u_{k-1}=t} \mathfrak{p}((0,\mathbf{x}),(0,\mathbf{z}_1);u_1)\,\mathfrak{p}((0,\mathbf{z}_1),(0,\mathbf{z}_2);u_2)\cdots\mathfrak{p}((0,\mathbf{z}_{n-1}),(0,\mathbf{y});u_{k-1})\,d\mathbf{u}.$$

When $p = 1$, if the left-most particle begins at 0 and ends at 0 then it was always at 0. It follows that these probabilities have the semigroup property. Thus after summing over $\mathbf{z}_1, \ldots, \mathbf{z}_{k-1}$ the integral becomes

$$\int_{u_1+\cdots+u_{k-1}=t} \mathfrak{p}((0,\mathbf{x}),(0,\mathbf{y}),t)\,d\mathbf{u} = \frac{t^{k-1}}{(k-1)!}\,\mathfrak{p}((0,\mathbf{x}),(0,\mathbf{y}),t).$$

Therefore, the kernel of $L_n^0(s)^k$ is

$$\int_0^\infty e^{-st}\,\frac{t^{k-1}}{(k-1)!}\,\mathfrak{p}((0,\mathbf{x}),(0,\mathbf{y}),t)\,dt.$$

It follows that the kernel of $\sum_{k=1}^{\infty}(\alpha\, L_n^0(s))^k$ is

$$\alpha \int_0^{\infty} e^{-st}\, e^{\alpha t}\, \mathfrak{p}((0,\mathbf{x}),(0,\mathbf{y}),t)\,dt = \alpha\, \widehat{\mathfrak{p}}((0,\mathbf{x}),(0,\mathbf{y});s-\alpha).$$

This gives $(I-\alpha\, L_n^0(s))^{-1} = 1+\alpha\, L_n^0(s-\alpha)$.

Once we have this result (and Corollary 1.2, Remark 1.1, and the determinant formula (A2) for $\mathfrak{p}(\mathbf{x},\mathbf{y};t)$), we have all the ingredients necessary to compute $\mathcal{P}_n(t)$ for small n. This is the probability that starting with no particles at time 0 there are exactly n particles at time t, which equals $\sum_{\mathbf{x}\in\mathcal{X}_n}\mathcal{P}_n(\mathbf{x};t)$. The computation of products of kernels involves summing geometric series.

C. SSEP

Proof of Theorem 2: A state of the system is a function $\eta:\mathbb{Z}^+\to\{0,1\}$ where $\eta_x=1$ means site x is occupied and $\eta_x=0$ means site x is not occupied. Recall that we defined $\Psi_n(x_1, x_2, \ldots, x_n; t)$ as the probability that sites $x_1,\ldots,x_n$ are occupied at time t. Thus,

$$\Psi_n(x_1,x_2,\ldots,x_n;\,t) = \langle \eta_{x_1}(t)\cdots\eta_{x_n}(t)\rangle. \tag{12}$$

The Markov generator $\mathcal{L}$ of ASEP on $\mathbb{Z}^+$ with an open boundary at zero is given[6] by

$$\mathcal{L}f(\eta) = \alpha\,(1-\eta_0)\left(f(\eta^0)-f(\eta)\right)+\beta\,\eta_0\left(f(\eta^0)-f(\eta)\right)$$

$$+\sum_{k=0}^{\infty}\left[p\,\eta_k\,(1-\eta_{k+1})+q\,(1-\eta_k)\,\eta_{k+1}\right]\left(f(\eta^{k,k+1})-f(\eta)\right). \tag{13}$$

Here f is an $\mathbb{R}$-valued function that depends on only finitely many sites, and

$$(\eta^k)_x = \begin{cases} 1-\eta_x & \text{if } x=k \\ \eta_x & \text{if } x\neq k, \end{cases} \qquad \left(\eta^{k,k+1}\right)_x = \begin{cases} \eta_{k+1} & \text{if } x=k \\ \eta_k & \text{if } x=k+1 \\ \eta_x & \text{if } x\neq k,k+1. \end{cases}$$

For SSEP the sum in (13) equals 1/2 times

$$\sum_{k=0}^{\infty}[\eta_k\,(1-\eta_{k+1})+(1-\eta_k)\,\eta_{k+1}]\,(f(\eta^{k,k+1})-f(\eta)).$$

