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On the diagonal susceptibility of the two-dimensional Ising model

Craig A. Tracy[1] and Harold Widom[2]
[1]*Department of Mathematics, University of California, Davis, California 95616, USA*
[2]*Department of Mathematics, University of California, Santa Cruz, California 95064, USA*

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We consider the diagonal susceptibility of the isotropic 2D Ising model for temperatures below the critical temperature. For a parameter k related to temperature and the interaction constant, we extend the diagonal susceptibility to complex k inside the unit disc, and prove the conjecture that the unit circle is a natural boundary. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4836779]

I. INTRODUCTION

For the 2D Ising model,[15,16,23] after the zero-field free energy[20] and the spontaneous magnetization,[21,28] the most important zero-field thermodynamic quantity is the magnetic susceptibility χ. Since the free energy is known only in zero magnetic field, the susceptibility is usually studied through its relation with the zero-field spin-spin correlation function,

$$\beta^{-1}\chi = \sum_{M,N\in\mathbb{Z}} \left\{\langle\sigma_{0,0}\sigma_{M,N}\rangle - \mathcal{M}^2\right\}, \tag{1}$$

where $\beta = (k_B T)^{-1}$, T is temperature, k_B is Boltzmann's constant, and $\mathcal{M}$ is the spontaneous magnetization. If T_c denotes the critical temperature, we recall that for the isotropic 2D Ising model, i.e., horizontal and vertical interaction constants have the same value J, the spontaneous magnetization is given for $T < T_c$ by

$$\mathcal{M} = (1-k^2)^{1/8}, \tag{2}$$

where $k := (\sinh 2\beta J)^{-2}$ and $\mathcal{M}$ is zero for $T > T_c$. (For $0 < T < T_c$ we have $0 < k < 1$.)

The analysis of $\chi = \chi(T)$ in the neighborhood of the critical temperature T_c has a long history. We refer the reader to McCoy *et al.*[17] for a review of these developments. The analysis of χ for *complex temperatures* was initiated by Guttmann and Enting[13] and by Nickel.[18,19] (For further developments see Refs. 10 and 22.) Nickel's analysis takes as its beginning the (commonly called) form-factor or particle expansion of the spin-spin correlation function.[27] For $T < T_c$ this expansion is an infinite sum whose nth summand is a $2n$-dimensional integral. From an asymptotic analysis of these integrals, Nickel was led to conjecture that $|k| = 1$ is a *natural boundary* for χ. As Nickel himself noted, the analysis is nonrigorous since one must show that there are no cancellations of singularities in the sum. This has turned out to be a difficult problem to resolve rigorously.

In Boukraa *et al.*,[7] these authors, building on results of Ref. 14, introduce a simplified model for χ, called the *diagonal susceptibility* χ_d, which is defined by having "a magnetic field which acts only on one diagonal of the lattice." (See Ref. 2 for further developments.) Thus, the analogue of (1) is

$$\beta^{-1}\chi_d = \sum_{N\in\mathbb{Z}} \left\{\langle\sigma_{0,0}\,\sigma_{N,N}\rangle - \mathcal{M}^2\right\}. \tag{3}$$

In this paper we consider χ_d only for $T < T_c$, in which case $k < 1$. Then we extend χ_d to k complex with $|k| < 1$. Using the Toeplitz determinant representation of the diagonal correlations,[15,24] we first derive the known representation of χ_d in terms of a sum of multiple integrals $\mathcal{S}_n$. The derivation is different from those in Refs. 8, 9, 14, and 26. As in Ref. 26 we use the identity of Geronimo-Case (GC)[12] and Borodin-Okounkov (BO)[5] relating a Toeplitz determinant to the Fredholm determinant of a product of Hankel operators. (For simplified proofs of the GCBO formula, see Refs. 3 and 6.) But here we go from there to the multiple integral representation directly using a general identity for the integral of a product of determinants[1] (see Eq. (1.3) in Ref. 25). For further background on the relationship between Toeplitz determinants and Ising correlations, see Refs. 4 and 11.

