Turbulent Liquid Crystals KPZ Universality and the Asymmetric Simple Exclusion Process

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Outline

- Stochastic growth processes

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- The Back Story: Random matrices & Random Permutations
- Exact solution of KPZ equation: Work of Amir-Corwin-Quastel & Sasamoto-Spohn
- Exact distribution from ASEP needed for KPZ analysis, C.T. & Widom (TW)

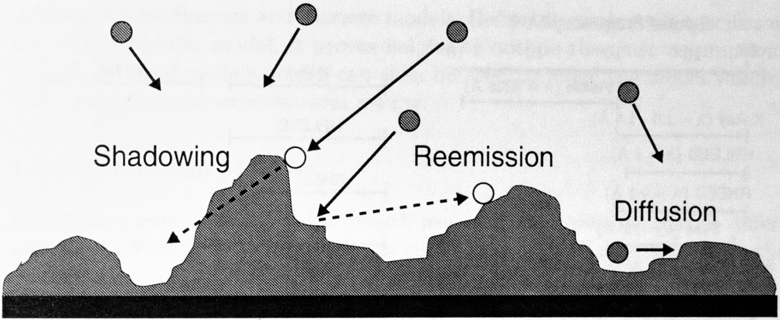


Fig. 1.3. Diagram of growth effects including diffusion, shadowing, and reemission that may affect surface morphology during thin film growth. The incident particle flux may arrive at the surface with a wide angular distribution depending on the deposition methods and parameters.

Figure: Want the (random) height function $h = h(x, t)$

Modelling Growth Processes

$$\frac{\partial h}{\partial t} = \Phi\left(h, x, t\right) + W(x, t)$$

$$h = h(x, t) = \text{height function}$$

$$\Phi \longrightarrow \text{captures growth effects to be modelled}$$

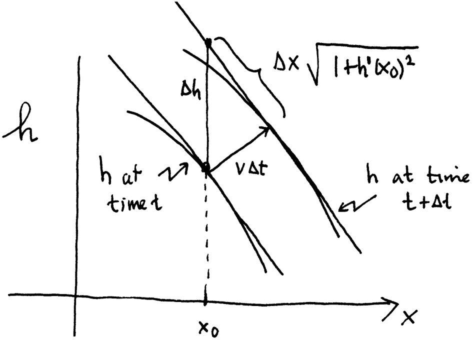
$$W \longrightarrow \text{noise term}$$

This is a **nonlinear stochastic PDE**

Discrete versions are also popular models

Kardar-Parisi-Zhang–1986

Growth occurs normal to the surface



$$
\begin{aligned}
\Delta h &= h(x_0, t+\Delta t) - h(x_0, t) \\
&= v\Delta t\sqrt{1 + h'(x_0)^2} \\
&\approx v\Delta t\left[1 + \frac{1}{2}h'(x_0)^2\right]
\end{aligned}
$$

Thus want $\left(\frac{\partial h}{\partial x}\right)^2$ term in Φ.

KPZ Equation

$$\frac{\partial h}{\partial t} = \nu \underbrace{\frac{\partial^2 h}{\partial x^2}}_{\text{diffusion}} + \lambda \underbrace{\left(\frac{\partial h}{\partial x}\right)^2}_{\text{growth}} + \underbrace{W}_{\text{noise}}$$

- Nonlinear stochastic PDE.
- Difficult to make rigorous sense due to nonlinear growth term.
- KPZ made important prediction as $t \to \infty$

$$h(x,t) = \underbrace{v_\infty t}_{\text{deterministic linear growth}} + \underbrace{t^{1/3}}_{\frac{1}{3}\text{ fluctuations}} \chi$$

Famous KPZ $\frac{1}{3}$ exponent. χ is a fluctuating quantity—no prediction from KPZ phenomenology.

Experiments

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- An early experiment (2003 Myllys, Timonen, ...) measured the "smouldering fronts in paper sheets" and determined that fluctuations were of order $1/3$ demonstrating growth is in KPZ universality class.