When $\eta_k=\eta_{k+1}$ the first factor equals 0; otherwise it equals 1. Since the second factor is zero when $\eta_k=\eta_{k+1}$, we can ignore the first factor, and we get

$$\sum_{k=0}^{\infty}(f(\eta^{k,k+1})-f(\eta)).$$

For the correlations we are interested in $f(\eta)=\eta_{x_1}\cdots\eta_{x_n}$, so the kth summand equals zero unless either $k=x_i$ for some i or $k=x_i-1$ for some i. (We will see that these cannot both happen for a nonzero summand.)

Suppose first that $k=x_i$. If $x_{i+1}=x_i+1$ then $k+1=x_{i+1}$ and the substitution $\eta\to\eta^{k,k+1}$ applied to $f(\eta)$ just interchanges η_{x_i} and $\eta_{x_{i+1}}$. Therefore, the kth summand is zero. It follows that for a nonzero summand we must have $x_{i+1}>x_i+1=k+1$, and the substitution $\eta\to\eta^{k,k+1}$ only affects the ith factor in $f(\eta)$. Therefore, the summand equals

$$\left(\eta_{x_1}\cdots\eta_{x_{i-1}}\,\eta_{x_i+1}\,\eta_{x_{i+1}}\cdots\eta_{x_n}-\eta_{x_1}\cdots\eta_{x_{i-1}}\,\eta_{x_i}\,\eta_{x_{i+1}}\cdots\eta_{x_n}\right)(1-\delta(x_{i+1}-x_i-1)). \tag{14}$$

Similarly if $k=x_i-1$ the summand equals

$$\left(\eta_{x_1}\cdots\eta_{x_{i-1}}\,\eta_{x_i-1}\,\eta_{x_{i+1}}\cdots\eta_{x_n}-\eta_{x_1}\cdots\eta_{x_{i-1}}\,\eta_{x_i}\,\eta_{x_{i+1}}\cdots\eta_{x_n}\right)(1-\delta(x_i-x_{i-1}-1)). \tag{15}$$

(This is to be multiplied by $1-\delta(x_1)$ when $i=1$.)

If $k = x_i$ for (14) and $k = x_{i'} - 1$ for (15) then $i' = i + 1$, so $x_{i+1} = x_{i'} = x_i + 1$ and (14) zero, and $x_{i'} = x_i + 1 = x_{i'-1} + 1$ so (15) is zero. Thus for the kth summand to be nonzero either $k = x_i$ for some i or $k = x_i - 1$ for some i, but not both.

It follows that for SSEP the expected value of the sum in (13) is equal to 1/2 times the sum over i of the expected values of the sum of (14) and (15). This equals $\mathcal{Q}_n(\langle \eta_{x_1} \cdots \eta_{x_n} \rangle)$.

Adding what we get from the first two terms of (13) we find that the differential equation for $\Psi_n = \Psi_n(t)$ is

$$\frac{d}{dt}\Psi_n(x_1, x_2, \ldots, x_n) = \mathcal{Q}_n\, \Psi_n(x_1, x_2, \ldots, x_n)$$

$$-(\alpha + \beta)\, \Psi_n(x_1, , x_2, \ldots, x_n)\, \delta(x_1) + \alpha\, \Psi_{n-1}(x_2, \ldots, x_n)\, \delta(x_1).$$

This is a well-known result for SSEP; namely, that the master equations for the n-point correlation functions (12) form a closed system of equations, see, e.g., Refs. 2,4, and 7. Observe that the constant functions $\Psi_n = (\alpha/(\alpha + \beta))^n$ satisfy the equations. Thus, the Bernoulli measure with density $\rho = \alpha/(\alpha + \beta)$ is stationary. This also follows from the results of Ref. 6 in which stationary measures were determined for general ASEP on $\mathbb{Z}^+$. Except for the change $\alpha - \beta \to -(\alpha + \beta)$, this is equation (6) without the B_n terms. Therefore, to complete the proof of Theorem 2 we need only make this change as we go through the rest of the proof of Theorem 1, which we need not do.