In Sec. IV, we show that for each root of unity $\epsilon \neq \pm 1$ a certain derivative of a certain $\mathcal{S}_n$ is unbounded as $k \to \epsilon$ radially, while the same derivative of the sum of the other $\mathcal{S}_n$ remains bounded (Lemma 4). Thus, the unit circle $|k| = 1$ is a natural boundary for χ_d. This proves the conjecture by Boukraa *et al.*[7] We note that in Ref. 7 the authors present an argument that the singularity of $\mathcal{S}_n$ at an nth root of unity ϵ is of the form $(k-\epsilon)^{2n^2-1}\log(k-\epsilon)$. Lemma 2 in Sec. IV formalizes this statement and fills in details of the proof.

II. TOEPLITZ DETERMINANT REPRESENTATION

It was shown in Refs. 15 and 24 that for $N > 1$ the diagonal correlation has a representation as an $N \times N$ Toeplitz determinant:

$$\langle \sigma_{0,0}\, \sigma_{N,N} \rangle = \det\,(\varphi_{m-n})_{1\le m,n\le N}\,.$$

Here

$$\varphi(\xi) = \left[\frac{1-k\xi^{-1}}{1-k\xi}\right]^{1/2},$$

and

$$\varphi_m = \frac{1}{2\pi i}\int \varphi(\xi)\,\xi^{-m-1}\,d\xi, \tag{4}$$

with integration over the unit circle. (We have $\langle \sigma_{0,0}^2\rangle = 1$.)

As in Ref. 26 we invoke the formula of Geronimo-Case[12] and Borodin-Okounkov[5] to write the Toeplitz determinant in terms of the Fredholm determinant of a product of Hankel operators. We have $\varphi(\xi) = \varphi_+(\xi)\,\varphi_-(\xi)$, where

$$\varphi_+(\xi) = (1-k\xi)^{-1/2} \quad \text{and} \quad \varphi_-(\xi) = (1-k\xi^{-1})^{1/2}.$$

Since $|k| < 1$ these extend analytically inside and outside the unit circle, respectively. The square roots are determined by $\varphi_+(0) = \varphi_-(\infty) = 1$.

The Hankel operator $H(\psi)$ is the operator on $\ell^2(\mathbb{Z}^+)$ with kernel $(\psi_{i+j+1})_{i,j\ge 0}$, where ψ_m given in analogy with (4). The operator $H_N(\psi)$ has kernel $(\psi_{N+i+j+1})$.

Using $\varphi_\pm(\xi) = 1/\varphi_\mp(\xi^{-1})$, we find that the formula of GCBO gives

$$\det\,(\varphi_{m-n})_{1\le m,n\le N} = \mathcal{M}^2\,\det\left(I - H_N\left(\frac{\varphi_-}{\varphi_+}\right) H_N\left(\frac{\varphi_+}{\varphi_-}\right)\right).$$

Thus, if we define

$$\Lambda(\xi) = \frac{\varphi_-(\xi)}{\varphi_+(\xi)} = \sqrt{(1-k\xi)(1-k/\xi)}\,, \qquad K_N = H_N(\Lambda)\,H_N(\Lambda^{-1}), \tag{5}$$

then

$$\beta^{-1}\chi_d = 1 - \mathcal{M}^2 + 2\mathcal{M}^2\sum_{N=1}^{\infty}[\det(I-K_N)-1] = 1 + \mathcal{M}^2(2\mathcal{S}-1),$$

where

$$\mathcal{S} = \sum_{N=1}^{\infty}[\det(I-K_N)-1]\,. \tag{6}$$

In what follows we extend Λ to be holomorphic in the complex plane cut along $[0, k] \cup [k^{-1}, \infty]$.

III. FORMULA FOR $\mathcal{S}$

We use a slightly different notation for Hankel operators here.

