Bethe Ansatz Methods in Stochastic Integrable Models

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Joint Work with Harold Widom
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Outline

- Brief history: Bethe Ansatz & quantum spin systems
- Modifications for interacting particle systems
- Asymmetric Simple Exclusion Process (ASEP) on integer lattice $\mathbb{Z}$
- Multi-species ASEP
- ASEP on half-line $\mathbb{Z}^+$

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- In 1966 Yang & Yang extended the Bethe Ansatz to study the spectral theory of the XXZ quantum spin chain. The Hamiltonian is defined on a Hilbert space $\bigotimes_{j=1}^{L} \mathbb{C}_j^2$

$$H_{XXZ} = -\sum_{1 \le j \le L} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right)$$

  where σ_j^α are Pauli spin matrices acting in slot j and the identity elsewhere. Assume periodic boundary conditions. (Bethe considered $\Delta = 1$.)

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- What are the essential ideas of Bethe Ansatz? Will explain in terms of H_{XXZ}.

- First note the operator $M = \sum_j \sigma_j^3$ commutes with H_{XXZ}. Not the case for H_{XYZ} Hamiltonian (see Baxter).

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- Let $\{e_X\}$ denote basis in subspace with m up spins,

$$e_X = \sigma_{x_1}^+ \ldots \sigma_{x_m}^+ |\downarrow \cdots \downarrow\rangle = |\cdots \underset{x_1}{\uparrow} \cdots \underset{x_2}{\uparrow} \cdots \underset{x_m}{\uparrow} \cdots\rangle$$

Expand

$$\Psi = \sum_X \psi(x_1, \ldots, x_m) e_X$$

- Bethe Ansatz gives an Ansatz for the coordinate eigenfunctions

$$\psi(x_1, \dots, x_m) = \sum_{\sigma \in \mathcal{S}_m} A_\sigma \, \mathrm{e}^{\mathrm{i} \sum_j x_j p_j}$$

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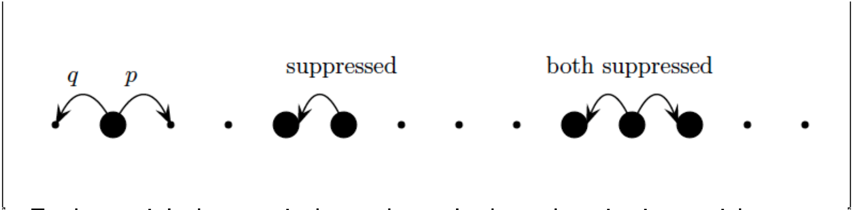
  - Issues related to the completeness of the eigenfunctions found via Bethe Ansatz
- Here we want to explain how these ideas get applied (and modified) to ASEP.

ASEP on Integer Lattice



- Each particle has an independent clock—when it rings with probability p (q) it makes a jump to the right (left) if site empty; otherwise, jump is suppressed.

Start with N-particle ASEP

A *state* $X = (x_1, \ldots, x_N)$ is specified by giving the location

$$x_1 < x_2 < \cdots < x_N, \quad x_i \in \mathbb{Z}$$

of the N particles on the lattice $\mathbb{Z}$. Want

$$\begin{aligned} P_Y(X;t) &= \text{probability of state } X \text{ at time } t \\ &\quad \text{given that we are in state } Y \text{ at } t = 0 \end{aligned}$$

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$P_Y(X;t)$ satisfies a differential equation, called the *Kolmogorov forward equation* or the *master equation*.

Formally,

$$P_Y(X;t) = \langle X | \, \mathrm{e}^{tL} \, | Y \rangle, \;\; P_Y(X;0) = \delta_{X,Y}.$$

$$L = \text{generator of the Markov process}$$

- Fact: The generator L is a similarity (not unitary) transformation of the XXZ spin Hamiltonian H_{XXZ} (observed in early 1990s). Suggests Bethe Ansatz ideas are relevant for ASEP. Consider ASEP on finite lattice with periodic boundary conditions.

$$P_Y(X;t) = \langle X|\, \mathrm{e}^{tL}\, |Y\rangle = \sum_n \langle X|\psi_n\rangle\langle\psi_n|Y\rangle\, \mathrm{e}^{tE_n}$$

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- New approach (first by G. Schütz for TASEP, 1997): Work on infinite lattice $\mathbb{Z}$ and avoid Bethe equations. This is the TW approach we now explain.