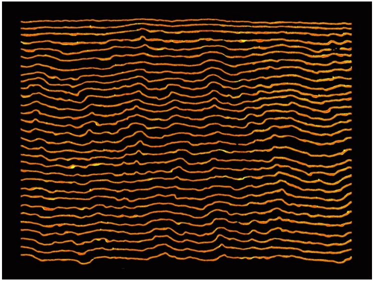


Figure: Digitized slow-combustion fronts with 10 s intervals. Courtesy of M. Myllys.

- Takeuchi & Sano, 2010–11: Convection of nematic liquid crystal driven by an electric field. They focus on the **interface between two turbulent states**. A thin square container is filled with a liquid crystal. The liquid crystal molecules, initially aligned perpendicular to the cell surfaces, strongly fluctuate when an AC voltage is applied leading to first turbulent state. A laser pulse nucleates a defect in the liquid crystal causing a second turbulent state.

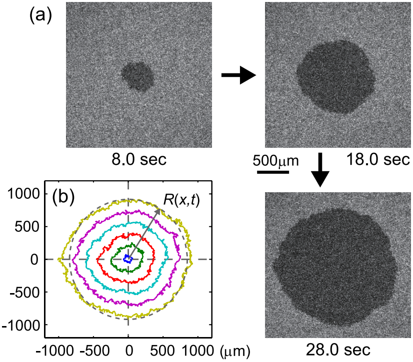
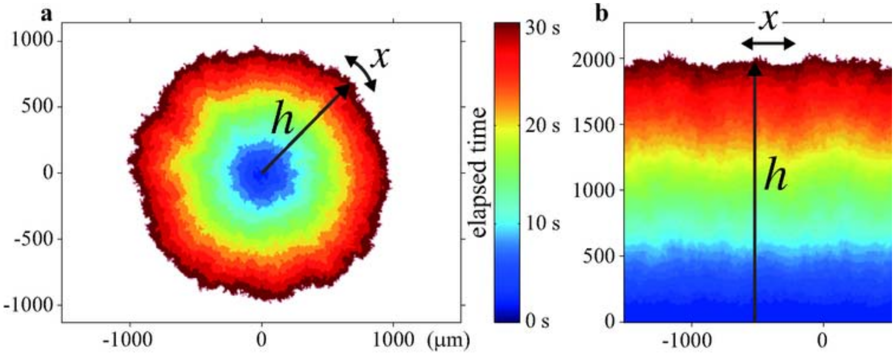


Figure: Growing droplet. Courtesy of K. Takeuchi.

- K. A. Takeuchi and M. Sano, *Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals*, Phys. Rev. Let. **104**, 230601 (2010)
- K. A. Takeuchi, M. Sano, T. Sasamoto and H. Spohn, *Growing interfaces uncover universal fluctuations behind scale invariance*, Scientific Reports 1:34 (2011)

Droplet initial conditions

Flat initial conditions



Binarised snapshots at successive times are shown with different colours. Indicated in the colour bar is the elapsed time after the laser emission. The local height $h(x, t)$ is defined in each case as a function of the lateral coordinate x along the mean profile of the interface (a circle for a and a horizontal line for b). See also **Supplementary Movies 1 and 2**.

Figure: The height function $h = h(x, t)$ for droplet and flat initial conditions