Proposition. Let $H_N(du)$ and $H_N(dv)$ be two Hankel matrices acting on $\ell^2(\mathbb{Z}^+)$ with i, j entries,

$$\int x^{N+i+j}\, du(x), \quad \int y^{N+i+j}\, dv(y), \tag{7}$$

respectively, where u and v are measures supported inside the unit circle. Set $K_N = H_N(du)\, H_N(dv)$. Then

$$\sum_{N=1}^{\infty} [\det(I - K_N) - 1]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \int \cdots \int \frac{\prod_i x_i y_i}{1 - \prod_i x_i y_i} \left(\det\left(\frac{1}{1 - x_i y_j}\right)\right)^2 \prod_i du(x_i)\, dv(y_i),$$

where indices in the integrand run from 1 to n.

Proof. The Fredholm expansion is

$$\det(I - K_N) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{p_1, \ldots, p_n \geq 0} \det(K_N(p_i, p_j)).$$

Therefore, its suffices to show that

$$\sum_{N=1}^{\infty} \sum_{p_1, \ldots, p_n \geq 0} \det(K_N(p_i, p_j))$$

$$= \frac{1}{n!} \int \cdots \int \frac{\prod_i x_i y_i}{1 - \prod_i x_i y_i} \left(\det\left(\frac{1}{1 - x_i y_j}\right)\right)^2 du(x_1) \cdots du(x_n)\, dv(y_1) \cdots dv(y_n).$$

We have

$$K_N(p_i, p_j) = \iint \frac{x^{N+p_i}\, y^{N+p_j}}{1 - xy}\, du(x)\, dv(y).$$

It follows by a general identity[1] (Eq. (1.3) in Ref. 25) that

$$\det(K_N(p_i, p_j)) = \frac{1}{n!} \int \cdots \int \det(x_i^{N+p_j})\, \det(y_i^{N+p_j}) \prod_i \frac{1}{1 - x_i y_i} \prod_i du(x_i)\, dv(y_i)$$

$$= \frac{1}{n!} \int \cdots \int \left(\prod_i x_i y_i\right)^N \det(x_i^{p_j})\, \det(y_i^{p_j}) \prod_i \frac{1}{1 - x_i y_i} \prod_i du(x_i)\, dv(y_i).$$

Summing over N gives

$$\sum_{N=1}^{\infty} \det(K_N(p_i, p_j)) =$$

$$\frac{1}{n!} \int \cdots \int \frac{\prod_i x_i y_i}{1 - \prod_i x_i y_i} \det(x_i^{p_j})\, \det(y_i^{p_j}) \prod_i \frac{1}{1 - x_i y_i} \prod_i du(x_i)\, dv(y_i).$$

(Interchanging the sum with the integral is justified since the supports of u and v are inside the unit circle.)

Now we sum over $p_1, \ldots, p_n \geq 0$. Using the general identity again (but in the other direction) gives

$$\sum_{p_1,\ldots,p_n\geq 0} \det(x_i^{p_j})\,\det(y_i^{p_j}) = n!\,\det\left(\sum_{p\geq 0} x_i^p\, y_j^p\right) = n!\,\det\left(\frac{1}{1-x_i y_j}\right).$$

We almost obtained the desired result. It remain to show that

$$\det\left(\frac{1}{1-x_i y_j}\right)\prod_i \frac{1}{1-x_i y_i}, \tag{8}$$

which we obtain in the integrand, may be replaced by

$$\frac{1}{n!}\left(\det\left(\frac{1}{1-x_i y_j}\right)\right)^2. \tag{9}$$

This follows by symmetrization over the x_i. (The rest of the integrand is symmetric.) For a permutation π, replacing the x_i by $x_{\pi(i)}$ multiplies the determinant in (8) by sgn π, so to symmetrize we replace the other factor by

$$\frac{1}{n!}\sum_\pi \operatorname{sgn}\pi\,\frac{1}{1-x_{\pi(i)}y_i} = \frac{1}{n!}\,\det\left(\frac{1}{1-x_i y_j}\right).$$