$N = 1$: The differential equation is

$$\frac{dP}{dt}(x;t) = pP(x-1;t) + qP(x+1;t) - P(x;t), \;\; x \in \mathbb{Z}.$$

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- DE is linear and separable in x and t
- Let $\xi \in \mathbb{C}$ be arbitrary. Easy to see there are solutions of the form

$$u(x;t) = f(\xi)\xi^x \, e^{t\varepsilon(\xi)}, \quad \varepsilon(\xi) = \frac{p}{\xi} + q\xi - 1.$$

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  $$u(x;t) = f(\xi)\xi^x \, e^{t\varepsilon(\xi)}, \quad \varepsilon(\xi) = \frac{p}{\xi} + q\xi - 1.$$

  where f is any function of ξ.
- DE is linear—take linear superposition

  $$\int f(\xi)\xi^x \, e^{t\varepsilon(\xi)} \, d\xi$$

- Set $f(\xi) = \xi^{-y-1}/(2\pi i)$ and choose contour of integration to be a circle centered at the origin of radius r

$$P_y(x;t) = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \xi^{x-y-1} \, \mathrm{e}^{t\varepsilon(\xi)} \; d\xi$$

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- This solves $N = 1$ ASEP. Solution is in Feller though not derived in the manner here.

- $N = 2$ ASEP: $X = (x_1, x_2)$
  If $x_2 > x_1 + 1$:

$$\frac{dP}{dt}(x_1, x_2) = pP(x_1 - 1, x_2) + qP(x_1 + 1, x_2) + pP(x_1, x_2 - 1) + qP(x_1, x_2 + 1) - 2P(x_1, x_2) \quad (1)$$

  If $x_2 = x_1 + 1$:

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- *Bethe's first idea:* Incorporate "hard equation" (2) into a boundary condition so that we have only to solve the "easy equation" (1).

- We now consider the "easy equation" on all of $X = (x_1, x_2) \in \mathbb{Z}^2$:

$$\begin{aligned} \frac{du}{dt}(x_1, x_2) &= pu(x_1 - 1, x_2) + qu(x_1 + 1, x_2) + \\ &\quad pu(x_1, x_2 - 1) + qu(x_1, x_2 + 1) - 2u(x_1, x_2) \end{aligned} \tag{3}$$

Require that the solution to (3) satisfy the boundary condition

$$pu(x_1, x_1) + qu(x_1 + 1, x_1 + 1) - u(x_1, x_1 + 1) = 0, \ \ x_1 \in \mathbb{Z} \tag{4}$$

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- Thus want solution to (3) that satisfies boundary condition (4) and initial condition $u(x_1, x_2; 0) = \delta_{x_1, y_1} \delta_{x_2, y_2}$.

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- Note that since (3) holds in all $\mathbb{Z}^2$ it is constant coefficient DE. How to find the solution that satisfies the boundary condition? *Bethe's second idea.*

- Let $\xi_1, \xi_2 \in \mathbb{C}$. Then

$$A_{12}(\xi)\, \xi_1^{x_1} \xi_2^{x_2}\; \mathrm{e}^{t(\varepsilon(\xi_1)+\varepsilon(\xi_2))}$$

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- But we can permute $\xi_1 \leftrightarrow \xi_2$ and still get a solution. Thus

$$\{A_{12}(\xi)\, \xi_1^{x_1} \xi_2^{x_2} + A_{21}(\xi)\, \xi_2^{x_1} \xi_1^{x_2}\} \; \mathrm{e}^{t(\varepsilon(\xi_1)+\varepsilon(\xi_2))}$$

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- Require this solution satisfy the boundary condition. Simple computation shows if we choose

$$A_{21} = S(\xi_2, \xi_1) A_{12}$$

  where

$$S(\xi, \xi') = -\frac{p + q\xi\xi' - \xi}{p + q\xi\xi' - \xi'}$$

  then boundary condition satisfied.

- Choose $A_{12} = \xi_1^{-y_1-1}\xi_2^{-y_2-1}$, then we have the solution

$$\int_{\mathcal{C}}\int_{\mathcal{C}} \left\{ \xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} + S(\xi_2,\xi_1)\,\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1} \right\} \mathrm{e}^{t(\varepsilon(\xi_1)+\varepsilon(\xi_2))}\, d\xi_1 d\xi_2$$

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- Does this solution satisfy the initial condition? If contour $\mathcal{C}_r$ is chosen to be a circle of radius r centered at the origin, then the first term satisfies the initial condition. This means the second term must vanish at $t = 0$.

