Integrable Models in Statistical Physics & Associated Universality Theorems & Conjectures

Aisenstadt Chair Lecture

March 6, 2009

Craig A. Tracy

UC Davis

Theme of Lecture

Simple models ➠ General Theory

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- Classical harmonic oscillator ➠ Theory of small oscillations

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- Classical harmonic oscillator ➠ Theory of small oscillations
- Quantum harmonic oscillator ➠ "This example is of importance for general theory, because it forms a corner-stone in the theory of radiation", P.A.M. Dirac

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- Ideal gas ➠ Formulation of statistical mechanics by Gibbs

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- Ideal gas ➠ Formulation of statistical mechanics by Gibbs
- Schrödinger's solution of the Hydrogen atom + Pauli principle ➠ Theory of atomic structure
- The problem, of course, is to choose the "right" simple model!

Three examples from statistical physics

- The 2D Ising Model
- Random Matrix Theory
- Asymmetric Simple Exclusion Process

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- Random Matrix Theory
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Integrable Structure

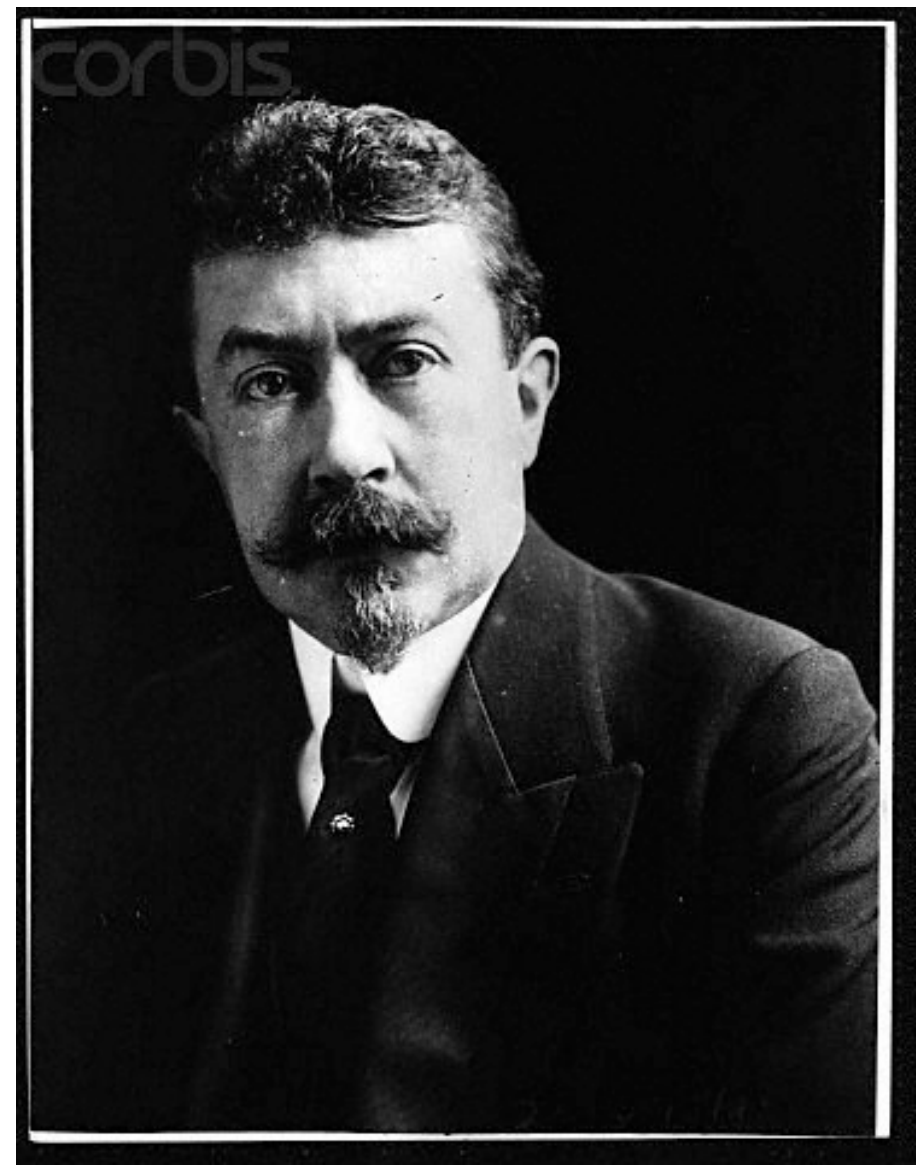
&

Universality Theorems/Conjectures

Integrable Structure

- Fredholm determinants
- Painlevé functions

Erik Ivar Fredholm
1866-1927



Paul Painlevé
1863-1933

2D Ising Model:
Simple model of a magnet

Lars Onsager
1903-1976

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- Renormalization group (K. Wilson)

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Spontaneous Magnetization

$$M = \left(1 - \left[\sinh\left(\log(1+\sqrt{2})\frac{T_{\rm C}}{T}\right)\right]^{-4}\right)^{1/8}$$

C. N. Yang

The ⅛ is the "universal" part for a large class of ferromagnetic 2D systems, critical exponent

Ising Correlations

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Probability of C = $\mathrm{Exp}(-\beta\, \mathcal{E}_\Lambda(C))/Z_\Lambda(\beta)$

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Gibbs measure =

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$\langle \sigma_0 \sigma_x \rangle$ = spin-spin correlation = $\lim_{\Lambda \to \infty} \mathbf{E}(\sigma_0 \sigma_x)$

Determinantal Structure of 2D Ising Model on Square Lattice

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Onsager, Montroll, Potts & Ward

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$\rightsquigarrow$ Strong Szegö limit theorem

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What about correlation functions near the critical temperature?

Scaling Limit

$\xi(T)$=correlation length$\rightarrow\infty$ as $T\rightarrow T_c^{\pm}$,

R=distance between two spins$\rightarrow\infty$

such that

$$r=R/\xi(T) \text{ is fixed}$$

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F_- = Fredholm det

F_+ = Fredholm det $\times$ extra factor

Theorem.(Wu, McCoy, Barouch & C.T., 1973–77)

$$F_\pm(r) \quad = \quad \left\{ \begin{array}{l} \sinh \psi(r)/2 \\ \cosh \psi(r)/2 \end{array} \right. \quad \times$$
$$\exp\left(-\frac{1}{4} \int_r^\infty \left(\frac{d\psi}{dy}\right)^2 - \sinh^2 \psi(y)\, dy \right)$$

where

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} \quad = \quad \frac{1}{2}\,\sinh(2\psi)$$
$$\psi(r) \quad \sim \quad \frac{2}{\pi}\, K_0(r), \;\; r \to \infty$$

Note: $\eta = \exp(-\psi)$ is a **Painlevé III** transcendent.