Thus, symmetrizing (8) gives (9). □

We apply this to the operator $K_N = H_N(\Lambda)\,H_N(\Lambda^{-1})$ given by (5). The matrix for $H_N(\Lambda)$ has i, j entry

$$\frac{1}{2\pi i}\int \Lambda(\xi)\,\xi^{-N-i-j-2}\,d\xi,$$

where the integration may be taken over a circle with radius in $(1, |k|^{-1})$. Setting $\xi = 1/x$ and using $\Lambda(1/x) = \Lambda(x)$ we see that the entries of $H_N(\Lambda)$ are given as in (7) with

$$du(x) = \frac{1}{2\pi i}\,\Lambda(x)\,dx,$$

and integration is over a circle $\mathcal{C}$ with radius in $(|k|, 1)$. Similarly $H_N(\Lambda^{-1}) = H_N(v)$ where in (7),

$$d\psi(y) = \frac{1}{2\pi i}\,\Lambda(y)^{-1}\,dy,$$

with integration over the same circle $\mathcal{C}$.

Hence, the proposition gives

$$\mathcal{S} = \sum_{n=1}^{\infty}\mathcal{S}_n, \tag{10}$$

where

$$\mathcal{S}_n = \frac{(-1)^n}{(n!)^2}\frac{1}{(2\pi i)^{2n}}\int\cdots\int \frac{\prod_i x_i y_i}{1-\prod_i x_i y_i}\left(\det\left(\frac{1}{1-x_i y_j}\right)\right)^2 \prod_i \frac{\Lambda(x_i)}{\Lambda(y_i)}\prod_i dx_i\,dy_i,$$

with all integrations over $\mathcal{C}$.

We deform $\mathcal{C}$ to the contour back and forth along the interval $[0, k]$, and then make the substitutions $x_i \to kx_i, y_i \to ky_i$. We obtain

$$\mathcal{S}_n = \frac{1}{(n!)^2}\frac{\kappa^{2n}}{\pi^{2n}}\int_0^1\cdots\int_0^1 \frac{\prod_i x_i y_i}{1-\kappa^n\prod_i x_i y_i}\left(\det\left(\frac{1}{1-\kappa x_i y_j}\right)\right)^2 \prod_i \frac{\Lambda_1(x_i)}{\Lambda_1(y_i)}\prod_i dx_i\,dy_i, \tag{11}$$

where we have set

$$\kappa = k^2, \quad \Lambda_1(x) = \sqrt{\frac{(1-x)(1-\kappa x)}{x}}.$$

Using the fact that the determinant in the integrand is a Cauchy determinant, we obtain the alternative expression

$$\mathcal{S}_n = \frac{1}{(n!)^2} \frac{\kappa^{n(n+1)}}{\pi^{2n}} \int_0^1 \cdots \int_0^1 \frac{\prod_i x_i y_i}{1 - \kappa^n \prod_i x_i y_i} \frac{\Delta(x)^2 \, \Delta(y)^2}{\prod_{i,j}(1-\kappa\, x_i y_j)^2} \prod_i \frac{\Lambda_1(x_i)}{\Lambda_1(y_i)} \prod_i dx_i\, dy_i, \quad (12)$$

where $\Delta(x)$ and $\Delta(y)$ are Vandermonde determinants. Clearly, $\mathcal{S}_n$ is holomorphic in κ for $|\kappa| < 1$. It is straightforward to prove that the sum (10) converges uniformly in κ for $|\kappa| \le r$ for all $0 < r < 1$; and hence, $\mathcal{S}$ is holomorphic in the κ unit disc.

IV. NATURAL BOUNDARY

Theorem. The unit circle $|\kappa| = 1$ is a natural boundary for $\mathcal{S}$.

There will be four lemmas. In these, $\epsilon \neq 1$ will be an nth root of unity and we consider the behavior of $\mathcal{S}$ as $\kappa \to \epsilon$ radially.

For $\ell \geq 0$ we use the representation (12) and look at

$$\int_0^1 \cdots \int_0^1 \frac{\prod_i x_i y_i}{(1 - \kappa^n \prod_i x_i y_i)^{\ell+1}} \frac{\Delta(x)^2 \, \Delta(y)^2}{\prod_{i,j}(1-\kappa\, x_i y_j)^2} \prod_i \frac{\Lambda_1(x_i)}{\Lambda_1(y_i)} \prod_i dx_i\, dy_i, \quad (13)$$

where all indices run from 1 to n. This will be the main contribution to $d^\ell \mathcal{S}_n / d\kappa^\ell$.

