The Asymmetric Simple Exclusion Process: Integrable Structure & Limit Theorems

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Joint work with Harold Widom

The *asymmetric simple exclusion process* (ASEP): Introduced in 1970 by Frank Spitzer in *Interaction of Markov Processes*

Called the "default stochastic model for transport phenomena" (H.-T. Yau)

ASEP is a model for interacting particles on a lattice

Frank Spitzer

Definition of Model

The ASEP is a model for **interacting particles** on a lattice S, say $S = \mathbb{Z}^d$.

1. A **state** η of the system is a map $\eta : S \to \{0, 1\}$ such that

$$\eta(x) = \begin{cases} 1 & \text{if site } x \in S \text{ is occupied by a particle,} \\ 0 & \text{if site } x \in S \text{ is vacant.} \end{cases}$$

   States $\Omega = \{0, 1\}^S$.

2. Introduce **dynamics**: $t \to \eta_t \in \Omega$:

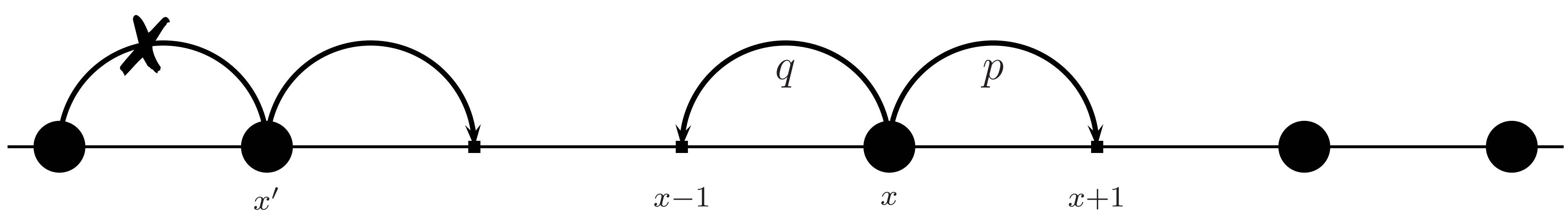
   (a) Each particle $x \in S$ waits exponential time with parameter 1, independently of all other particles;

   (b) at the end of that time, it chooses a $y \in S$ with probability $p(x, y)$; and

   (c) if y is vacant, it goes to y, while if y is occupied, it stays at x and the clock starts over.

ASEP on integer lattice $\mathbb{Z}$

- Asymmetric condition $q > p$, drift to the left
- Continuous time: Zero probability of two clocks going off at same time
- Must specify initial configuration