This is the first appearance of **Painlevé functions** in statistical physics

T. T. Wu

B. M. McCoy

Theory Emerging from 2D Ising Model

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- Notion of a **τau-function** & "rediscovery" of the importance of **isomonodromic deformations**: M. Sato, T. Miwa and M. Jimbo (1977-1981).

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- **Connection formulas** for Painlevé functions, MTW, M. Ablowitz, H.Segur, M. Jimbo, A. Its, A. Kapaev, . . .

M. Sato

M. Jimbo

T. Miwa

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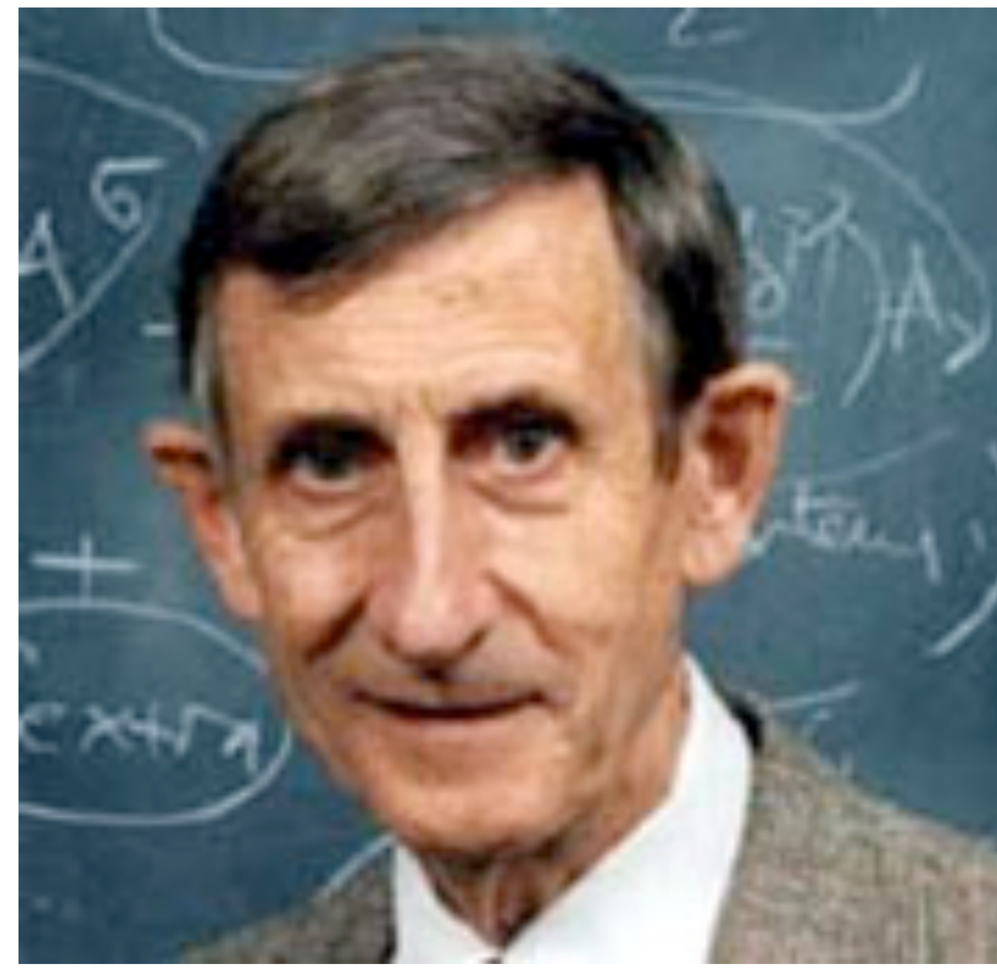
- Problem: Classify 2D ferromagnetic systems whose scaling functions are $F_{\pm}$.

Random Matrix Theory

In the beginning . . .

John Wishart
1898–1956

Eugene Wigner
1902–1995



Freeman Dyson

Gaussian Ensembles

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- Basic fact is that the distribution of $\lambda_{max}(A)$ is a Fredholm determinant.
- Want limit theorem as $N \to \infty$

Limit Theorem

H. Widom & T.

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- F_β have Painlevé representation

Painlevé II Representation of F_2

$$
\begin{aligned}
F_2(s) &= \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right) \\
\frac{d^2q}{dx^2} &= xq + 2q^3, \quad \text{Painlevé II} \\
q(x) &\sim \mathrm{Ai}(x), \ \ x \to \infty
\end{aligned}
$$

This q is called the *Hastings-McLeod* solution.

Universality Theorems

Invariant measures

Riemann-Hilbert approach:

Unitary Case: Its, Bleher, Deift, Kriecherbauer, McLaughlin, Venakides, Zhou

Orthogonal & Symplectic: Deift, Gioev

Replace Gaussian measure with

$\exp(-N\,V(A))\,dA/Z_N$

where V is a polynomial, Gaussian $V(x)=x^2$

Have orthogonal, unitary & symplectic cases

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Have orthogonal, unitary & symplectic cases

Generically behavior is same as Gaussian cases

Universality Theorems

Wigner Symmetric/Hermitian Matrices

$$A_N = \frac{1}{\sqrt{N}} \left(a_{ij}\right)_{i,j=1}^{N}$$

algebraically independent a_{ij} are iid random variables, odd moments zero, even moments decay at least as fast as Gaussian moments. This is "**nonintegrable**" in that there is no Fredholm determinant repr. of distr. function.

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A. Soshnikov: Limiting distribution of largest eigenvalue same as in Gaussian case

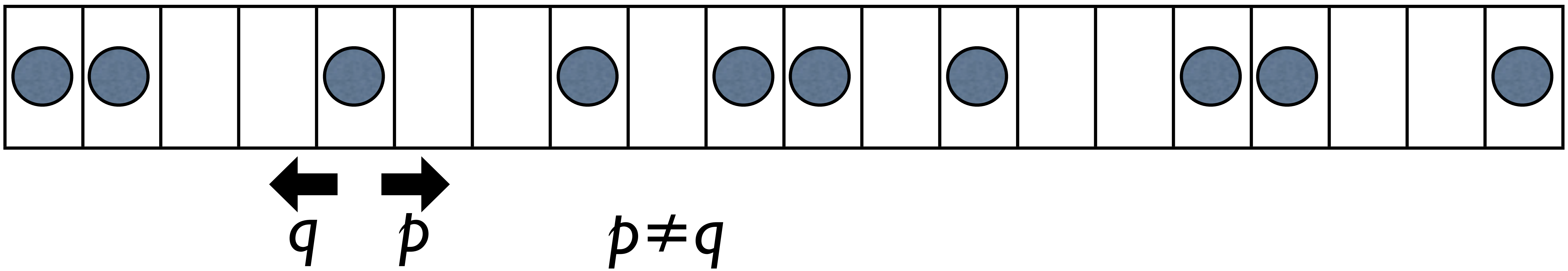
The **asymmetric simple exclusion process** (ASEP): Introduced in 1970 by Frank Spitzer in Interaction of Markov Processes.