Lemma 1. The integral (13) is bounded when $\ell < 2n^2 - 1$ and it is of the order $\log(1 - |\kappa|)^{-1}$ when $\ell = 2n^2 - 1$.

Proof. First we establish the first part of the statement. The numerator in the first factor is bounded and the denominator in the second factor is bounded away from zero as $\kappa \to \epsilon$, since $\epsilon \neq 1$.

If $\prod_i x_i y_i < 1 - \delta$ then the rest of the integrand is bounded except for the last quotient, and the integral of that is $O(1)$.

If $\prod_i x_i y_i > 1 - \delta$ then each $x_i, y_i > 1 - \delta$ and the integrand has absolute value at most a constant times

$$\frac{\Delta(x)^2 \, \Delta(y)^2}{|\kappa^{-n} - \prod_i x_i y_i|^{\ell+1}} \prod_i \sqrt{\frac{1-x_i}{1-y_i}}.$$

We assumed that $\kappa \to \epsilon$ along a radius, so $\kappa^{-n} > 1$. Therefore, we get an upper bound if we replace κ^{-n} by 1. Then in the integral we make the substitutions $x_i = 1 - \xi_i, y_i = 1 - \eta_i$ (so $\xi_i, \eta_i < \delta$), and we obtain

$$\frac{\Delta(\xi)^2 \, \Delta(\eta)^2}{(1 - \prod_i (1-\xi_i)(1-\eta_i))^{\ell+1}} \prod_i \sqrt{\frac{\xi_i}{\eta_i}}.$$

Whenever $z_i \in [0, 1]$ $(i = 1, \ldots, m)$ we have $z_1 \cdots z_m \le z_i$ for each i, and so averaging gives

$$z_1 \cdots z_m \le \Big(\sum_j z_j\Big)/m,$$

and therefore,

$$1 - z_1 \cdots z_m \ge \sum_j (1 - z_j)/m.$$

It follows that

$$1-\prod_i(1-\xi_i)(1-\eta_i)\geq\sum_i(\xi_i+\eta_i)/2n. \tag{14}$$

Therefore, the integrand above is at most $(2n)^{\ell+1}$ times

$$\frac{\Delta(\xi)^2\,\Delta(\eta)^2}{(\sum_i(\xi_i+\eta_i))^{\ell+1}}\prod_i\sqrt{\frac{\xi_i}{\eta_i}}.$$

This is homogeneous of degree $2n(n-1)-\ell-1$. We first integrate over the region $\sum_i(\xi_i+\eta_i)=r$ and then over r. The resulting integral is at most a constant times

$$\int_0^{2n\delta}r^{2n^2-\ell-2}\,dr.$$

This is finite when $\ell<2n^2-1$, and so the first statement of the lemma is established. We note that the $(2n-1)$-dimensional volume of the region $\sum_i(\xi_i+\eta_i)=1$ is $1/\Gamma(2n)$, another nice factor which we can use if needed. But it will not be.

We now consider the integral when $\ell=2n^2-1$. As before, the integral over the region $\prod_i x_iy_i<1-\delta$ is $O(1)$, so we assume $\prod_i x_iy_i>1-\delta$. In particular, each $x_i,y_i>1-\delta$. The factors $1-\kappa\,x_iy_j$ in the second denominator equal $1-\kappa(1+O(\delta))=(1-\kappa)(1+O(\delta))$ since κ is bounded away from 1. From this we see that if we factor out κ^{2n^3} from the first denominator and $(1-\kappa)^{n^2}$ from the second, the integrand becomes

$$\frac{\Delta(x)^2\,\Delta(y)^2}{(\kappa^{-n}-\prod_i x_iy_i)^{2n^2}}\prod_i\sqrt{\frac{1-x_i}{1-y_i}}\,(1+O(\delta)).$$