Clock for particle at x' rings
Jump to left suppressed

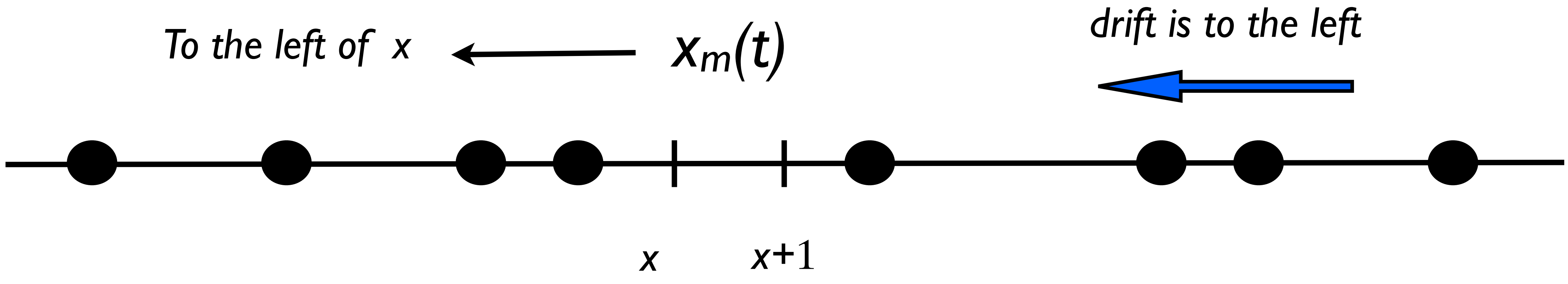
Clock for particle at x rings

Current Fluctuations

$J(x,t)$ = net number of particles through $[x,x+1]$ in time t

$$\{ J(x,t) \geq m\} = \{ x_m(t) \leq x \}$$

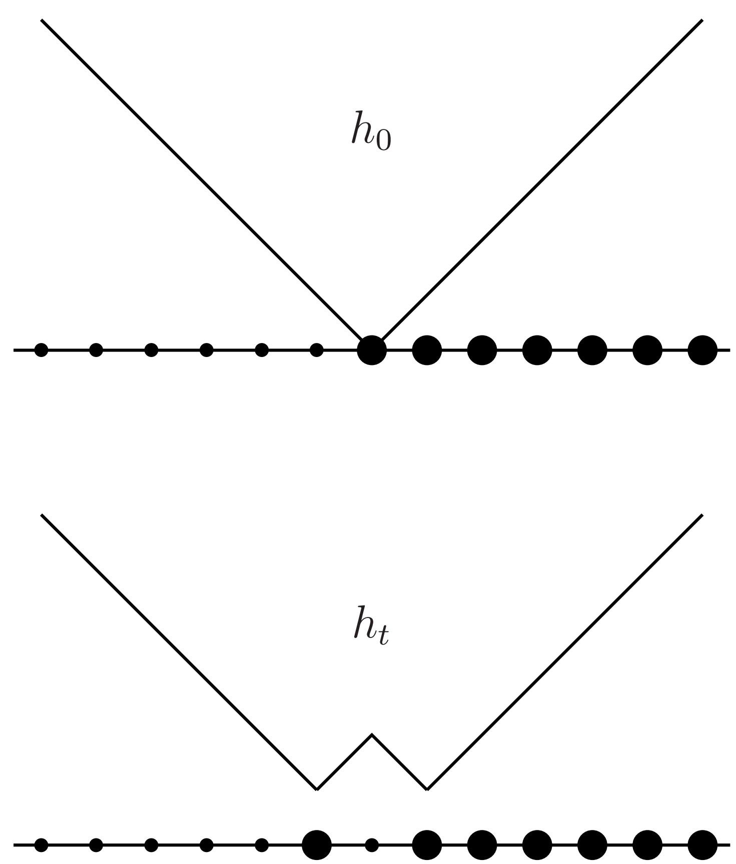
Thus current fluctuations are related to fluctuations in the position of the m^{th} particle



Growth Processes & ASEP

Height function: $h_t(x)$

The rule to construct h_t is: if $\eta_t(x) = 1$ then $h_t(x)$ in the interval $[x, x+1]$ increases with slope $+1$ whereas if $\eta_t(x) = 0$ then $h_t(x)$ decreases in that interval with slope -1.



Step initial condition

$$h_t(x+1) - h_t(x) = 2\eta_t(x) - 1$$

KPZ Equation & Growth Processes

$\hookrightarrow$ Kardar, Parisi & Zhang

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \lambda \left(\frac{\partial h}{\partial x} \right)^2 + w$$

diffusion growth noise

$$u(x,t) = \frac{\partial h}{\partial x} \longrightarrow \text{Noisy Burgers eqn}$$

- Physicists expect KPZ eqn to describe a large class of stochastically growing interfaces: 1+1 KPZ universality class.

- KPZ difficult to handle mathematically

- Natural to make space discrete

- ASEP is expected to be in the KPZ universality class in the long time and large space asymptotic limits.

- Thus asymptotic results for ASEP are expected to have a "universal character"

T(totally)ASEP

TASEP is the case when particles can jump only to the right ($p = 1$) or only to the left ($q = 1$).

The first limit law is due to **Kurt Johansson** (2000) in the case of *step initial condition*:

$$Y = \{1, 2, 3, \ldots, \} = \text{initial location of particles}, \;\; q = 1$$

Let $x_m(t)$ denote the position of the mth particle from the left, set $0 < \sigma = m/t < 1$ then there exist explicit constants c_1 and c_2 (depending upon σ) such that as $m, t \to \infty$

$$\frac{x_m(t) - c_1 t}{c_2 t^{1/3}} \longrightarrow F_2$$

where convergence is in distribution and F_2 is the GUE largest eigenvalue distribution function.

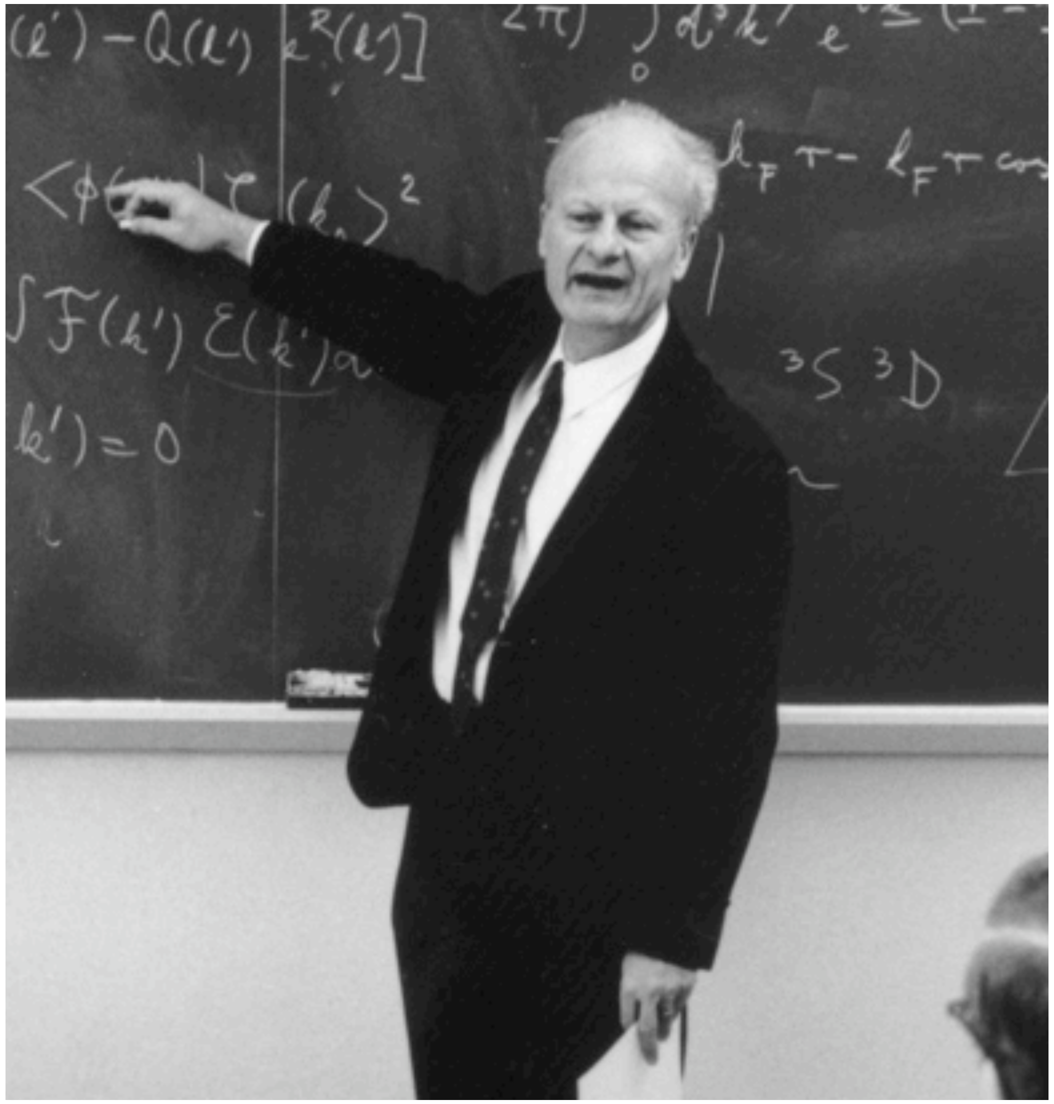
- Note the $\frac{1}{3}$ in the scaling: KPZ Universality Class
- TASEP is a *determinantal process*. Can adapt methods of random matrix theory to prove limit theorem
- Methods do not work for ASEP but do we get the same limit law?