Corollaries 2.1 and 2.2: Just as (8) is obtained from Theorem 1, we obtain now for $n > 0$,

$$\widehat{\Psi}_n(s) = \sum_{k=0}^{n} \alpha^{n-k}\, M_n(s)L_n(s)A_n \cdots M_{k+1}(s)L_{k+1}(s)A_{k+1}\, M_k(s)L_k(s)\, \Psi_k(0).$$

Corollary 2.1 follows. For Corollary 2.2 we have $\Psi_n(0) = \rho^n$, and we use the fact that $L_n(1) = \sum_{\mathbf{y}\in\mathcal{X}_n} \widehat{\mathfrak{p}}(\mathbf{x}, \mathbf{y};\, s) = s^{-1}$. Corollary 2.3 is the case $\rho = 0$ of Corollary 2.2.

$\Delta N(t)$, the net number of particles that entered the system: We assume that we have SSEP with Bernoulli initial condition. Corollary 2.2 when $n = 1$ gives, for the Laplace transform of $\langle \eta_x(t) \rangle$,

$$\widehat{\langle \eta_x \rangle}(s) = \widehat{\Psi}_1(x; s) = s^{-1}\, (\rho\, M_1(s)\, 1 + \alpha\, M_1(s)L_1(s)A_1\, 1).$$

By Remark 1.1(a) applied here, we have (since the definition of $M_k(s)$ is different now we must replace $\alpha - \beta$ by $-(\alpha + \beta)$ when using the remark)

$$M_1(s)L_1(s)A_1\, 1 = \frac{\widehat{\mathfrak{p}}(x, 0; s)}{1 + \gamma\, \widehat{\mathfrak{p}}(0, 0; s)}, \tag{16}$$

where we set

$$\gamma = \alpha + \beta.$$

Similarly, from (9) we obtain

$$M_1(s)\, 1 = 1 - \gamma\, \frac{\widehat{\mathfrak{p}}(x, 0; s)}{1 + \gamma\, \widehat{\mathfrak{p}}(0, 0; s)}.$$

Combining the two gives

$$\widehat{\langle \eta_x \rangle}(s) = \frac{1}{s}\Big[\rho + \frac{\alpha - \gamma\, \rho}{1 + \gamma\, \widehat{\mathfrak{p}}(0, 0; s)}\, \widehat{\mathfrak{p}}(x, 0; s)\Big].$$

Subtracting ρ/s from both sides and summing over $x \geq 0$ we get for the Laplace transform of $\langle \Delta N(t) \rangle$

$$\widehat{\langle \Delta N(t) \rangle}(s) = \frac{1}{s^2}\, \frac{\alpha - \gamma\rho}{1 + \gamma\, \widehat{\mathfrak{p}}(0, 0; s)}.$$

From (10) we have for SSEP

$$\widehat{\mathfrak{p}}(0, 0; s) = \frac{2}{s + \sqrt{s^2 + 2s}} \sim \sqrt{\frac{2}{s}} \quad \text{as } s \to 0.$$

Hence

$$\langle \widehat{\Delta N(t)}\rangle(s) \sim \frac{\alpha-\gamma\,\rho}{\sqrt{2}\,\gamma}\, s^{-3/2} \ \text{ as } s\to 0.$$

By the Tauberian theeorem this implies

$$\langle \Delta N(t)\rangle \sim \sqrt{\frac{2}{\pi}}\,\frac{\alpha-\gamma\,\rho}{\gamma}\, t^{1/2} \ \text{ as } t\to\infty.$$

As stated in the last section the second moment $\langle \Delta N(t)^2\rangle$ is finite. To show this we use that the second moment is equal to

$$\lim_{N\to\infty}\sum_{x_1,x_2<N}\langle(\eta_{x_1}(t)-\eta_{x_1}(0))\,(\eta_{x_2}(t)-\eta_{x_2}(0))\rangle.$$

We can show that the sum is a polynomial of degree two in ρ, and that each of the three coefficients of the powers of ρ has a limit as $N\to\infty$. The argument is quite involved, and we do not include it.