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- S is the Yang-Yang S-matrix in ASEP variables.

$P_Y(X;t)$ for N-particle ASEP

- $X \in \mathbb{Z}^N$
$$X_i^{\pm} = \{x_1, \dots, x_{i-1}, x_i \pm 1, x_{i+1}, \dots, x_N\}$$
The "free equation" on $\mathbb{Z}^N \times \mathbb{R}$ is
$$\frac{du}{dt}(X) = \sum_{i=1}^{N} \left(pu(X_i^-; t) + qu(X_i^+; t) - u(X; t) \right)$$

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- The boundary conditions are

  $$pu(x_1, \ldots, x_i, x_i, \ldots, x_N; t) + qu(x_1, \ldots, x_i + 1, x_i + 1, \ldots, x_N) \\ = u(x_1, \ldots, x_i, x_i + 1, \ldots, x_N, \quad i = 1, 2, \ldots, N-1$$

  This boundary condition comes when particle at x_i is neighbor to particle at $x_{i+1} = x_i + 1$

$P_Y(X;t)$ for N-particle ASEP

- $X \in \mathbb{Z}^N$

$$X_i^{\pm} = \{x_1, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_N\}$$

  The "free equation" on $\mathbb{Z}^N \times \mathbb{R}$ is

$$\frac{du}{dt}(X) = \sum_{i=1}^{N} \left(pu(X_i^-; t) + qu(X_i^+; t) - u(X; t) \right)$$

- The boundary conditions are

$$pu(x_1, \ldots, x_i, x_i, \ldots, x_N; t) + qu(x_1, \ldots, x_i + 1, x_i + 1, \ldots, x_N)$$
$$= u(x_1, \ldots, x_i, x_i + 1, \ldots, x_N, \quad i = 1, 2, \ldots, N-1$$

  This boundary condition comes when particle at x_i is neighbor to particle at $x_{i+1} = x_i + 1$

- Check that no new boundary conditions are needed, e.g. when 3 or more particles are all adjacent.

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This boundary condition comes when particle at x_i is neighbor to particle at $x_{i+1} = x_i + 1$
- Check that no new boundary conditions are needed, e.g. when 3 or more particles are all adjacent.
- Require initial condition $u(X;0) = \delta_{X,Y}$ in physical region.

- Look for solutions of the form (*Bethe's second idea*)

$$u(X;t) = \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \sum_{\sigma \in \mathbb{S}_N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} \; e^{t \sum_i \varepsilon(\xi_i)} \; d\xi_1 \cdots d\xi_N$$

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- Find boundary conditions are satisfied if the A_σ satisfy

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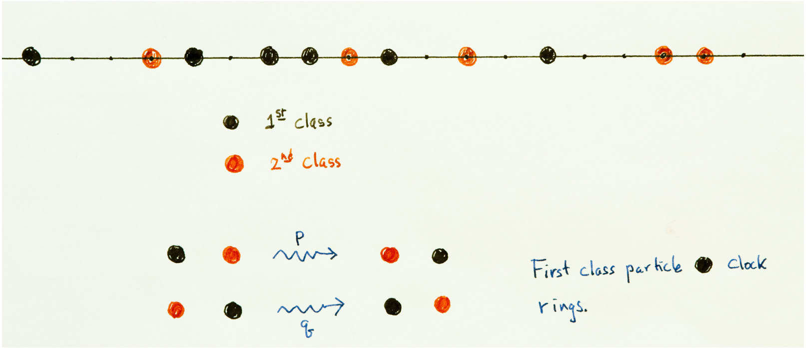
- **Final step: Show $u(X;t)$ satisfies the initial condition.** As before, the term corresponding to the identity permutation gives $\delta_{X,Y}$. We must show the sum of the $N! - 1$ other terms sum to zero in the physical region! True if r is chosen so that all singularities coming from the A_σ lie *outside* the contour $\mathcal{C}_r$ (we assume $p \neq 0$). Our original article had an error (see the erratum).
  Morally, the *initial value problem* is the same issue as *completeness of eigenfunctions*. We return at the end of the lecture to a sketch of the proof.