Integrable Structure of ASEP

We solve the Kolmogorov forward equation ("master equation") for the transition probability Y→X:

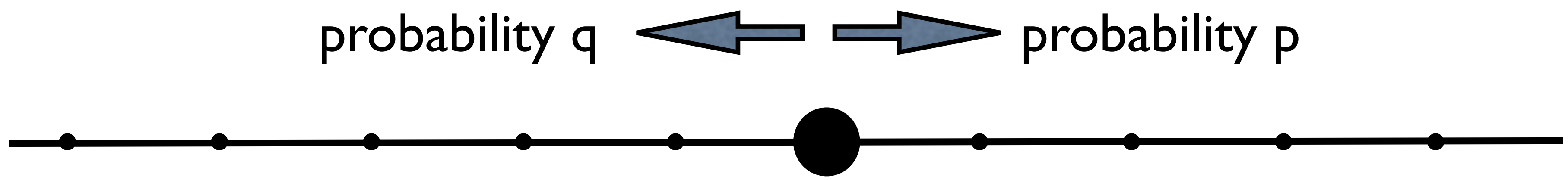
$$P_Y(X;t)$$

Main idea comes from the *Bethe Ansatz* (1931)



Hans Bethe in 1967

N=1 ASEP



$$u(x,t) = \mathbb{P}(\eta_t(x) = 1, \eta_t(y) = 0, y \neq x)$$

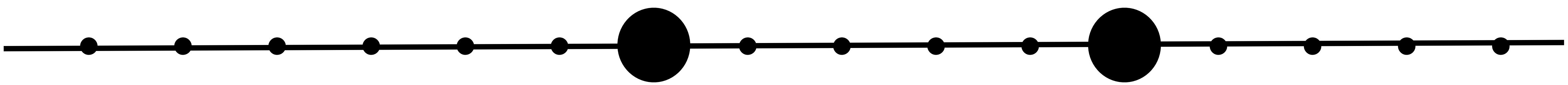
Master equation:

$$\frac{du(x;t)}{dt} = p\,u(x-1;t) + q\,u(x+1;t) - u(x;t)$$

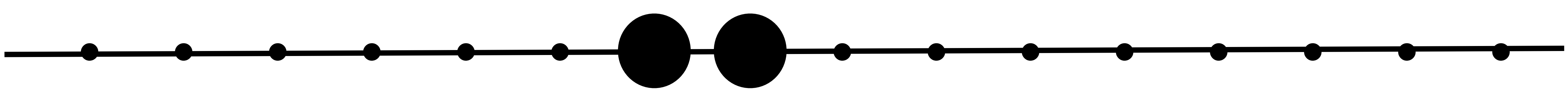
$$u(x;t) = \frac{1}{2\pi i}\int_C \xi^{x-y-1} e^{t\varepsilon(\xi)}\,d\xi$$

$$\varepsilon(\xi) = \frac{p}{\xi} + q\xi - 1$$

N=2 ASEP



Master equation takes simple form for this configuration



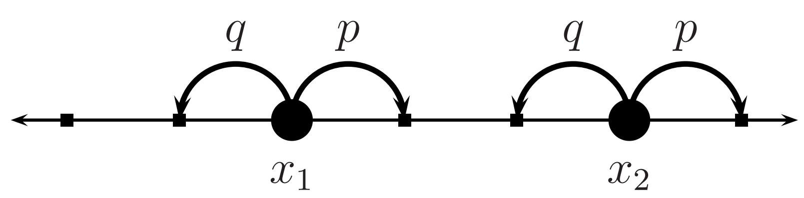
Master equation reflects exclusion for this configuration

Impose boundary conditions for first equation so that if satisfied the second equation is automatically satisfied

Differential Equations for $\mathbb{P}_Y(X;t)$

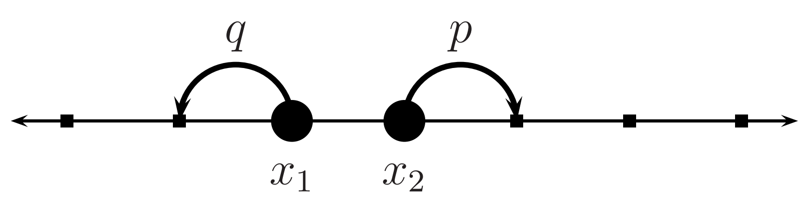
Configuration $X = \{x_1, x_2, \ldots, x_N\}$.