The second moment of $\Delta N(t)$ when $\rho = 0$: Now it is mainly a question of determining the asymptotics of $\sum_{x_1<x_2}\langle\eta_{x_1}\eta_{x_2}\rangle$. The Laplace transform of $\Psi_2(\mathbf{x};t)=\langle\eta_{x_1}\,\eta_{x_2}\rangle$ is given by Corollary 2.2 as

$$\widehat{\Psi}_2(\mathbf{x};s)=\alpha^2\,s^{-1}\,M_2(s)L_2(s)A_2M_1(s)L_1(s)A_1\,1.$$

We computed $M_1(s)L_1(s)A_1\,1$ in (16). Combining this with Remark 1.1(a) gives

$$\widehat{\Psi}_2(\mathbf{x};s)=\alpha^2\,s^{-1}\,L^0_{2,1}(s)\,(I+\gamma\,L^0_1(s))^{-1}\frac{\widehat{\mathfrak{p}}_0}{1+\gamma\,\widehat{\mathfrak{p}}_0(0)},$$

where $\widehat{\mathfrak{p}}_0$ is the function $x\to\widehat{\mathfrak{p}}(x,0;s)$.

To obtain the Laplace transform $\sum_{x_1<x_2}\widehat{\langle\eta_{x_1}\eta_{x_2}\rangle}$ we sum over $\mathbf{x}\in\mathcal{X}_2$. If we recall that the kernel of $L^0_{2,1}(s)$ is $\widehat{\mathfrak{p}}(\mathbf{x},(0,y))$ and that the sum of this over $\mathbf{x}\in\mathcal{X}_2$ is s^{-1} we see that the desired sum is the inner product of the remaining function with the constant function s^{-1}. Thus,

$$\sum_{x_1<x_2}\widehat{\langle\eta_{x_1}\eta_{x_2}\rangle}=\frac{\alpha^2 s^{-2}}{1+\gamma\,\widehat{\mathfrak{p}}_0(0)}((I+\gamma\,L^0_1(s))^{-1}\widehat{\mathfrak{p}}_0,\,1). \tag{17}$$

(Here we use the fact that $\widehat{\mathfrak{p}}_0$ belongs to L^1 and that $L^0_1(s)$ is a bounded operator on this space.)

What follows is not rigorous. We want to rescale $(I+\gamma\,L^0_1(s))^{-1}$ as $s\to 0$, and we refer to formula (A1) given in the Appendix for $\mathfrak{p}(\mathbf{x},\mathbf{y};t)$. After taking Laplace transforms in the case $n=2$ we find that $L^0_1(s)$ has kernel

$$L^0_1(x,y;s)=\sum_{\sigma\in\mathbb{B}_2}\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_R}\int_{\mathcal{C}_R}A_\sigma\,\frac{\xi^x_{\sigma(2)}\,\xi_2^{-y}}{s-\varepsilon(\xi_1)-\varepsilon(\xi_2)}\,\frac{d\xi_1\,d\xi_2}{\xi_1\,\xi_2}, \tag{18}$$

where $\mathcal{C}_R$ is a circle with radius R with R large. (Some integrals are taken over two pairs of different contours and the results averaged.) To begin with, s is so large that taking Laplace transforms under the integral sign is valid.

If we ignore the poles of the A_σ we can move both contours to the unit circle $\mathcal{C}_1$. Then the range of $\varepsilon(\xi_1)+\varepsilon(\xi_2)$ is $[-4,0]$, so we may take any $s>0$. As $s\to 0$ the main contribution comes from a neighborhood of $\xi_1=\xi_2=1$ because the denominator vanishes there when $s=0$. If we set $\xi_1=e^{iv_1},\ \xi_2=e^{iv_2}$ then the integral with its factor becomes to first order

$$\frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}A_\sigma(e^{iv_1},e^{iv_2})\,\frac{e^{i\,(v_{\sigma(2)}\,x-v_2\,y)}}{s+(v_1^2+v_2^2)/2}\,dv_1\,dv_2$$

$$=\frac{1}{2\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}A_\sigma(e^{i\sqrt{2s}\,v_1},e^{i\sqrt{2s}\,v_2})\,\frac{e^{i\,\sqrt{2s}\,(v_{\sigma(2)}\,x-v_2\,y)}}{1+v_1^2+v_2^2}\,dv_1\,dv_2.$$

This becomes, after the scaling $x \to x/\sqrt{2s},\ y \to y/\sqrt{2s}$,

$$\frac{1}{2\pi^2\sqrt{2s}}\int_{\mathbb{R}}\int_{\mathbb{R}} A_\sigma(e^{i\sqrt{2s}\,v_1}, e^{i\sqrt{2s}\,v_2})\,\frac{e^{i\,(v_{\sigma(2)}\,x-v_2\,y)}}{1+v_1^2+v_2^2}\,dv_1\,dv_2,$$

which acts on functions on $\mathbb{R}^+$.