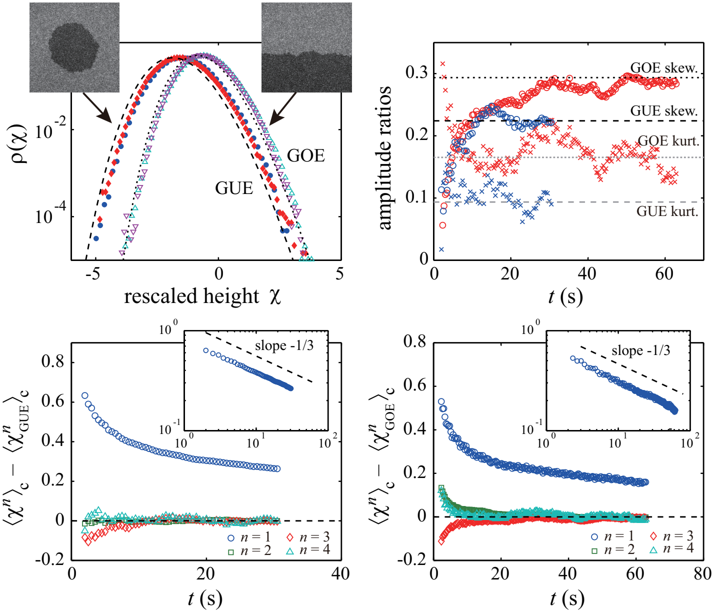


Figure: Comparison with theoretical predictions. Courtesy of K. Takeuchi.

Random Matrices

$$\Omega_N = \{A : N \times N \ \text{ complex hermitian or real symmetric matrix}\}$$

Gaussian Measure:

$$e^{-\mathrm{tr}(A^2)}\ dA$$

Gaussian **U**nitary **E**nsemble for hermitian matrices (GUE)

Gaussian **O**rthogonal **E**nsemble for real symmetric matrices (GOE)
Eigenvalues are random variables

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N$$

Want distribution of **largest eigenvalue** λ_N.

Actually, want *limiting distribution* as size $N \to \infty$

Must center and normalize random variable λ_N to obtain a *limit law*.

Largest Eigenvalue Distributions (TW, 1990s)

- Unitary symmetry (GUE, $\beta = 2$)

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx \right)$$

- Orthogonal symmetry (GOE, $\beta = 1$)

$$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty \left(q(x) + (x-s)q^2(x)\right)\,dx \right)$$

- $q = q(x)$ is the unique solution to Painlevé II

$$q'' = xq + 2q^3$$

satisfying

$$q(x) \sim \mathrm{Ai}(x), \;\; x \to \infty$$

RMT Universality Theorems

- **Invariant measures**:

$$e^{-\mathrm{tr}(A^2)} \longrightarrow e^{-\mathrm{tr}(V(A))}, \quad V(A) = A^{2n} + \sum_{i<2n} c_i A^i$$

  - **Hermitian case**: Limit law F_2, Deift, Kriecherbauer, McLaughlin, Venakides & Zhou
  - **Real symmetric case**: Limit law F_1, Deift & Gioev
  - Early special case $V(A) = A^4 - gA^2$: Its, Bleher, Stojanovic

- **Wigner Matrices**: Entries are iid random variables
  - Soshnikov (1999)—odd moments zero & conditions on decay of distribution
  - Tao & Vu (2010)—needs only third moment to vanish to get limit law

Random Permutations

$$\mathcal{S}_N = \text{Permutations of } \{1, 2, \dots, N\}$$

Assign equal probability to each $\sigma \in \mathcal{S}_N$.

$$\ell_N(\sigma) = \text{Length of longest increasing subsequence in } \sigma$$

$$\sigma = (\mathbf{2}, 8, \mathbf{4}, 1, \mathbf{5}, 3, 9, \mathbf{7}, \mathbf{10}, 6) \in \mathcal{S}_{10}$$

$$\ell_{10}(\sigma) = 5$$

Theorem (Baik, Deift, Johansson, 1999):

$$\lim_{N \to \infty} \mathbb{P}\left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \leq x \right) = F_2(x)$$

The **Baik-Deift-Johansson theorem** opened a new chapter in growth processes and their connections with RMT largest eigenvalue distributions.

Building on work of **Baik** and **Rains**, **Prähofer & Spohn**, in the context of a discrete growth model called PNG (polynuclear growth), predicted that in KPZ growth

- Growth from a droplet: F_2 and at the process level Airy_2 *process*.
- Growth from a flat substrate: F_1 and (later) the Airy_1 *process*.

But what does the original KPZ equation predict?