$\mathbf{N = 2}$ particles and $x_1 + 1 < x_2$:



$$\begin{aligned}\frac{d}{dt}u(x_1,x_2) &= p\,u(x_1-1,x_2) + q\,u(x_1+1,x_2)\\ &\quad + p\,u(x_1,x_2-1) + q\,u(x_1,x_2+1) - 2\,u(x_1,x_2).\end{aligned}$$

For $x_1 + 1 = x_2$:



$$\frac{d}{dt}u(x_1,x_2) = p\,u(x_1-1,x_2) + q\,u(x_1,x_2+1) - u(x_1,x_2).$$

Formally subtract when $x_2 = x_1 + 1$ to obtain.

$$0 = p\,u(x_1,x_1) + q\,u(x_1+1,x_1+1) - u(x_1,x_1+1).$$

Treat this as a **boundary condition**. Consider $(x_1,x_2) \in \mathbb{Z}^2$, then

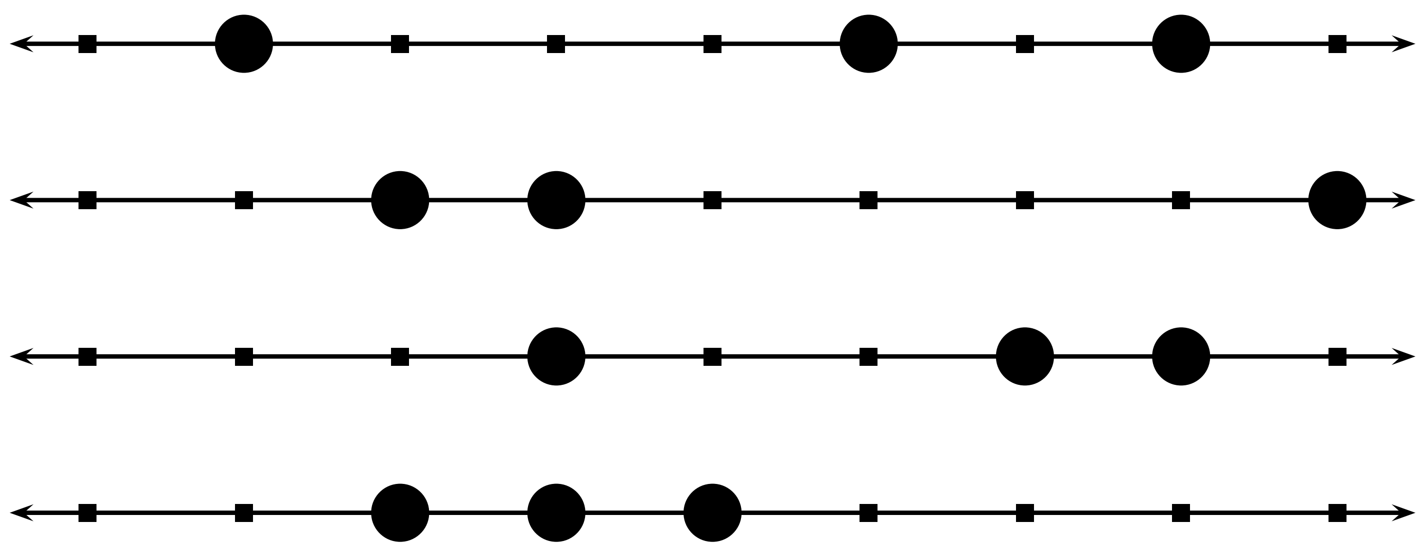
First Equation + Boundary Condition $\Rightarrow$ Second Equation.

Find a solution in $\mathbb{Z}^2$ satisfying first DE, BC and the initial condition in region $x_1 < x_2 \Longrightarrow$ found the probability $\mathbb{P}_Y(X;t)$.

Important Point:
Additional boundary conditions arise when, say,

3 particles are adjacent
4 particles are adjacent
etc.

For example, for $N = 3$ we have four cases



These *new conditions* are automatically satisfied by the boundary conditions arising when **2 particles are adjacent**. The underlying algebraic reason is that the "S-matrix" satisifes the

Yang-Baxter equations

Bethe Ansatz Solution

$$\varepsilon(\xi) := \frac{p}{\xi} + q\,\xi - 1$$

For any $\xi_1, \ldots, \xi_N \in \mathbb{C}\backslash\{0\}$ a solution of the DE is

$$\prod_j \left(\xi_j^{x_j}\; e^{\varepsilon(\xi_j)\,t} \right).$$

For any $\sigma \in \mathcal{S}_N$, another solution is

$$\prod_j \xi_{\sigma(j)}^{x_i} \prod_j e^{\varepsilon(\xi_j)\,t}$$

or any linear combination of these, or any integral of a linear combination.