Multispecies ASEP on Integer Lattice



Multispecies ASEP with M distinct species: Configurations

$$\mathcal{X} = (X, \pi), \;\; X = (x_1, \ldots, x_N), \;\; \pi : [1, N] \to [1, M]$$

Higher species number means higher priority, i.e. $\pi = (1, 2, 2, 2)$ has left-most particle 2nd class and other three first class.

Again try Bethe Ansatz

$$P_{\mathcal{Y}}(\mathcal{X};t) = \sum_{\sigma \in \mathcal{S}_N} \int_{\mathcal{C}_r^N} A_\sigma^\pi(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i (\xi_i^{-y_i-1} \, \mathrm{e}^{t\varepsilon(\xi_i)}) \, d^N \xi$$

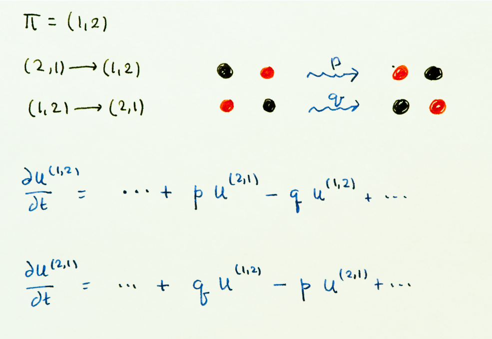
For 1-species ASEP

$$\begin{aligned}\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \{ & pu(x_i-1)\,(1-\delta(x_i-x_{i-1}-1)) + qu(x_i+1)\,(1-\delta(x_{i+1}-x_i-1)) \\ & -pu(x_i)\,(1-\delta(x_{i+1}-x_i-1)) - qu(x_i)\,(1-\delta(x_i-x_{i-1}-1))\}\end{aligned}$$

For multispecies ASEP must account for additional possibilities. Let T_i denote transposition operator, e.g.

$$T_3(3,1,\mathbf{4},\mathbf{6},2,5) = (3,1,\mathbf{6},\mathbf{4},2,5)$$

Additional contributions to forward equation



$$\alpha_i(\pi) = \begin{cases} 0 & \text{if } \pi_i = \pi_{i+1} \\ p & \text{if } \pi_i < \pi_{i+1} \\ q & \text{if } \pi_i > \pi_{i+1} \end{cases}$$

and $\beta_i(\pi) = \alpha_i(\pi)_{p\leftrightarrow q}$, then term that must be added to above DE is

$$\sum_{i=1}^{N-1} \left\{ \alpha_i(\pi) u^{T_i \pi}(x_i, x_{i+1}) - \beta_i(\pi) u^{\pi}(x_i, x_{i+1}) \right\} \delta(x_{i+1} - x_i - 1)$$

Bethe's first idea: Consider the free equation

$$\frac{\partial u^\pi}{\partial t} = \sum_{i=1}^{N} \{ p u^\pi(x_i - 1) + q u^\pi(x_i + 1) - u^\pi(x_i) \}$$

with BC to take care of the interaction $\longrightarrow$ Modify the BC to incorporate the new additional terms

$$\begin{aligned} p u^\pi(x_i, x_i) + q u^\pi(x_i + 1, x_i + 1) - u^\pi(x_i, x_i + 1) & \\ -\alpha_i(\pi) u^{T_i \pi}(x_i, x_{i+1}) + \beta_i(\pi) u^\pi(x_i, x_i + 1) &= 0 \end{aligned}$$

with initial condition

$$u^\pi(X; 0) = \delta_Y(X) \delta_\nu(\pi)$$

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Choose A^π_σ to satisfy BC: First set

$$A^\pi_\sigma := h^\pi_\sigma A_\sigma$$

Find equations

$$h^\pi_{T_i\sigma} = h^\pi_\sigma + \left(1 + S(\xi_{\sigma(i)}, \xi_{\sigma(i+1)})\right) \left[\alpha_i(\pi) h^{T_i\sigma}_\sigma - \beta_i(\pi) h^\pi_\sigma\right]$$

Must show that these formulas together with $h^\pi_{\mathbf{id}} = \delta_\nu(\pi)$ define h^π_σ consistently.