The "default stochastic model for transport phenomena". The ``Ising model of nonequilibrium phenomena".

ASEP is a model for interacting particles on a lattice.

Frank Spitzer
1926-1992

ASEP on Integer Lattice



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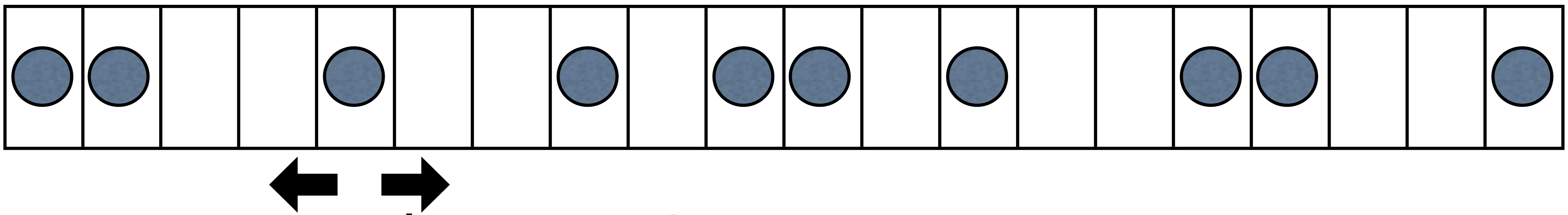
ASEP on Integer Lattice



$q \quad p \qquad\qquad p \neq q$

- Each particle has an alarm clock -- exponential distribution with parameter one

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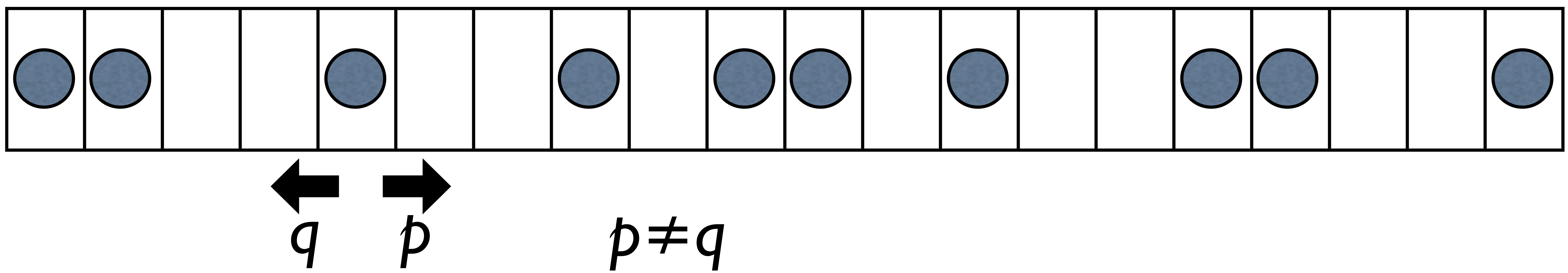

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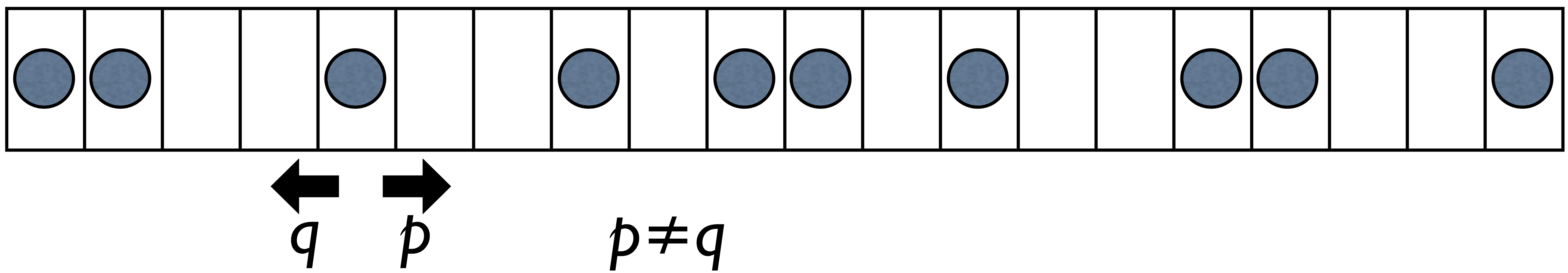

- Each particle has an alarm clock -- exponential distribution with parameter one
- When alarm rings particle jumps to right with probability p and to the left with probability q
- Jumps are suppressed if neighbor is occupied

Initial Conditions

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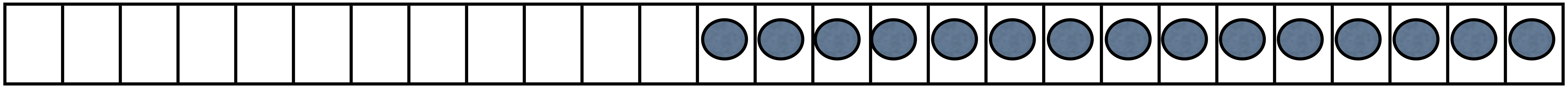

Step Initial Condition, q>p

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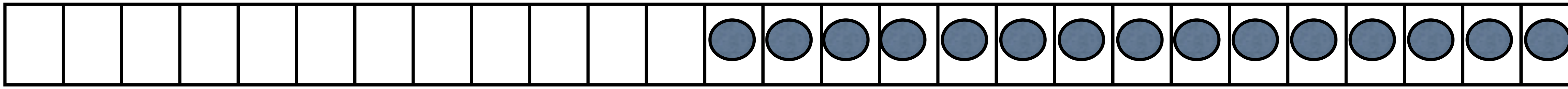