We again make the substitutions $x_i=1-\xi_i,y_i=1-\eta_i$ and set $r=\sum_i(\xi_i+y_i)$. Then since $\prod_i(1-\xi_i)(1-\eta_i)=1-r+O(r^2)$ the integrand becomes

$$\frac{\Delta(\xi)^2\,\Delta(\eta)^2}{(\kappa^{-n}-1+r+O(r^2))^{2n^2}}\prod_i\sqrt{\frac{\xi_i}{\eta_i}}\,(1+O(\delta)).$$

The integration domain $\prod_i x_iy_i>1-\delta$ becomes $r+O(r^2)<\delta$, which is contained in $r<2\delta$ and contains $r<\delta/2$. The integral without the $O(\delta)$ term is at least a constant times

$$\int_0^{\delta/2}\frac{r^{2n^2-1}}{(\kappa^{-n}-1+2r)^{2n^2}}\,dr,$$

which is asymptotically a constant independent of δ times $\log(\kappa^{-n}-1)^{-1}$ as $\kappa\to\epsilon$. Similarly the integral of the $O(\delta)$ term is at most a constant independent of δ times $\delta\log(\kappa^{-n}-1)^{-1}$. Since δ is arbitrarily small, this proves the lemma. □

Lemma 2. We have

$$\left(\frac{d}{d\kappa}\right)^{2n^2-1}\mathcal{S}_n\approx\log(1-|\kappa|)^{-1}.$$

Proof. To compute the derivative of the integral in (12) one integral we get is a constant depending on n times (13) with $\ell=2n^2-1$. The other integrals are similar but in each the power in the denominator is less than $2n^2-1$ while we get extra factors obtained by differentiating the rest of the integrand for $\mathcal{S}_n$. These factors are of the form $(1-\kappa x_iy_i)^{-1},(1-\kappa x_i)^{-1}$, or $(1-\kappa y_i)^{-1}$. By an obvious modification of the first statement of Lemma 1 we see that these other integrals are all bounded. The lemma follows. □

Lemma 3. If $\epsilon^m \neq 1$ then

$$\left(\frac{d}{d\kappa}\right)^{2n^2-1} \mathcal{S}_m = O(1).$$

Proof. If $\epsilon^m \neq 1$ all integrands obtained by differentiating the integral in (12) are bounded as $\kappa \to \epsilon$. □

Lemma 4. We have

$$\sum_{m>n} \left(\frac{d}{d\kappa}\right)^{2n^2-1} S_m = O(1).$$

Proof. We shall show that for κ sufficiently close to ϵ all integrals we get by differentiating the integral for S_m are at most $A^m\, m^m$, where A is some constant. Note that the value of A will change with each of its appearances. In may depend on n, but not on m. Because of the $1/(m!)^2$ appearing in front of the integrals this will show that the sum is bounded.

As before, we first use (12) (with n replaced by m) and consider the integral we get when the first factor in the integrand is differentiated $2n^2 - 1$ times. All indices in the integrands now run from 1 to m.

First,

$$|1-\kappa^m \prod_i x_i y_i| = |\kappa^m|\,|\kappa^{-m} - \prod_i x_i y_i| \geq |\kappa|^m\,(1-\prod_i x_i y_i).$$

Next, $|1 - \kappa x_i| \leq 2$. Since $y_i \in [0, 1]$ and $\kappa \in [0, \epsilon]$ we also have $\kappa y_i \in [0, \epsilon]$. Therefore, $|1 - \kappa y_i| \geq a$, where $a = \text{dist}(1, [0, \epsilon])$. Hence, the integrand in (12) after differentiating the first factor has absolute value at most A^m times

$$\frac{1}{(1-\prod x_i y_i)^{2n^2}}\, \frac{\Delta(x)^2\, \Delta(y)^2}{\prod_{i,j} |1-\kappa x_i y_j|^2} \prod_i \sqrt{\frac{1-x_i}{1-y_i}\,\frac{y_i}{x_i}}. \tag{15}$$

Since we also have

$$|1-\kappa x_i y_j| \geq a, \tag{16}$$

(15) is at most

$$a^{-m^2}\, \frac{\Delta(x)^2\, \Delta(y)^2}{(1-\prod x_i y_i)^{2n^2}} \prod_i \sqrt{\frac{1-x_i}{1-y_i}\,\frac{y_i}{x_i}}.$$