Each A_σ has absolute value 1 on $\mathcal{C}_1\times\mathcal{C}_1$, and each $A_\sigma(e^{i\sqrt{2s}v_1}, e^{i\sqrt{2s}v_2})$ has limit 1 as $s\to 0$ except when $v_1 = v_2$ (mod 2π). Thus, we replace the above by the approximation

$$\frac{1}{2\pi^2\sqrt{2s}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{i\,(v_{\sigma(2)}\,x-v_2\,y)}}{1+v_1^2+v_2^2}\,dv_1\,dv_2.$$

This depends only on $\sigma(2)$. If we use the fact that the denominator is even in each v_i, and that $v_{-i} = -v_i$, we see that the sum over $\sigma\in\mathbb{B}_2$ of the integrals equals

$$\frac{1}{\pi^2\sqrt{2s}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{i\,v_2\,(x-y)}+e^{i\,v_2\,(x+y)}}{1+v_1^2+v_2^2}\,dv_1\,dv_2+\frac{2}{\pi^2\sqrt{2s}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{i\,(v_1\,x-v_2\,y)}}{1+v_1^2+v_2^2}\,dv_1\,dv_2.$$

In the first integral we integrate first with respect to v_1, and we obtain

$$\frac{1}{\pi\sqrt{2s}}\int_{\mathbb{R}}\frac{e^{i\,v\,(x-y)}+e^{i\,v\,(x+y)}}{\sqrt{1+v^2}}\,dv+\frac{2}{\pi^2\sqrt{2s}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{i\,(v_1\,x-v_2\,y)}}{1+v_1^2+v_2^2}\,dv_1\,dv_2. \tag{19}$$

(The integrals are K_0 Bessel functions, but this fact is not useful.)

These are of the order $1/\sqrt{s}$ as $s\to 0$. Now we indicate why the contributions from the poles of the A_σ, when we shrink the contours, are of lower order.

Consider the permutations $(\pm 2\,1)$, when $A_\sigma = S(\xi_2,\xi_1)$. (Times ξ_1^{-1} when $\sigma = (-2\,1)$; this has no effect on what follows.) With the ξ_2-integration over $\mathcal{C}_R$, we shrink the ξ_1-contour to $\mathcal{C}_1$. Then when we shrink the ξ_2-contour we pass the pole at $\xi_2 = 2-\xi_1^{-1}$ for all $\xi_1\in\mathcal{C}_1$. The residue is a constant times

$$\int_{\mathcal{C}_1}\left(\frac{\xi-1}{\xi}\right)^2\frac{\xi^x\,(2-\xi^{-1})^{-y-1}}{s-\frac{(\xi-1)^2}{2\xi-1}}\,\frac{d\xi}{\xi}, \tag{20}$$

where we replaced ξ_1 by ξ. With either branch of $\xi^{1/2}$ we may write

$$\frac{(\xi-1)^2}{2\xi-1}=\frac{(\xi^{1/2}-\xi^{-1/2})^2}{2-\xi^{-1}}.$$

On $\mathcal{C}_1$ the numerator is negative real (except when $\xi = 1$) while the denominator lies in the right half-plane. Thus the quotient lies in the left half-plane (except when $\xi = 1$) so we may take any $s > 0$ in the integral. Since again the main contribution comes from near $\xi = 1$, we set $\xi = e^{iv}$ and make the replacements $2-\xi^{-1}\to 2-(1-iv) = 1+iv\to e^{iv}$, and we get

$$\int_{\mathbb{R}}\frac{v^2}{s+v^2}\,e^{iv(x-y)}\,dv=\sqrt{s}\int_{\mathbb{R}}\frac{v^2}{1+v^2}\,e^{i\sqrt{s}\,v(x-y)}\,dv.$$

After the scaling this becomes independent of s.