Bethe Ansatz:

$$u(X;t) = \int \sum_{\sigma \in \mathcal{S}_N} F_\sigma(\xi) \prod_j \xi_{\sigma(j)}^{x_j} \prod_j e^{\varepsilon(\xi_j)\,t}\, d^N \xi.$$

Want the boundary conditions to be satisfied.

Look for F_σ such that the integrand satisfies the BCs *pointwise*: Find condition

$$\frac{F_{T_i \sigma}}{F_\sigma} = -\frac{p + q\xi_{\sigma(i)}\xi_{\sigma(i+1)} - \xi_{\sigma(i+1)}}{p + q\xi_{\sigma(i)}\xi_{\sigma(i+1)} - \xi_{\sigma(i)}}, \; T_i = \text{transposition operator at sites } i, i+1$$

Recognize this as the Yang-Yang S-matrix in the XXZ spin Hamiltonian

$$S_{\alpha\beta} = -\frac{p + q\xi_\alpha \xi_\beta - \xi_\alpha}{p + q\xi_\alpha \xi_\beta - \xi_\beta}$$

Upshot: If we define

$$A_\sigma = \operatorname{sgn}\sigma \, \frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i \xi_j - \xi_i)}$$

then

$e^{t\varepsilon(\xi_i)}$

$$u(X;t) = \sum_\sigma \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} d^N\xi$$

satisfies the master equation + boundary conditions.

The $\sigma = id$ summand satisfies initial condition.

Theorem (TW): If $p \neq 0$ and r is small enough then

$$\mathbb{P}_Y(X;t) = \sum_{\sigma \in \mathcal{S}_N} \int_{\mathcal{C}_r^N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i - 1} \, e^{\varepsilon(\xi_i)\, t} \right) d^N \xi.$$

where

$$A_\sigma = \operatorname{sgn} \sigma \, \frac{\prod_{i<j} (p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j} (p + q\xi_i \xi_j - \xi_i)}$$

and satisfies

$$\mathbb{P}_Y(X;0) = \delta_Y(X).$$

Remarks:

- There is no Ansatz in our work!
- Usual Bethe Ansatz calculates the spectrum of the operator. This leads to transcendental equations for the eigenvalues and issues of completeness of the eigenfunctions.
- We compute the semigroup directly. No spectral theory.

Marginal Distributions
$P(x_m(t) \leq x)$

Case $m=1$:

Fix $x_1=x$, sum $P_Y(X;t)$ over allowed $x_2, x_3, x_4, \ldots$

Can do this since contours are small: $|\xi_i| < 1$

Result is an expression involving N! terms.
Use first *miraculous identity* to reduce sum to one term!

Here's the identity:

First Identity

$$\sum_{\sigma\in\mathcal{S}_N} \operatorname{sgn}\sigma \left(\prod_{i<j}(p+q\xi_{\sigma(i)}\xi_{\sigma(j)}-\xi_{\sigma(i)}) \right.$$

$$\left. \times \frac{\xi_{\sigma(2)}\xi^2_{\sigma(3)}\cdots\xi^{N-1}_{\sigma(N)}}{(1-\xi_{\sigma(2)}\xi_{\sigma(3)}\cdots\xi_{\sigma(N)})(1-\xi_{\sigma(3)}\cdots\xi_{\sigma(N)})\cdots(1-\xi_{\sigma(N)})}\right)$$

$$= p^{N(N-1)/2}\,\frac{(1-\xi_1\cdots\xi_N)\,\prod_{i<j}(\xi_j-\xi_i)}{\prod_i(1-\xi_i)}$$

- Using this identity we get for m=1 an expression for $P(x_1(t) \le x)$ as a single m-dimensional integral with a product integrand. This expression is for finite N ASEP

$$I(x, Y, \xi) \quad = \quad \prod_{i<j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \frac{1 - \xi_1 \cdots \xi_N}{(1 - \xi_1) \cdots (1 - \xi_N)}$$

$$\prod_i \left(\xi_i^{x - y_i - 1} e^{\varepsilon(\xi_i) t} \right)$$

$$\mathbb{P}(x_1(t) = x) \quad = \quad p^{N(N-1)/2} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} I(x, Y, \xi)\, d\xi_1 \cdots d\xi_N$$

$$(p \neq 0)$$

- Sum of N! integrals has been reduced to one integral
- However form is not so useful to take N→∞
- We now expand contour outwards -- only residues that contribute come from ξ=1.
- Can take N→∞ in resulting expression to obtain

$$
\begin{aligned}
\sigma(S): &= \sum_{i\in S} i \\
\mathbb{P}(x_1(t)=x) &= \sum_S \frac{p^{\sigma(S)-|S|}}{q^{\sigma(S)-|S|(|S|+1)/2}} \times \\
&\quad \int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_R} I(x,Y_S,\xi)\, d^{|S|}\xi
\end{aligned}
$$

The sum is over all nonempty subsets of $\mathbb{Z}$

When p=0 only one term is nonzero, S=$\{1\}$.