KPZ & Stochastic Heat Equation

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  1. Define solution to KPZ equation through a **Hopf-Cole transformation**

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where Z satisfies the stochastic heat equation

$$\frac{\partial Z}{\partial T} = \frac{1}{2}\frac{\partial^2 Z}{\partial X^2} - Z\,W$$

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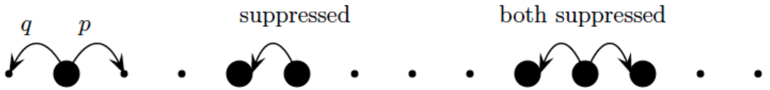
  2. $Z(T, X)$ is obtained from a weakly asymmetric simple exclusion process (WASEP)
- For wedge initial conditions, in 2010 Sasamoto/Spohn and Amir/Corwin/Quastel carried this program out which required new theorems about the relation between stochastic heat equation and WASEP. Both groups used the ASEP results of TW which required a very delicate asymptotic analysis of the TW formula.

ASEP on Integer Lattice



- Each particle has an independent clock—when it rings with probability p (q) it makes a jump to the right (left) if site empty; otherwise, jump is suppressed.

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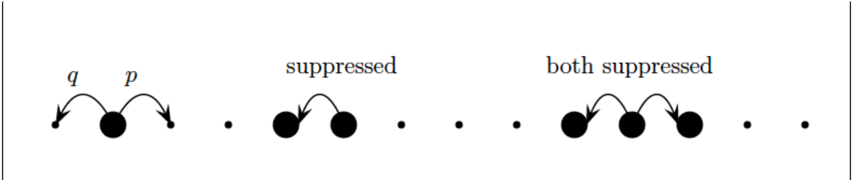

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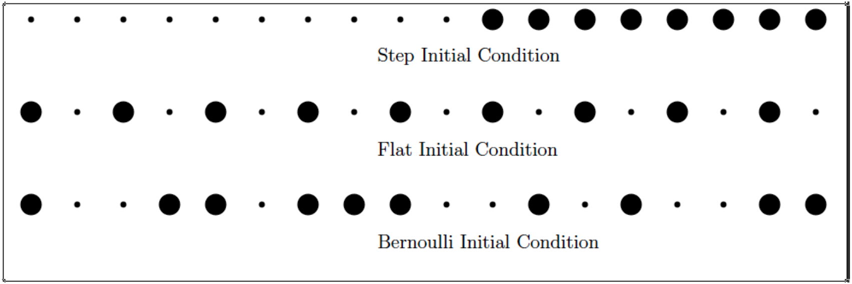
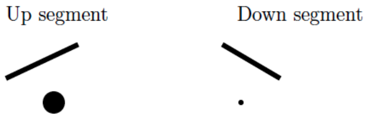
- Initial conditions:



Mapping to Growing Interface



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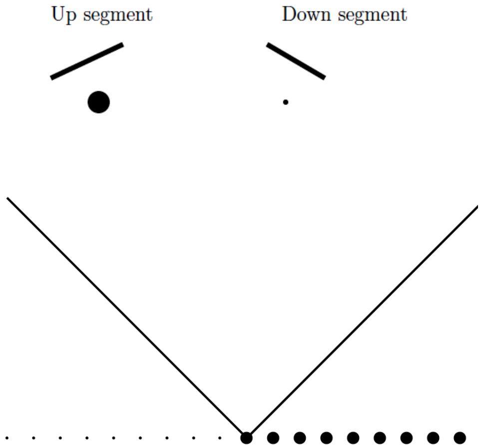

Initial height function corresponding to step initial condition

$$h(x, 0) = |x|$$

Discrete $Z_\varepsilon(T, X)$

BERTINI & GIACOMIN, SASAMOTO & SPOHN, AMIR, CORWIN & QUASTEL:

$$Z_\varepsilon(T,X) = \frac{1}{2}\varepsilon^{-1/2}\exp\left[-\lambda_\varepsilon h(\frac{t}{\gamma}, x) + \nu_\varepsilon \varepsilon^{-1/2} t\right]$$

where

$$t = \varepsilon^{-3/2}T,\ \ x = \varepsilon^{-1}X,\ \ \gamma = q - p = \varepsilon^{1/2}$$

$$\nu_\varepsilon = \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2,\ \ \lambda_\varepsilon = \varepsilon^{1/2} + \frac{1}{3}\varepsilon^{3/2}$$