With $x_i = 1 - \xi_i, y_i = 1 - \eta_i$, and $r = \sum_i(\xi_i + \eta_i)$ again, we first integrate over $r < \delta$, where the small δ will be chosen below. Using (14) again, we see that the integrand is at most A^m times

$$a^{-m^2}\, \frac{\Delta(\xi)^2\, \Delta(\eta)^2}{(\sum_i(\xi_i+\eta_i))^{2n^2}} \prod_i \sqrt{\xi_i/\eta_i}.$$

(The factor m^{2n^2} coming from using (14) and a bound for $\prod \sqrt{y_i/x_i}$ appearing in (15) were absorbed into A^m.) When $\xi_i, \eta_i < 1$ we have $\Delta(\xi)^2, \Delta(\eta)^2 < 1$, so integrating with respect to r over $r < \delta$, using homogeneity, gives at most a constant times

$$a^{-m^2} \int_0^\delta r^{2m(m-1)-2n^2+2m-1}\, dr = a^{-m^2} \int_0^\delta r^{2m^2-2n^2-1}\, dr.$$

(The integral of the last factor over $r = 1$ equals $(\pi/2)^m/\Gamma(2m)$.) The integral is $O(\delta^{2m^2})$ since $m > n$, and so the above is exponentially small in m if we choose $\delta^2 < a$.

There remains the integral over the region $r > \delta$, and for this we use the representation (11). We are led to (15) with the second factor replaced by the absolute value of

$$\left(\det\left(\frac{1}{1-\kappa x_i y_j}\right)\right)^2.$$

From (14) we see that in this region the first factor in (15) is at most $(2m/\delta)^{2n^2}$. By (16) and the Hadamard inequality, the square of the determinant has absolute value at most $a^{-2m}\, m^m$. Therefore, the integral over this region has absolute value at most

$$\left(\frac{2m}{\delta}\right)^{2n^2} a^{-2m}\, m^m \int_0^1 \cdots \int_0^1 \prod_i \sqrt{\frac{1-x_i}{1-y_i}\,\frac{y_i}{x_i}}\; \prod_i dx_i\, dy_i.$$

The integral here is A^m, and so we have shown that the integral in the region $r > \delta$ is at most $A^m\, m^m$.

This is a bound for only one term we get when we differentiate $2n^2 - 1$ times the integrand for $\mathcal{S}_m$. The number of factors in the integrand involving κ is $O(m^2)$ so if we differentiate $2n^2 - 1$ times we get a sum of $O(m^{4n^2})$ terms. In each of the other terms the denominator in the first factor has a power no larger than $2n^2$ and at most $2n^2$ extra factors appear which are of the form $(1 - \kappa x_i y_i)^{-1}$, $(1 - \kappa x_i)^{-1}$, or $(1 - \kappa y_i)^{-1}$. Each has absolute value at most a^{-1}, so their product is $O(1)$. It follows that we have the bound $A^m\, m^m$ for the sum of these integrals. Lemma 4 is established. □

Proof of the theorem. Suppose $\kappa \to \epsilon$ radially, where $\epsilon \neq 1$ is a root of unity. It is a primitive nth root of unity for some n. Then $\epsilon^m \neq 1$ when $m < n$ so Lemma 3 applies for these m. Combining this with Lemmas 4 and 2 gives

$$\left(\frac{d}{d\kappa}\right)^{2n^2-1} \mathcal{S} \approx \log(1 - |\kappa|)^{-1}.$$

Therefore, $\mathcal{S}$ cannot be analytically continued beyond any such ϵ, and these are dense in the unit circle. □

Remark. From the proofs of Lemmas 3 and 4, with $2n^2 - 1$ replaced by ℓ in both, one can see that $\mathcal{S}$ extends to a C^ℓ function of κ up to the boundary except for the mth roots of unity with $m \leq \sqrt{(\ell+1)/2}$. In particular, $\mathcal{S}$ extends to a function of class C^6 up to the boundary except for $\kappa = 1$.

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