- To go beyond the left most particle, m=1, there are new complications

- These come from the fact that we must sum over all $x_j > x_m$ and all $x_i < x_m$. Some contours must be small (former) and some must be large (latter) to obtain convergence of geometric series

- This involves finding a new identity

Second Identity

S ranges over subsets of $\{1, 2, \ldots, N\}$

$$\sum_{|S|=m} \prod_{i\in S,\, j\in S^c} \frac{p + q\xi_i\xi_j - \xi_i}{\xi_j - \xi_i} \cdot (1 - \prod_{j\in S^c} \xi_j)$$

$$= q^m \begin{bmatrix} N \\ m \end{bmatrix} (1 - \prod_{j=1}^{N} \xi_j).$$

$$[N] = \frac{p^N - q^N}{p - q}, \qquad [N]! = [N]\,[N-1]\,\cdots\,[1],$$

$$\begin{bmatrix} N \\ m \end{bmatrix} = \frac{[N]!}{[m]!\,[N-m]!}, \qquad (q - \text{binomial coefficient}),$$

Final series result for case $Y = \mathbb{Z}^+$

$$\begin{aligned} \mathbb{P}\left(x_m(t) \leq x\right) &= (-1)^m \sum_{k \geq m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix}_\tau p^{(k-m)(k-m+1)/2} q^{km+(k-m)(k+m-1)/2} \\ &\times \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{1}{(1-\xi_i)(q\xi_i - p)} \\ &\times \prod_i \left(\xi_i^x e^{\varepsilon(\xi_i)t} \right) d\xi_1 \cdots d\xi_k \end{aligned}$$

- For *p=0* only *k=m* term is nonzero

- Recognize double product as a determinant whose entries are a kernel, i.e. $K(\xi_i,\xi_j)$

- Result can then be expressed as a contour integral whose integrand is a Fredholm determinant

Fredholm determinant

- Let K(x,y) be a kernel function
- Fredholm expansion of det(I-λK):

$$\frac{(-1)^n}{n!}\int\cdots\int \det\left(K(\xi_i,\xi_j)\right)_{1\le i,j\le n}\, d\xi_1\cdots d\xi_n =$$

$$\int_{\mathcal{C}} \det\left(I-\lambda K\right)\,\frac{d\lambda}{\lambda^{n+1}}$$

•Can then do sum over k (q-Binomial theorem):

Final expression for mth particle distribution fn. Step initial condition

Set $\gamma = q - p > 0$, $\tau = q/p$ and define an integral operator K on circle C_R:

$$K(\xi, \xi') = q\,\frac{\xi'^{x} e^{t\varepsilon(\xi')/\gamma}}{p + q\xi\xi' - \xi}$$

Then

$$\mathbb{P}\left(x_m(t/\gamma) \le x\right) = \int \frac{\det(I - \lambda K)}{\prod_{k=0}^{m-1}(1 - \lambda\tau^{k-1})}\,\frac{d\lambda}{\lambda}$$

↑ k-1→k

The contour in the λ-plane encloses all of the singularities of the integrand.

Asymptotic analysis

We now transform the operator K so that we can perform a steepest descent analysis.

Recall that the generic behavior for the coalescence of two saddle points leads to the Airy function *Ai(x)*

George Airy

$$\xi \longrightarrow \frac{1-\tau\eta}{1-\eta}, \quad \tau = \frac{p}{q} < 1,$$

$$K(\xi,\xi') \longrightarrow K_2(\eta,\eta') = \frac{\varphi(\eta')}{\eta' - \tau\eta}$$

$$\varphi(\eta) = \left(\frac{1-\tau\eta}{1-\eta}\right)^x e^{\left[\frac{1}{1-\eta} - \frac{1}{1-\tau\eta}\right]\mathbf{t}}$$

$$\text{Introduce:} \quad K_1(\eta,\eta') = \frac{\varphi(\tau\eta)}{\eta' - \tau\eta}$$

Proposition:

Let Γ be any closed curve going around $\eta=1$ once counterclockwise with $\eta=1/\tau$ on the outside. Then the Fredholm determinant of $K(\xi,\xi')$ acting on C_R has the same Fredholm determinant as $K_1(\eta,\eta')-K_2(\eta,\eta')$ acting on Γ.

Proposition:

Suppose the contour Γ is star-shaped with respect to $\eta=0$. Then the Fredholm determinant of K_1 acting on Γ is equal to

$$\prod_{k=0}^{\infty}(1-\lambda\tau^k)$$

Denote by R the resolvent kernel of K_1

Factor determinant:

$\det(I-\lambda K)=\det(I-\lambda K_1)\det(I+K_2(I+R))$

Set $\lambda=\tau^{-m}\mu$ so formula for distr. fn becomes

$$\int \prod_{k=0}^{\infty}(1-\mu\tau^k)\det\big(I+\tau^{-m}\mu K_2(I+R)\big)\ \frac{d\mu}{\mu}$$

μ runs over a circle of radius $> \tau$

By a perturbative expansion of R, followed by a deformation of operators, we show

$$
\begin{aligned}
\det\left(I+\lambda K_2(I+R)\right) &= \det\left(I+\mu J\right)\\
J(\eta,\eta') &= \int \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')}\,\frac{\zeta^m}{(\eta')^{m+1}}\,\frac{f(\mu,\zeta/\eta')}{\zeta-\eta}\,d\zeta\\
\varphi_\infty(\eta) &= (1-\eta)^{-x}\,e^{\frac{\eta t}{1-\eta}}\\
f(\mu,z) &= \sum_{k=-\infty}^{\infty}\frac{\tau^k}{1-\tau^k\mu}\,z^k
\end{aligned}
$$

The kernel $J(\eta,\eta')$, which acts on a circle centered at 0 with radius less than τ, is analyzed by the steepest descent method.

Note: *m* now appears inside the kernel!