Need ASEP formula for $h(t,x)$ and then let $\varepsilon \to 0$

$h(t,x)$ for ASEP
Step I—Transition probability for N-particle system

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- First compute for N-particle ASEP

$$\begin{aligned}\mathbb{P}_Y(X;t) \;&=\; \text{probability of configuration } X \text{ at time } t \\ &\quad\;\; \text{given the initial configuration is } Y \text{ at } t=0.\end{aligned}$$

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- Write **master equation** (forward equation) for $\mathbb{P}_Y(X;t)$ and use ideas from **Bethe Ansatz**: Incorporate the interaction between particles into the boundary conditions of a **free particle system**.
- Want solution to master equation that obeys the **initial condition**

$$\mathbb{P}_Y(X;0) = \delta_{X,Y}$$

Satisfying the initial condition is the hard part!

$\mathcal{S}_N$ denotes the permutation group on N symbols, $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathcal{S}_N$

Theorem (TW):

$$\mathbb{P}_Y(X;t) = \sum_{\sigma \in \mathcal{S}_N} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} A_\sigma(\xi) \prod_{i=1}^{N} \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{t\varepsilon(\xi_i)} \, d^N\xi$$

where

$$\begin{aligned}
\varepsilon(\xi_i) &= \frac{p}{\xi_i} + q\xi_i - 1 \\
A_\sigma(\xi) &= \prod_{\text{inversions } (\beta,\alpha) \text{ of } \sigma} S(\xi_\beta, \xi_\alpha) \\
S(\xi, \xi') &= -\frac{p + q\xi\xi' - \xi}{p + q\xi\xi' - \xi'} \\
\mathcal{C} &= \text{sufficiently small circle about zero} \\
&\quad \text{i.e. all poles of } A_\sigma \text{ lie outside of } \mathcal{C}
\end{aligned}$$

and each factor $d\xi_i$ carries a factor $\frac{1}{2\pi i}$.

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$$\sum_\sigma \mathrm{sgn}(\sigma) \Bigg(\prod_{i<j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)}) \times$$

$$\frac{\xi_{\sigma(2)}\xi_{\sigma(3)}^2 \cdots \xi_{\sigma(N)}^{N-1}}{(1-\xi_{\sigma(1)}\xi_{\sigma(2)}\cdots\xi_{\sigma(N)})(1-\xi_{\sigma(2)}\cdots\xi_{\sigma(N)})\cdots(1-\xi_{\sigma(N)})}\Bigg)$$

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  where $f(\xi, \xi') = p + q\xi\xi' - \xi$
- Surprisingly this equals

$$p^{N(N-1)} \frac{\prod_{i<j}(\xi_j - \xi_i)}{\prod_i (1-\xi_i)}$$

Story behind proof of identity

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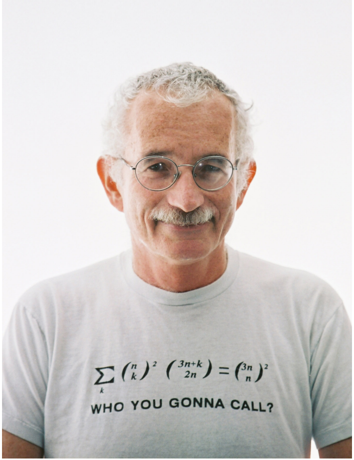
- 

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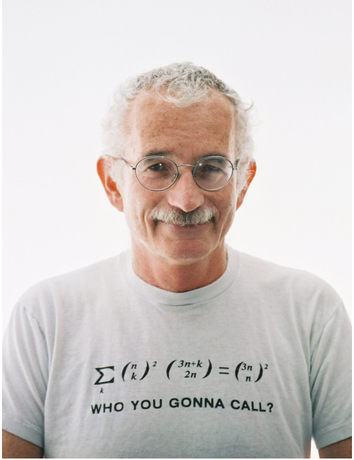
- 

- Doron saw the identity when it was still a conjecture and suggested to the authors that an identity of I. Schur (Problem VII.47 in Polya & Szegö) had a similar look about it and might be proved in a similar way. This led to the proof.