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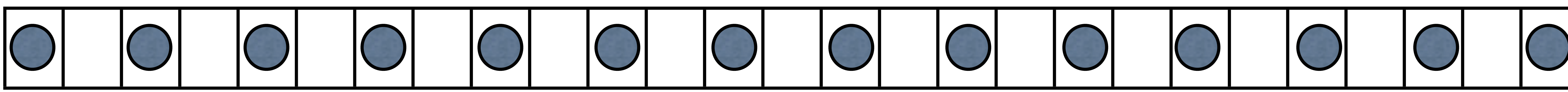

Flat Initial Condition

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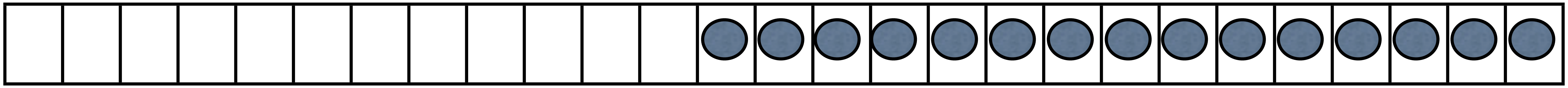

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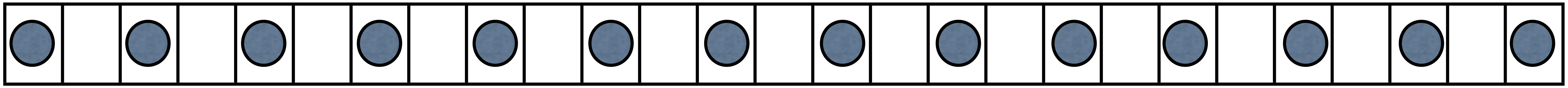

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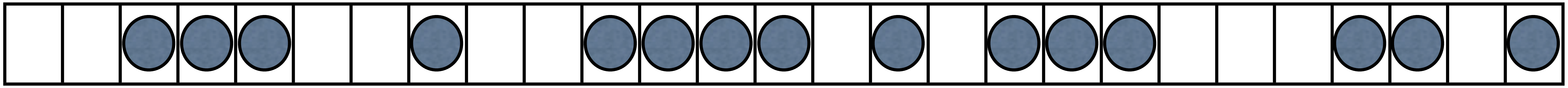
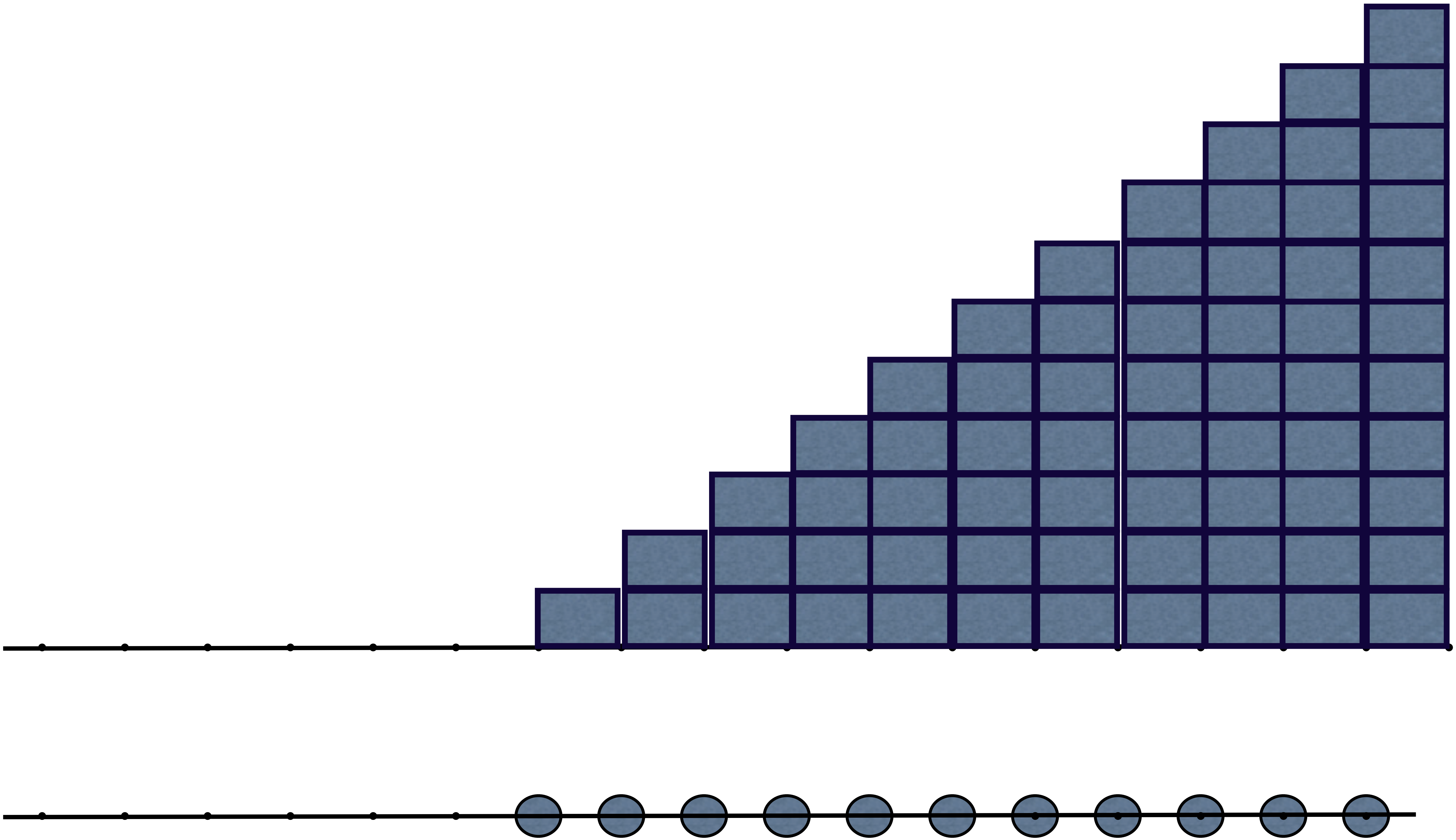