The factor $(\xi\ -\ 1)^2$ in the integrand in (20) was important. It also appears in the residues for the other integrals, which also become independent of s by similar computations. We omit the details.

Thus when we scale the contributions from the poles of the A_σ are independent of s, and so of lower order than the main terms (19).

Set

$$J_1(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{i\,v\,(x-y)}+e^{i\,v\,(x+y)}}{\sqrt{1+v^2}}\,dv,$$

$$J_2(x,y)=\frac{1}{\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{i\,(v_1\,x-v_2\,y)}}{1+v_1^2+v_2^2}\,dv_1\,dv_2.$$

We showed that $1 + \gamma L_1^0(s)$, when scaled, is equal to $2\gamma/\sqrt{2s}\,(J_1 + J_2)$ plus an operator independent of s. Therefore, we presume that to a first approximation the scaled operator $(1 + \gamma L_1^0(s))^{-1}$ is equal to $(2\gamma)^{-1}\sqrt{2s}\,(J_1 + J_2)^{-1}$. Also, we see from (10) and the symmetry of $\widehat{\mathfrak{p}}(\mathbf{x}, \mathbf{y}; s)$ that

$$\widehat{\mathfrak{p}}_0(x) = \widehat{\mathfrak{p}}(x, 0; s) \sim \sqrt{\frac{2}{s}}\, e^{-\sqrt{2s}\,x} \quad \text{as } s \to 0.$$

This gives the **Conjecture**

$$((I + \gamma L_1^0(s))^{-1}\,\widehat{\mathfrak{p}}_0, 1) \sim \frac{1}{\gamma\sqrt{2s}}((J_1 + J_2)^{-1}\,e^{-x}, 1) \quad \text{as } s \to 0. \tag{21}$$

We shall show that the inner product equals $2/\pi$. Assume this, and the conjecture, for now. Then from (17) and the asymptotics $\widehat{\mathfrak{p}}(0, 0; s) \sim \sqrt{2/s}$ as $s \to 0$ we find that

$$\sum_{x_1<x_2} \widehat{\langle \eta_{x_1}\eta_{x_2}\rangle} \sim \frac{1}{\pi}\frac{\alpha^2}{\gamma^2}\, s^{-2} \quad \text{as } s \to 0.$$

Therefore by the Tauberian theorem,

$$\sum_{x_1<x_2} \langle \eta_{x_1}\,\eta_{x_2}\rangle \sim \frac{1}{\pi}\frac{\alpha^2}{\gamma^2}\, t \quad \text{as } t \to \infty.$$

Since

$$\langle \Delta N(t)^2\rangle = \sum_{x_1,x_2\geq 0} \langle \eta_{x_1}\,\eta_{x_2}\rangle = 2\sum_{x_1<x_2} \langle \eta_{x_1}\,\eta_{x_2}\rangle + \langle \Delta N(t)\rangle,$$

and $\Delta N(t) = o(t)$, we get the result stated in Sec. II,

$$\langle \Delta N(t)^2\rangle \sim \frac{2}{\pi}\frac{\alpha^2}{\gamma^2}\, t \quad \text{as } t \to \infty.$$

Now we show that the inner product in (21) equals $2/\pi$. We first use an observation about Wiener-Hopf plus Hankel operators, of which J_1 is one. (Recall that our operators act on functions on $\mathbb{R}^+$.) Suppose we have such an operator with kernel $J(x - y) + J(x + y)$, so that the result of its action on a function f on $\mathbb{R}^+$ is

$$(Jf)(x) = \int_0^\infty (J(x - y) + J(x + y))\, f(y)\, dy.$$

If we extend f to an even function on $\mathbb{R}$, then this equals

$$\int_{-\infty}^\infty J(x - y)\, f(y)\, dy.$$

So the operator becomes simply convolution by J.