Main Result

We set

$$\sigma = \frac{m}{t}, c_1 = -1+2\sqrt{\sigma}, c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}, \gamma = q-p$$

Theorem (TW). When $0 \le p < q$ we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_m(t/\gamma) - c_1 t}{c_2 t^{1/3}} \le s\right) = F_2(s)$$

This theorem establishes KPZ Universality for ASEP with step initial condition

Painlevé II Representation of F_2

$$
\begin{aligned}
F_2(s) &= \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right) \\
\frac{d^2q}{dx^2} &= xq + 2q^3, \quad \text{Painlevé II} \\
q(x) &\sim \mathrm{Ai}(x), \ \ x \to \infty
\end{aligned}
$$

This q is called the *Hastings-McLeod* solution.

First Appearance of F_2 in growth processes
Baik, Deift & Johansson

Patience Sorting (Aldous & Diaconis)

$$\sigma = \{\ 6 \quad 7 \quad 1 \quad 8 \quad 5 \quad 4 \quad 10 \quad 9 \quad 2 \quad 3\ \}$$

First Appearance of F_2 in growth processes
Baik, Deift & Johansson

Patience Sorting (Aldous & Diaconis)

σ = { }

	2		
	4		
1	5	3	9
6	7	8	10

$$\ell(\sigma) = \text{ Number of piles } = 4$$

Baik-Deift-Johansson Theorem

Theorem. Given a random permutation $\sigma \in \mathcal{S}_n$, let $\ell(\sigma)$ equal the number of piles resulting from the patience sorting algorithm. Then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma) - 2\sqrt{n}}{n^{1/6}} \le s\right) = F_2(s).$$

- Johannsson showed F_2 arises in a last passage percolation model (corner growth) which includes TASEP with step initial condition.
- TASEP with flat initial conditions leads to F_1, Sasamoto, Borodin, Ferrari, ...

Can also study limit for fixed *m*, *t*→∞

Theorem: Assume $0 < p < q$. For fixed m the limit

$$\mathbb{P}\left(\frac{x_m(t/\gamma)+t}{\sqrt{\gamma}\sqrt{t}} \le s\right)$$

equals

$$\int \frac{\det(I-\lambda K)}{\prod_{k=0}^{m-1}(1-\lambda\tau^k)}\,\frac{d\lambda}{\lambda}$$

where now K has kernel (acting on $\mathbb{R}$)

$$K(z,z') = \frac{q}{\sqrt{2\pi}}\exp\left(-(p^2+q^2)(z^2+z'^2)/4 + pqzz'\right)$$

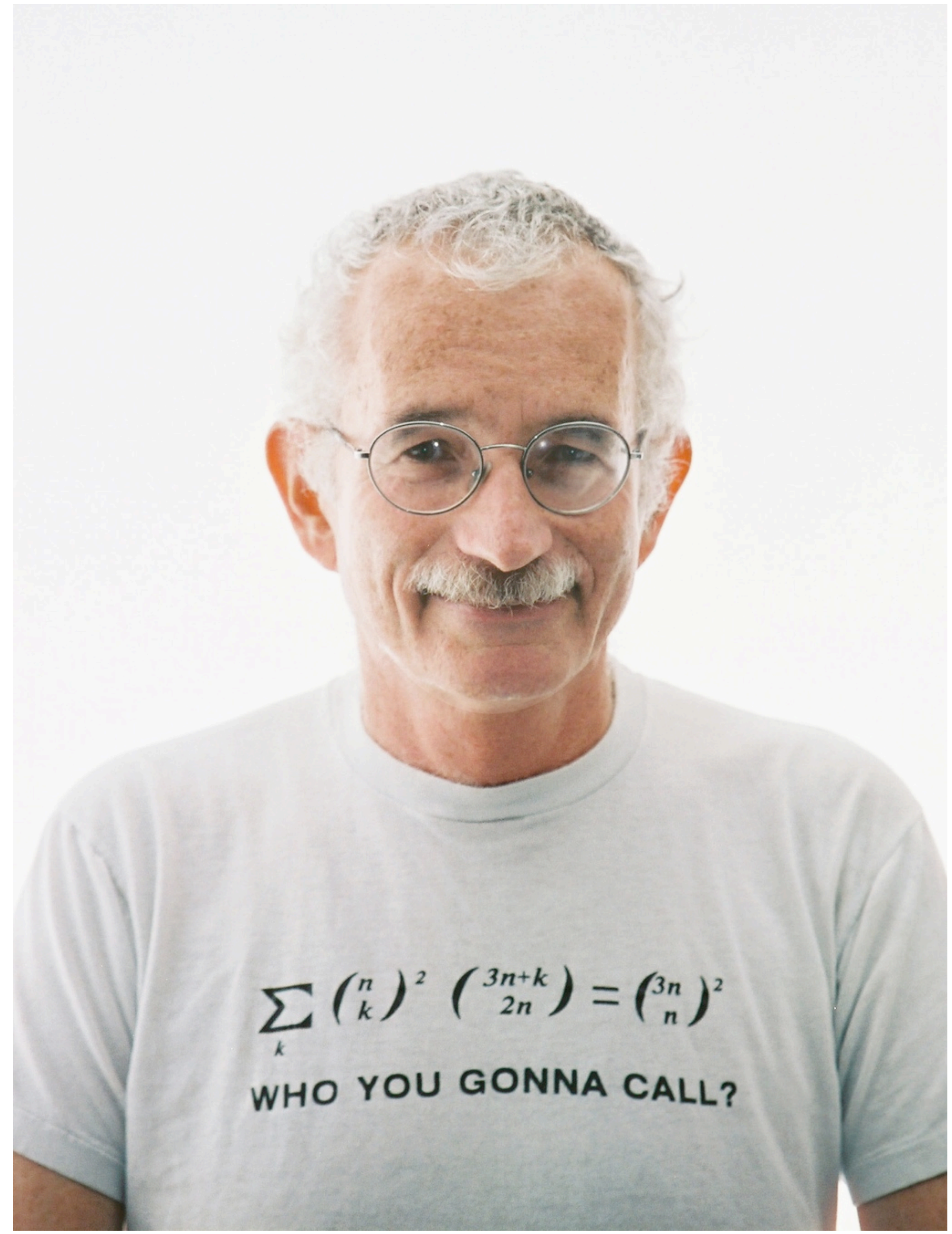
- Would like a conceptual understanding of why identities & cancellations appear in ASEP proofs.