- In this formulation we have a representation of transposition operators T_i
- Let $\mathcal{H}_0$ denote set of all functions $h : \pi \to \text{function of } \xi$ and $\mathcal{H} = \mathcal{S}_N \times \mathcal{H}_0$. Define

$$T_i^0(\sigma, h) = h + \left(1 + S(\xi_{\sigma(i)}, \xi_{\sigma(i+1)})\right) [\alpha_i \cdot (h \circ T_i) - \beta_i \cdot h]$$

Must show these T_i^0 satisfy the braid relations

$$\begin{aligned} T_i T_i &= I \\ T_i T_j &= T_j T_i \text{ when } |i-j| > 1 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \end{aligned}$$

- First two are easy to verify and for third enough to check for $N = 3$. Do this by computer verification.

Remarks

- Can reformulate problem in terms of **Yang-Baxter equations**. This multispecies model was shown to be "Yang-Baxter solvable" by F. Alcaraz and R. Bariev in 2000.

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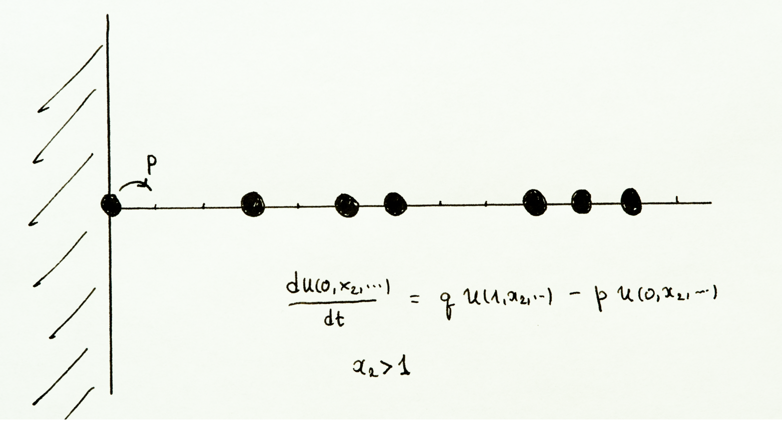
- Note: We don't have closed formulas for A_σ^π except in a few very specific cases.
- Analysis of marginals and thermodynamic limit are completely open problems.

ASEP on Nonnegative Integer Lattice



Must impose **additional BC** on free equation on $\mathbb{Z}$:

$$-p\, u(-1, x_2, \dots, x_N) + q\, u(0, x_2, \dots, x_N) = 0$$

- Known since Gaudin's work in 1971 on the Bose gas, that Bethe Ansatz has to be modified for half-line problems

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- Identify B_N with the group of **signed permutations**, e.g.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -2 & 4 & -5 & -1 & 6 & 3 \end{pmatrix}$$

with

$$\sigma(-i) = -\sigma(i)$$

Order of group is $2^N N!$.

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- **Inversions** in B_N: A pair $(\pm\sigma(i), \sigma(j))$ with $i < j$ such that $\pm\sigma(i) > \sigma(j)$, e.g. if $\sigma = (-3, 1, -2)$ inversions are

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- Let $\tau = p/q$ and note that $\varepsilon(\xi) = p/\xi + q\xi - 1$ is unchanged when $\xi \to \tau/\xi$. Define $\xi_{-a} = \tau/\xi_a$

Bethe Ansatz for solution of forward equation for half-line ASEP:

$$P_Y(X;t) = \sum_{\sigma \in B_N} \int_{\mathcal{C}_r^N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i (\xi_i^{-y_i-1} \, e^{t\varepsilon(\xi_i)}) \, d^N\xi$$

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- How to choose A_σ to satisfy usual ASEP BC *and* new BC from half-line restriction?
- BC satisfied if

$$\sum_{\sigma \in B_N} A_\sigma (1 - \tau \xi_{\sigma(1)}^{-1}) = 0$$

or

$$\sum_{\sigma \in B_N} A_\sigma (1 - \xi_{-\sigma(1)}) = 0$$

Pair permutations σ and σ' where $\sigma'(1) = -\sigma(1)$. Then BC satisfied if

$$\frac{A_{\sigma'}}{1 - \xi_{\sigma'(1)}} = -\frac{A_\sigma}{1 - \xi_{\sigma(1)}}$$

- Then check for $\sigma \in B_N$ that

$$A_\sigma = \prod_{\sigma(i)<0} r(\xi_{\sigma(i)}) \prod_{\text{inversions }(b,a)} S(\xi_b, \xi_a)$$

satisfies both BCs where

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- **Problem of initial condition (choice of contours).** For $N = 1$

$$P_y(x;t) = \int_{\mathcal{C}} \left[\xi^{x-y-1} - \left(\frac{1-\tau/\xi}{1-\xi} \right) \tau^x \xi^{-x-y-1} \right] e^{t\varepsilon(\xi)} \, d\xi$$

Small contours do *not* satisfy initial condition since second term (with $t = 0$) does not vanish for $x, y \geq 0$. Does vanish if contour $\mathcal{C} = \mathcal{C}_R$ with $R \gg 1$. So choose large contours?