Also, it follows from the fact that $J_2(x, y)$ is even in y that if f, defined on $\mathbb{R}^+$, is extended to be even on $\mathbb{R}$ then

$$(J_2\, f)(x) = \frac{1}{2}\int_{-\infty}^\infty J_2(x, y)\, f(y)\, dy.$$

To recapitulate, define

$$K_1(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i\,v\,(x-y)}}{\sqrt{1+v^2}}\, dv,$$

$$K_2(x, y) = \frac{1}{2\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{i\,(v_1\,x - v_2\,y)}}{1+v_1^2+v_2^2}\, dv_1\, dv_2,$$

acting on functions on $\mathbb{R}$. Then on $\mathbb{R}^+$ we have $(J_1 + J_2)f = (K_1 + K_2)f$, where the f on the right is the even extension of the f on the left.

It follows from this that for an even function f on $\mathbb{R}$, $(J_1 + J_2)^{-1}f$ is the restriction to $\mathbb{R}^+$ of $(K_1 + K_2)^{-1}f$. (This uses that $K_1 + K_2$ commutes with the operator $f(x) \to f(-x)$.) Therefore, the inner product in (21), which is over $\mathbb{R}^+$, is equal to

$$\frac{1}{2}((K_1 + K_2)^{-1}\, e^{-|x|}, 1), \tag{22}$$

where this inner product is over $\mathbb{R}$.

Because of the forms of the kernels of K_1 and K_2, the operators simplify when we conjugate with the Fourier transform. The operator K_1 becomes $\widehat{K}_1$, which is multiplication by the function $1/\sqrt{1+v^2}$, and the operator K_2 becomes $\widehat{K}_2$, which has kernel

$$\widehat{K}_2(u, v) = \frac{1}{\pi}\frac{1}{1+u^2+v^2}.$$

Since the Fourier transform of 1 with factor $1/2\pi$ outside the integral is δ_0 and the Fourier transform of $e^{-|x|}$ with factor 1 is $2/(1+v^2)$, we see that (22) (which equals the inner product in (21)) equals

$$((\widehat{K}_1 + \widehat{K}_2)^{-1}(1+v^2)^{-1}, \delta_0).$$

If we use

$$(\widehat{K}_1 + \widehat{K}_2)^{-1} = \widehat{K}_1^{-1/2}\,(1 + \widehat{K}_1^{-1/2}\,\widehat{K}_2\,\widehat{K}_1^{-1/2})^{-1}\,\widehat{K}_1^{-1/2}$$

and the fact that $\widehat{K}_1^{-1/2}\delta_0 = \delta_0$, we see that the above equals

$$((I + \widehat{K})^{-1}\,\psi, \delta_0), \tag{23}$$

where

$$\widehat{K}(u, v) = \frac{1}{\pi}\frac{(1+u^2)^{1/4}\,(1+v^2)^{1/4}}{1+u^2+v^2}, \qquad \psi(v) = (1+v^2)^{-3/4}.$$

If we conjugate with the unitary operator $f(u) \to (\cosh x)^{1/2} f(\sinh x)$ we find that (23) equals $((I + \tilde{K})^{-1}\,\widetilde{\psi}, \delta_0)$, where

$$\tilde{K}(x, y) = (\cosh x)^{1/2}\,\widehat{K}(\sinh x, \sinh y)\,(\cosh y)^{1/2} = \frac{1}{\pi}\frac{\cosh x\ \cosh y}{1+\sinh^2 x + \sinh^2 y},$$

$$\widetilde{\psi}(x) = (\cosh x)^{1/2}\,\psi(\sinh x) = \operatorname{sech} x.$$

From

$$\cosh x\ \cosh y = \frac{1}{2}(\cosh(x+y) + \cosh(x-y)),$$

$$1 + \sinh^2 x + \sinh^2 y = \cosh(x+y)\,\cosh(x-y),$$

one sees that

$$\tilde{K}(x, y) = \frac{1}{2\pi}(\operatorname{sech}(x-y) + \operatorname{sech}(x+y)).$$

Now $\widetilde{\psi}$ is even, and when $\tilde{K}$ is restricted to the space of even functions it equals the operator with convolution kernel $\operatorname{sech}(x - y)/\pi$. Conjugating with the Fourier transform, this operator becomes multiplication by $\operatorname{sech}(\pi\xi/2)$. The Fourier transform of $\operatorname{sech} x$ with factor $1/2\pi$ is $(1/2)\operatorname{sech}(\pi\xi/2)$, and the Fourier transform of δ_0 with factor 1 is 1. Therefore,

$$((I + \tilde{K})^{-1}\,\widetilde{\psi}, \delta_0) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{1+\operatorname{sech}(\pi\xi/2)}\operatorname{sech}(\pi\xi/2)\,d\xi = \frac{2}{\pi}.$$

Thus the inner product in (21) is $2/\pi$ as claimed.