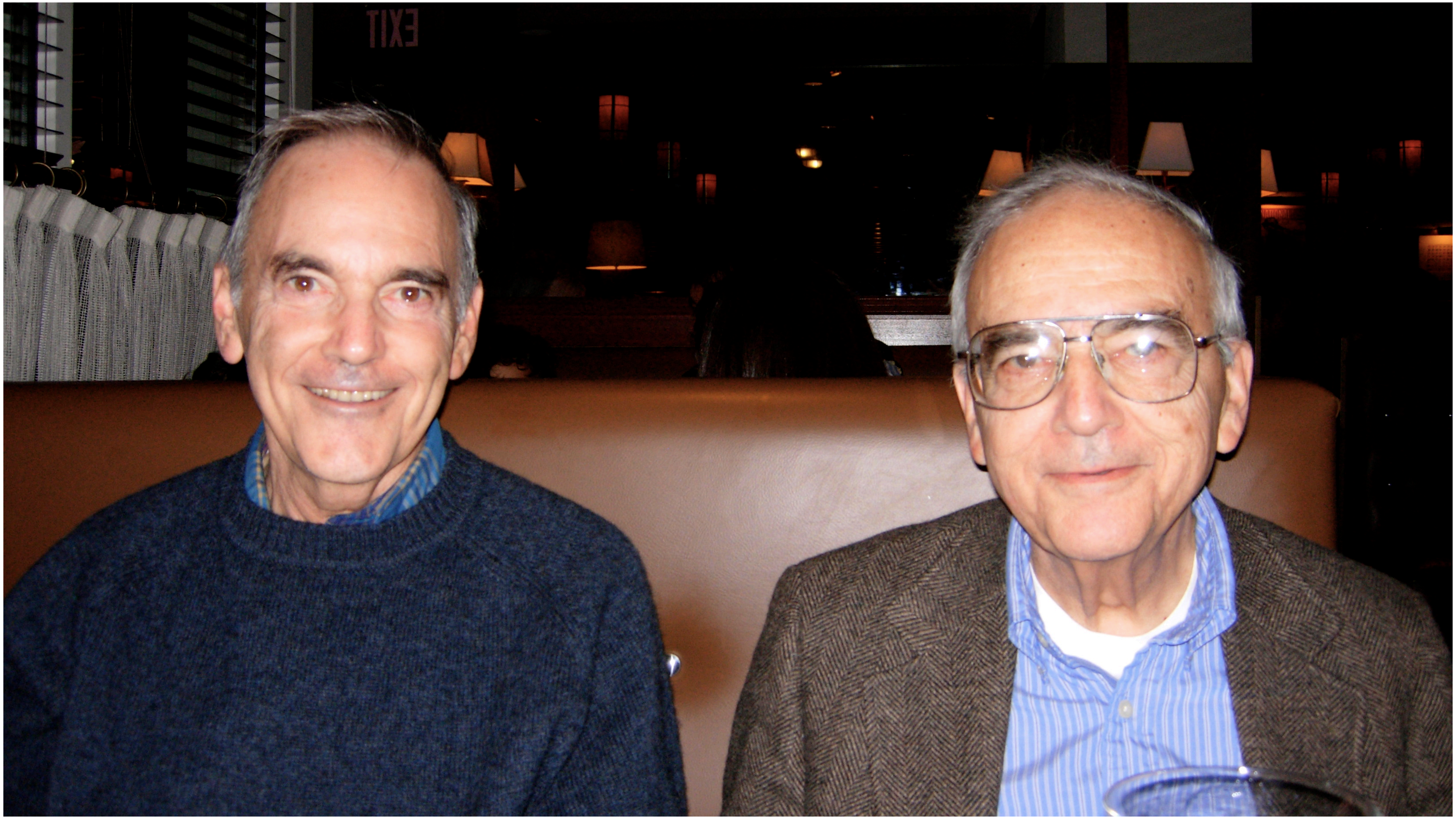
- Extend ASEP results to other initial conditions, e.g. flat initial conditions. Do we see F_1 as in TASEP?

- Can we apply Bethe Ansatz methods to other growth models?

- Ultimately we want universality theorems not to rely upon integrable stucture of ASEP. For ⅓ exponent progress by Balázs, Seppäläinen, Quastel & Valkó.

Thanks to Anne Schilling & Doron Zeilberger for advice with the combinatorial identities





Harold Widom (left) and his brother Ben