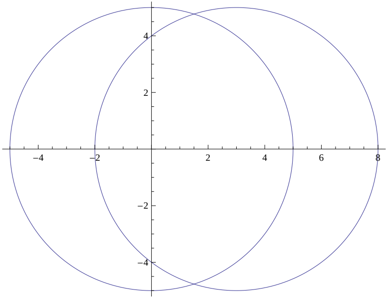
Choice of Contours

- Suppose we have inversion $(a, -b)$ in σ where $a > 0, b > 0$: Get factor

$$S(\xi_a, \xi_{-b}) = S(\xi_a, \tau/\xi_b) = -\frac{\xi_a + \xi_b - p^{-1}\xi_a\xi_b}{\xi_a + \xi_b - q^{-1}}$$

and if $\xi_a, \xi_b \in \mathcal{C}_R$ with $R \gg 1$ have

$$\mathcal{C}_R \cap (q^{-1} - \mathcal{C}_R) \neq \varnothing$$



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- With this domain of integration we show the initial condition is satisfied.

ASEP on $\mathbb{Z}$: Sketch of Proof of Initial Condition

Let $\mathcal{C}_r$ be the circle with center zero, radius r.

Theorem: If $p \neq 0$ and r is small enough then

$$P_Y(X;t) = \sum_{\sigma} \int_{\mathcal{C}_r^N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \xi_i^{-y_i-1} \, e^{t\varepsilon(\xi_i)} \, d^N\xi.$$

If $I(\sigma)$ is the σ-summand with $t = 0$ we have to show

$$\sum_{\sigma \neq id} I(\sigma) = 0.$$

Proof by induction on N.

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Proof by induction on N.

For those σ with $\sigma(N) = N$ use the induction hypothesis.

If $\sigma(N) < N$ make the substitution

$$\xi_N \longrightarrow \frac{\eta}{\prod_{i<N} \xi_i}$$

The product of S-factors involving ξ_N becomes

$$\prod_{\text{inversions}\,(N,j)} S\left(\frac{\eta}{\prod_{i<N} \xi_i}, \xi_j\right).$$

and the product of powers of the ξ_i becomes

$$\eta^{x_{\sigma^{-1}(N)} - y_N - 1} \prod_{i<N} \xi_i^{x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(N)} + y_N - y_i - 1}.$$

This is zero at $\xi_j = 0$ for all inversions (N, j).

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If $\sigma(N) = N - 1$ there is only one inversion (N, j), the j-factor is analytic for ξ_j inside $\mathcal{C}_r$ except for a simple pole at zero, so $I(\sigma) = 0$.

If $\sigma(N) = N - 2$ consider all permutations with inversions (N, j) and (N, k). Integrate with respect to ξ_j by shrinking the contour. There is a pole from the k-factor. Integrate the residue with respect to ξ_k by shrinking the contour. There is a pole from the j-factor. We get an

$(N - 2)$-dimensional integral in which $\xi_j = \xi_k$.

Pair σ and σ' if they are the same except that the positions of j and k are interchanged. Then the integrands for σ and σ' are negatives of each other since $S(\xi_j, \xi_k)$ is a factor for one and not the other and it equals -1 when $\xi_j = \xi_k$. (The other factors involving ξ_j or ξ_k are equal when $\xi_j = \xi_k$.)

Therefore $I(\sigma) + I(\sigma') = 0$.

If $\sigma^{-1}(N) = N - 3$ consider all permutations with inversion $(N, j),\ (N, k),\ (N, \ell)$. All $I(\sigma)$ are sums of integrals in which $\xi_j = \xi_k,\ \xi_j = \xi_\ell$, or $\xi_k = \xi_\ell$.

For each of these, pair permutations as before. The corresponding integrands are negatives of each other.

And so on, for general $\sigma(N)$.

Thus $\sum_{\sigma \neq id} I(\sigma) = 0$, as claimed.