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APPENDIX: THE FORMULAS FOR $\mathfrak{p}(\mathbf{x}, \mathbf{y}; t)$

The Weyl group $\mathbb{B}_n$ is the group of *signed permutations*, functions σ: $[1, n] \to [-n, -1] \cup [1, n]$ such that $|\sigma| \in \mathbb{S}_n$. An *inversion* in $\mathbb{B}_n$ is a pair $(\pm\sigma(i), \sigma(j))$ with $i < j$ such that $\pm\sigma(i) > \sigma(j)$. We write $\tau = p/q$.

We define

$$S(\xi, \xi') = -\frac{p + q\,\xi\xi' - \xi}{p + q\,\xi\xi' - \xi'}, \quad r(\xi) := \frac{\xi - 1}{1 - \tau\,\xi^{-1}}, \quad \varepsilon(\xi) = p\,\xi^{-1} + q\,\xi - 1,$$

and then define

$$A_\sigma = \prod_{\sigma(i)<0} r(\xi_{\sigma(i)}) \times \prod \{S(\xi_a,\ \xi_b) : (a, b) \text{ is an inversion in } \mathbb{B}_n\},$$

with the convention $\xi_{-a} = \tau/\xi_a$.

The formula, valid when $q \neq 0$, is

$$\mathfrak{p}(\mathbf{x}, \mathbf{y}; t) = \frac{1}{n!} \sum_{\sigma \in \mathbb{B}_n} \frac{1}{(2\pi i)^n} \int \cdots \int A_\sigma(\xi) \prod_i \left(\xi_{\sigma(i)}^{x_i}\, \xi_i^{-y_i - 1} e^{\varepsilon(\xi_i)\,t} \right) d\xi_1 \cdots d\xi_n, \tag{A1}$$

where $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, x_n\}$. The domain of integration is

$$\bigcup_{\mu \in \mathbb{S}_n} \mathcal{C}_{\mu(1)} \times \cdots \times \mathcal{C}_{\mu(n)},$$

where the $\mathcal{C}_a$ are circles with center $1/2q$ and distinct radii R_a. The R_a should be so large that $S(\xi, \xi')$ is analytic for ξ, ξ' on and outside $\mathcal{C}_a$. (We cannot simply take $\mathcal{C} \times \cdots \times \mathcal{C}$ with $\mathcal{C}$ a circle with large radius, because then there would be nonintegrable singularities of $S(\xi_a, \xi_b)$ on the contour when $a > 0, b < 0$. However, by taking the $R_a \to R$ we can interpret each integral as a symmetric distribution supported on $\mathcal{C} \times \cdots \times \mathcal{C}$ applied to the product in the integrand.)

In the special case $n = 1$,

$$\mathfrak{p}(x, y; t) = \frac{1}{2\pi i} \int_{\mathcal{C}_R} \left[\xi^{x-y-1} + \frac{\tau - \xi}{1 - \xi}\, \tau^x\, \xi^{-x-y-2} \right] e^{\varepsilon(\xi)\,t}\, d\xi,$$

where $\mathcal{C}_R$ is a circle around 0 of radius $R > 1$.

The formulas do not hold for $p = 1$ TASEP on $\mathbb{Z}^+$. But then the probability is the same as for TASEP on $\mathbb{Z}$, and we have then[5] (or Ref. 8, p. 820)

$$\mathfrak{p}(\mathbf{x}, \mathbf{y}; t) = \det\left(\int_{\mathcal{C}_r} (1 - \xi)^{j-i} \xi^{x_i - y_j - 1} e^{t\varepsilon(\xi)}\, d\xi \right), \tag{A2}$$

where $\mathcal{C}_r$ is a circle with center 0 and radius $r < 1$.

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