Random: Product Bernoulli measure

Growth Processes & ASEP



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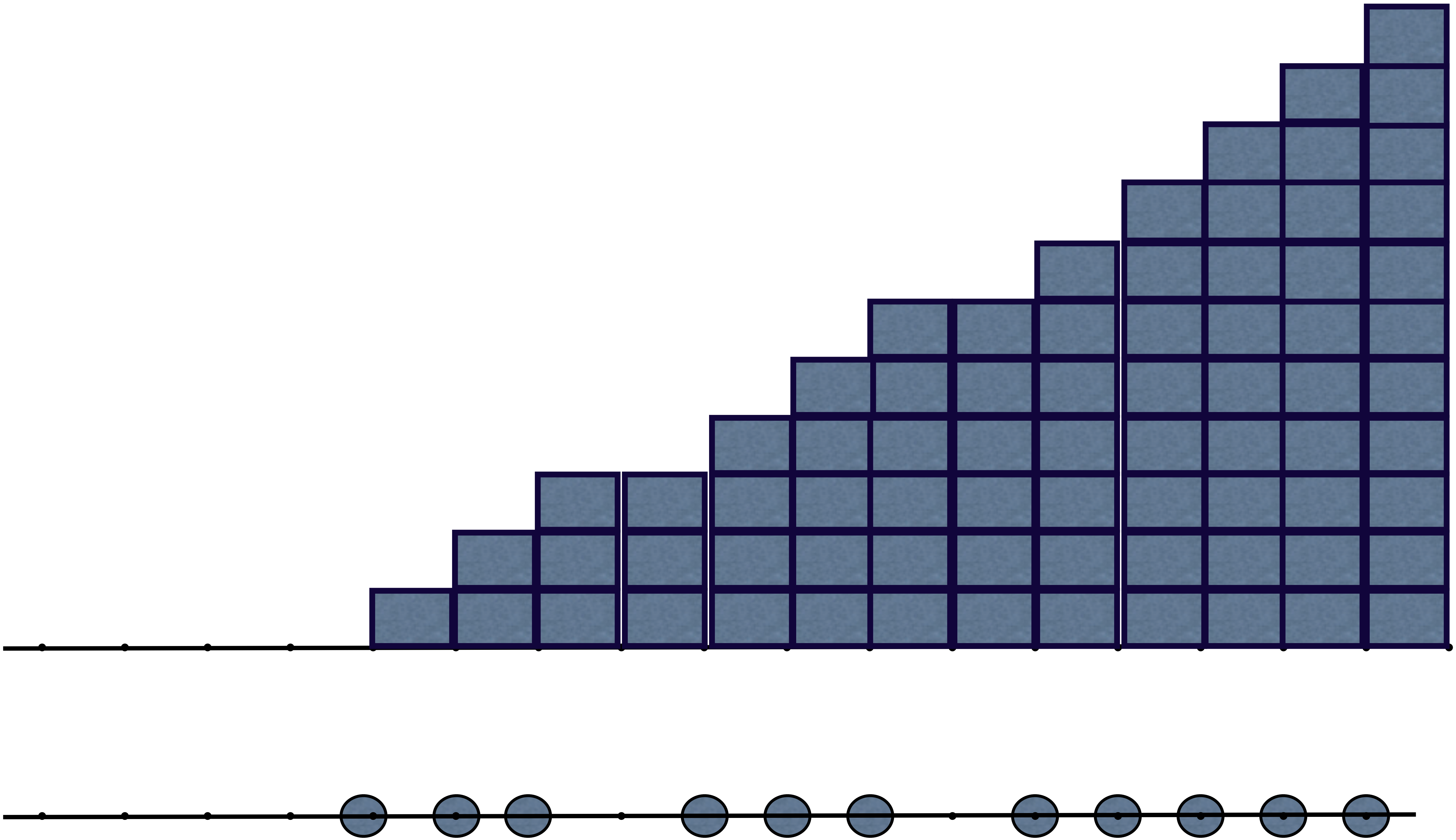

KPZ Equation & Growth Processes

Kardar, Parisi & Zhang

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \lambda \left(\frac{\partial h}{\partial x} \right)^2 + w$$

diffusion growth noise

$$u(x,t) = \frac{\partial h}{\partial x} \longrightarrow$$ Noisy Burgers eqn

Remarks

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- Physicists expect KPZ equation to describe a large class of stochastically growing interfaces: KPZ Universality
- KPZ eqn mathematically difficult to handle so make discrete space approximation.
- ASEP is one discrete version of KPZ; thus expect ``universal behavior" in limit theorems
- TASEP is ASEP with jumps only to left or jumps only to right. TASEP is a simpler model (determinantal process)

Total Current $I(x,t)$

Step Initial Condition

Take $q>p$ net drift to left

$I(x,t)$ = # of particles $\leq x$ at time t, $x \leq 0$

Let $\eta(x,t)=1$ if particle at x at time t

otherwise 0

so $I(x,t)=\sum_{y\leq x} \eta(y,t)$

Current & Position of mth particle

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Event:

$$\{I(x,t)=m\}=\{x_m(t)\leq x,\ x_{m+1}(t)>x\}$$

From this and exclusion property:

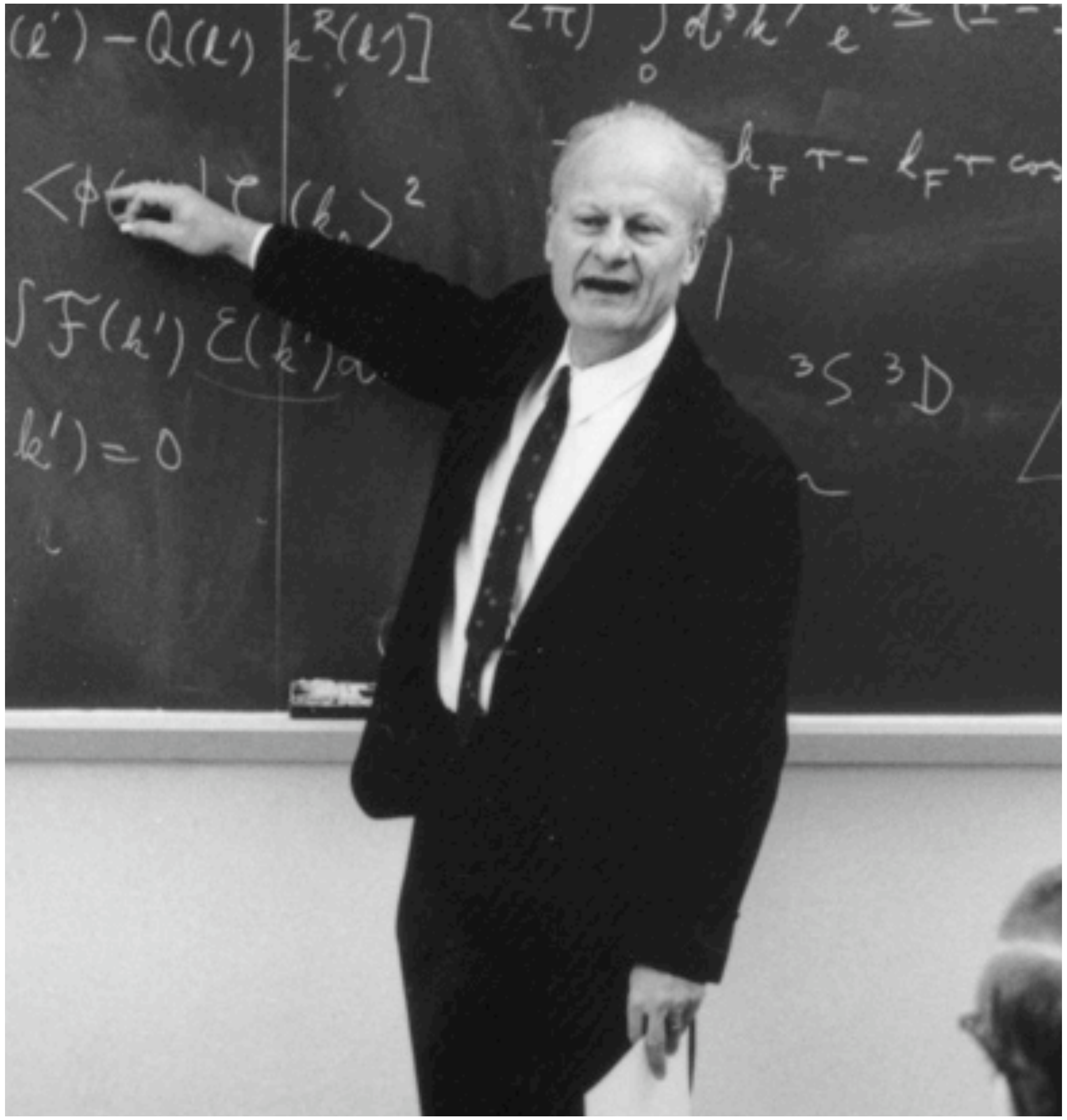
$$\mathrm{Prob}(I(x,t)\leq m) = 1-\mathrm{Prob}(x_{m+1}(t)\leq x)$$