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- For *step initial condition* and a final symmetrization of the integrand leads to

$$
\begin{aligned}
\mathbb{P}(x_m(t) \le x) \quad &= \quad (-1)^m \sum_{k \ge m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix}_\tau \tau^{m(m-1)/2 - mk + k/2} (pq)^{k^2/2} \\
&\times \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \prod_{i \ne j} \frac{\xi_j - \xi_i}{f(\xi_i, \xi_j)} \prod_i \frac{\xi_i^x e^{t\varepsilon(\xi_i)}}{(1-\xi_i)(q\xi_i - p)} \, d\xi_1 \cdots d\xi_k
\end{aligned}
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  where $\left[{n \atop k}\right]_\tau$ is the τ-binomial coefficient and $\mathcal{C}_R$ is a large contour about zero, i.e. no poles outside of contour.
- Unfortunately, we are unable to perform an asymptotic analysis at this stage. Have similar formulas for other initial conditions.

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- With this determinant identity, recognize the kth term to be the kth term in the Fredholm expansion times some coefficients. This together with the τ-binomial theorem gives

$$\mathbb{P}_{\mathbb{Z}^+}(x_m(t) \leq x) = \int \frac{\det(I - \lambda K)}{\prod_{k=0}^{m-1}(1 - \lambda\tau^k)} \frac{d\lambda}{\lambda}$$

$$K(\xi, \xi') = q \frac{\xi^x e^{t\varepsilon(\xi)}}{f(\xi, \xi')}$$

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- However, still unable to do asymptotic analysis! The operators K have exponentially large norms as $t \to \infty$.
- The idea is to replace K with operators with the same Fredholm determinant but better behaved norms.

Limit Theorems

Theorem (TW) Let $m=[\sigma t]$, $\gamma=q-p$ fixed, then

$$\lim_{t\to\infty}\mathbb{P}_{\mathbb{Z}^+}\left(x_m(t/\gamma)\le c_1(\sigma)t+c_2(\sigma)\,s\,t^{1/3}\right)=F_2(s)$$

uniformly for σ in compact subsets of $(0,1)$ where $c_1(\sigma)=-1+2\sqrt{\sigma}$, $c_2(\sigma)=\sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}$.

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Theorem (ACQ, SS) Let

$$Z_\varepsilon(T,X) = p(T,X) e^{F_\varepsilon(T,X)}, \;\; p = \text{heat kernel}$$

then

$$F_T(s) = \lim_{\varepsilon\to 0} \mathbb{P}(F_\varepsilon(T,X) + \frac{T}{4!} \le s) = \text{KPZ crossover distribution}$$

Remark: Explicit formulas for $F_T(s)$.

Corollary(ACQ, SS)

$$\lim_{T \to \infty} F_T \left(2^{-1/3} T^{1/3} s \right) = F_2(s)$$

The KPZ Equation is in the KPZ Universality Class!

Corollary(ACQ, SS)

$$\lim_{T\to\infty} F_T\left(2^{-1/3}T^{1/3}s\right) = F_2(s)$$

The KPZ Equation is in the KPZ Universality Class!

Summary of KPZ Universality

- **Scaling exponent** $\frac{1}{3}$ does not depend upon initial configuration
- **Droplet initial conditions**: Long time one-point fluctuations described by $\mathbf{F_2}$.
- **Flat initial conditions**: Long time one-point fluctuations described by $\mathbf{F_1}$. Not (yet) a rigorous proof of this for KPZ equation.

Thank you for your attention