Integrable Structure of ASEP

We solve the Kolmogorov forward equation ("master equation") for the transition probability $Y \to X$:

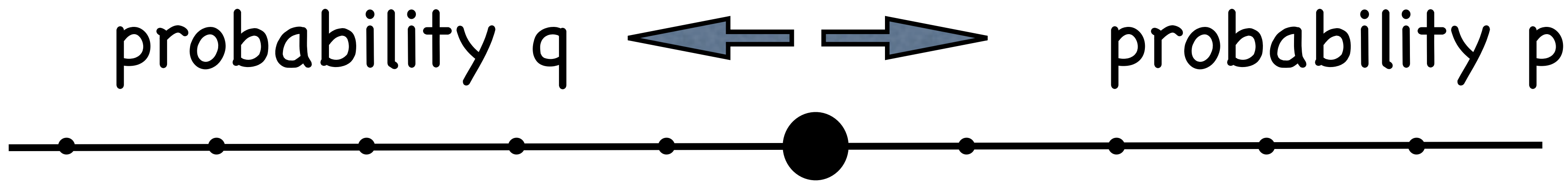
$$P_Y(X;t)$$

Main idea comes from the **Bethe Ansatz** (1931)



Hans Bethe
1906-2005

N=1 ASEP



Let $P_Y(X;t)$=probability $Y \rightarrow X$ at time t.

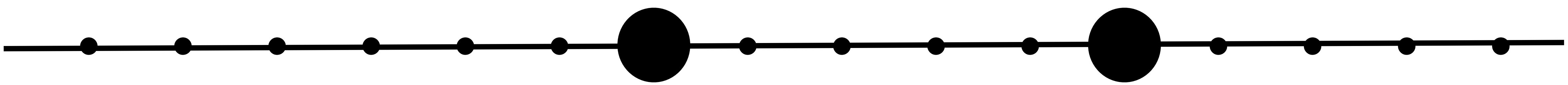
Master equation:

$$\frac{dP}{dt} = p\,P(x-1;t) + q\,P(x+1;t) - P(x;t)$$

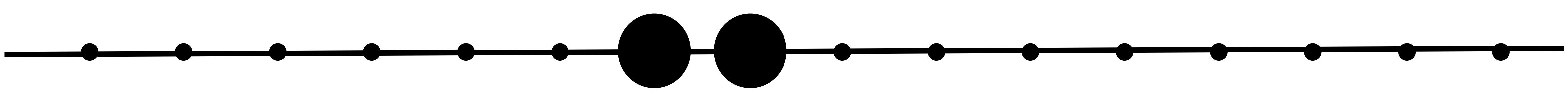
$$P_y(x;t) = \int_{\mathcal{C}} \xi^{x-y-1} e^{t\varepsilon(\xi)}\, d\xi$$

$$\varepsilon(\xi) = \frac{p}{\xi} + q\,\xi - 1$$

N=2 ASEP

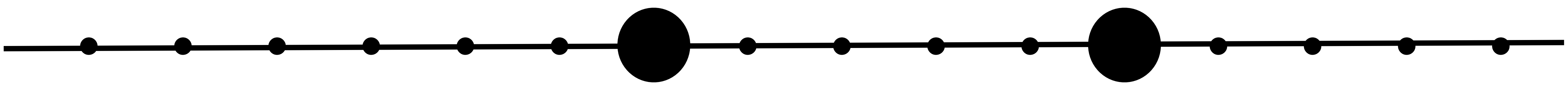


Master equation takes simple form for this configuration



Master equation reflects exclusion for this configuration

N=2 ASEP



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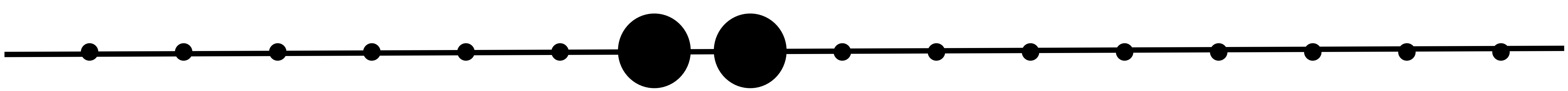
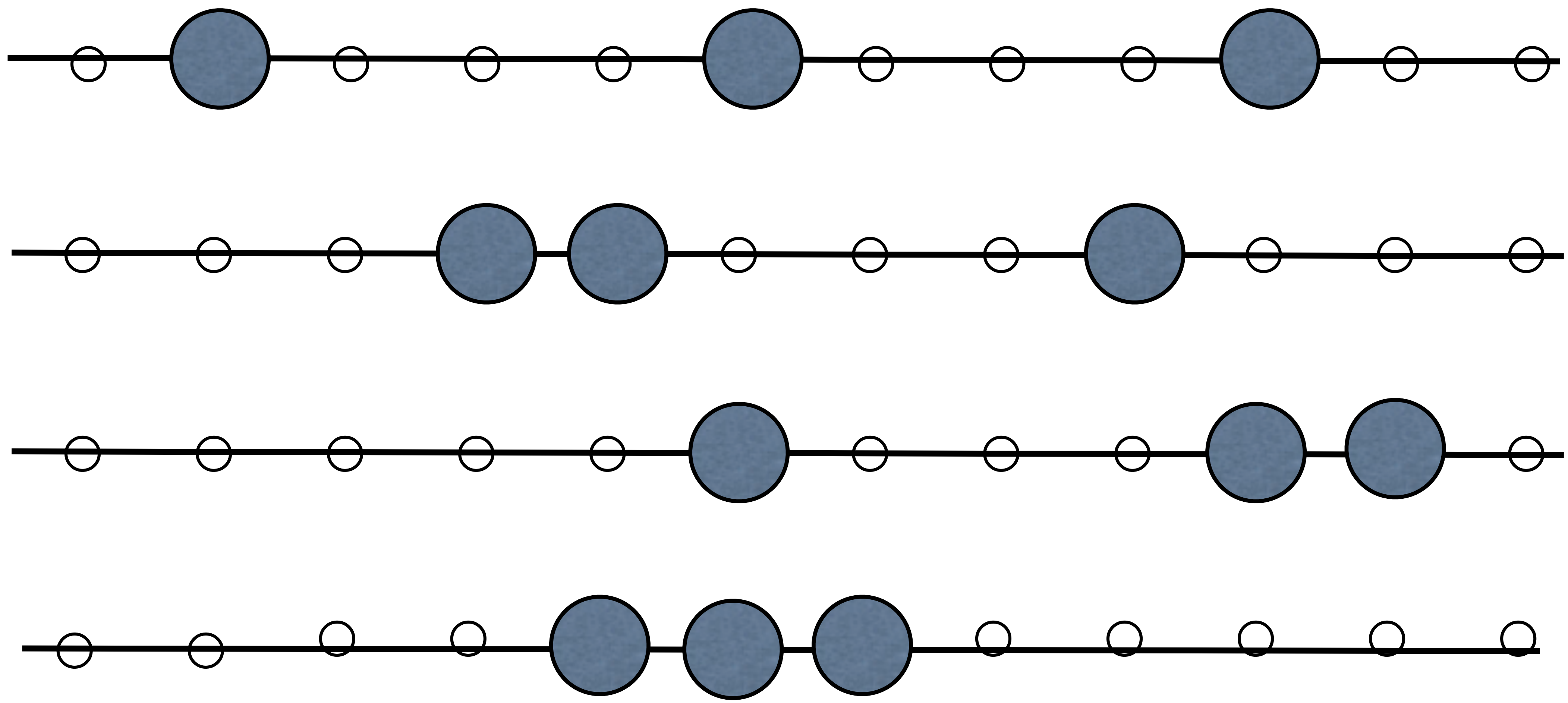


Master equation reflects exclusion for this configuration

Impose boundary conditions for first equation so that if satisfied the second equation is automatically satisfied --- Bethe's Idea

Important Point

New boundary conditions arise for N=3, 4,...



Last configuration requires new BC -- automatically satisfied by 2-particle BC

Bethe Ansatz Solution of Master Equation

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For any $\xi_1,\ldots,\xi_N \in \mathbf{C}\backslash\{0\}$ and any permutation σ a solution is $$\prod_j \xi_{\sigma(j)}^{x_j}\, e^{t\varepsilon(\xi_j)}$$

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$$\prod_j \xi_{\sigma(j)}^{x_j} \, e^{t\varepsilon(\xi_j)}$$

Can take linear combination or integral of a linear combination & have a solution:

$$\int \sum_{\sigma \in \mathcal{S}_N} F_\sigma(\xi) \prod_j \xi_{\sigma(j)}^{x_j} \prod_j e^{t\varepsilon(\xi_j)} \, d^N \xi$$

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- Gives condition on coefficients with result (this part same as the Yang & Yang analysis of XXZ spin Hamiltonian)

$$\text{If } A_\sigma = \mathrm{sgn}(\sigma) \frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i\xi_j - xi_i)}$$

then solution to DE that satisfies BC is

$$\sum_\sigma \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{t\varepsilon(\xi_i)} \, d^N\xi$$

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- To recap: We have found a solution to the master equation satisfying the BCs.
- Must show the solution at $X=\{x_1,...x_N\}$ & $t=0$ reduces to $\delta_{X,Y}$ where $Y=\{y_1,...,y_N\}$ is the initial configuration of N particles.
- Term for σ=id satisfies initial condition. Must show remaining $N!-1$ terms give zero at $t=0$. This is the new part of the problem.

Remarks

- To recap: We have found a solution to the master equation satisfying the BCs.
- Must show the solution at $X=\{x_1,\ldots x_N\}$ & $t=0$ reduces to $\delta_{X,Y}$ where $Y=\{y_1,\ldots,y_N\}$ is the initial configuration of N particles.
- Term for $\sigma=\mathrm{id}$ satisfies initial condition. Must show remaining $N!-1$ terms give zero at $t=0$. This is the new part of the problem.
- Have not yet specified the contours.

Theorem (TW): If $p \neq 0$ and r is small enough then

$$\mathbb{P}_Y(X;t) = \sum_{\sigma \in \mathcal{S}_N} \int_{\mathcal{C}_r^N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i - 1} \, e^{\varepsilon(\xi_i)\, t} \right) d^N \xi.$$

where

$$A_\sigma = \operatorname{sgn} \sigma \, \frac{\prod_{i<j} (p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j} (p + q\xi_i \xi_j - \xi_i)}$$

and satisfies

$$\mathbb{P}_Y(X;0) = \delta_Y(X).$$

Remarks:

- There is no Ansatz in our work!
- Usual Bethe Ansatz calculates the spectrum of the operator. This leads to transcendental equations for the eigenvalues and issues of completeness of the eigenfunctions.
- We compute the semigroup directly. No spectral theory.

Marginal Distributions

$$P(x_m(t) \leq x)$$

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Several "miraculous identities" occur that allow one to take $N \to \infty$ to obtain marginal distribution as a single contour integral whose integrand is a Fredholm determinant

$$P(x_m(t) \leq s) = \oint \det(I - \lambda K)\, f(\lambda)\, d\lambda$$

K is an integral operator on circle

Asymptotic analysis

We now transform the operator K so that we can perform a steepest descent analysis.

Recall that the generic behavior for the coalescence of two saddle points leads to the Airy function Ai(x)

George Airy
1801-1892

Main Result

We set

$$\sigma = \frac{m}{t}, c_1 = -1+2\sqrt{\sigma}, c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}, \gamma = q-p$$

Theorem (TW). When $0 \le p < q$ we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_m(t/\gamma) - c_1 t}{c_2 t^{1/3}} \le s\right) = F_2(s)$$

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Theorem also has a current fluctuation formulation

Total Current Fluctuations

I(x,t) = # of particles ≤ x at time t, x ≤0

Theorem (TW).

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{I([-vt], t/\gamma) - a_1 t}{a_2 t^{2/3}} \le s\right) = 1 - F_2(-s)$$

where

$$0 \le v < 1,\ \ a_1 = \frac{1}{4}(1-v)^2,\ \ a_2 = 2^{-4/3}(1-v^2)^{2/3}$$

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- Theorems generalize Kurt Johansson's results for TASEP to ASEP.
- Coefficients a_1 & c_1 known since 1980s, Liggett, Rost.
- These theorems establish KPZ Universality at the fluctuation level for ASEP.
- Balázs & Seppäläinen; Quastel & Valkó prove $t^{1/3}$ fluctuations for ASEP with Bernoulli product initial condition -- general probabilistic methods

First Appearance of F_2 in growth processes

Baik, Deift & Johansson

Patience Sorting (Aldous & Diaconis)

$$\sigma = \{ \; 6 \quad 7 \quad 1 \quad 8 \quad 5 \quad 4 \quad 10 \quad 9 \quad 2 \quad 3 \; \}$$

First Appearance of F_2 in growth processes

Baik, Deift & Johansson

Patience Sorting (Aldous & Diaconis)

$\sigma = \{ \qquad \qquad \qquad \qquad \qquad \qquad \}$

	2		
	4		
1	5	3	9
6	7	8	10

$$\ell(\sigma) = \text{ Number of piles } = 4$$

Baik-Deift-Johansson Theorem

Baik–Deift–Johansson Theorem

Theorem. Given a random permutation $\sigma \in \mathcal{S}_n$, let $\ell(\sigma)$ equal the number of piles resulting from the patience sorting algorithm. Then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma) - 2\sqrt{n}}{n^{1/6}} \le s\right) = F_2(s).$$

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- TASEP with flat initial conditions leads to F_1, Sasamoto, Borodin, Ferrari, ...


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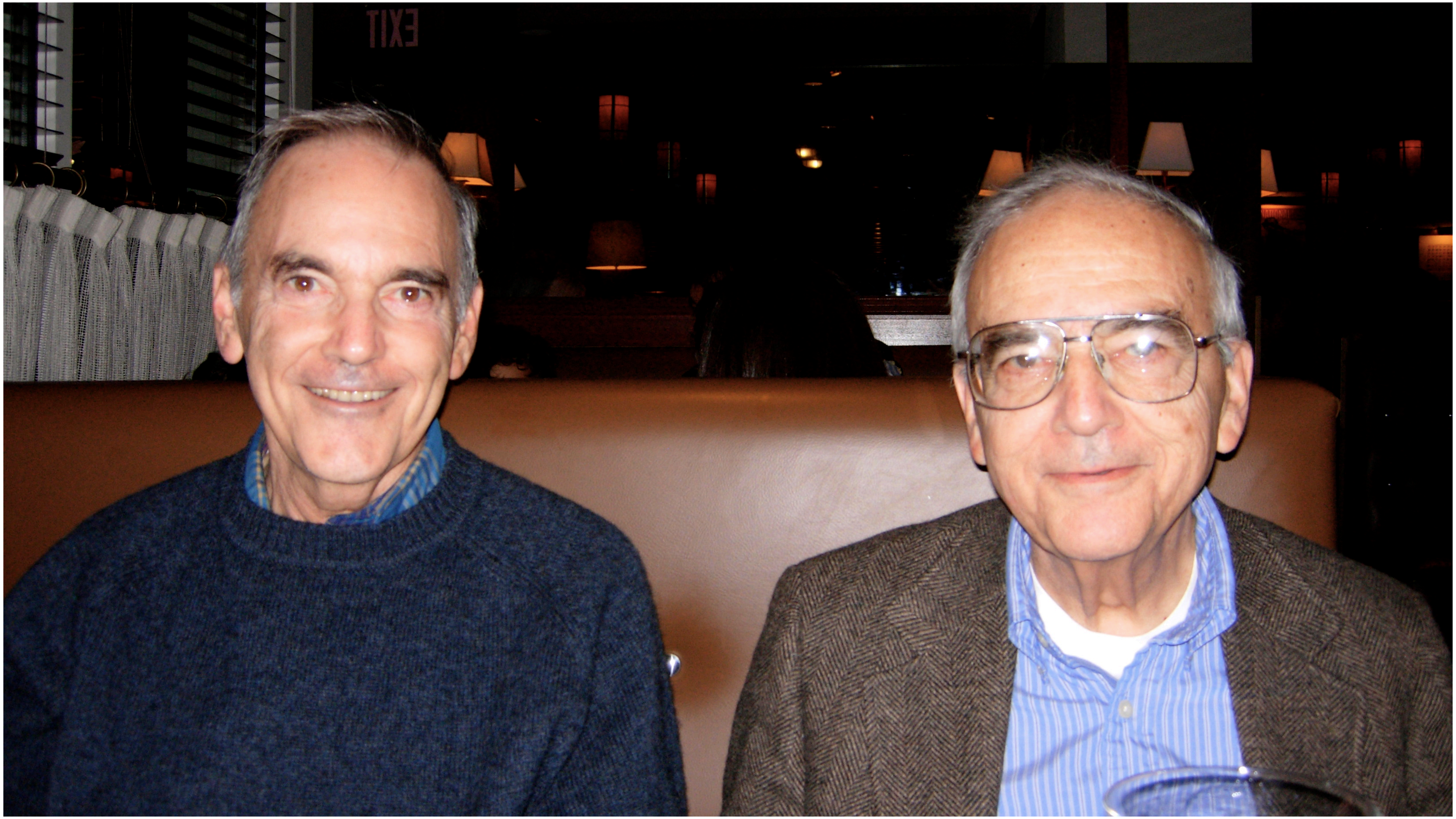
• Ultimately we want universality theorems not to rely upon integrable stucture of ASEP. For ⅓ exponent progress by Balázs, Seppäläinen, Quastel & Valkó.



Harold Widom (left) and his brother Ben

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Link to ASEP papers: arXiv