Asymmetric Simple Exclusion Process: Integrable Structure[a]

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The *asymmetric simple exclusion process* (ASEP): Introduced in 1970 by **Frank Spitzer** (1926–1992) in *Interaction of Markov Processes*.

Figure 1: Frank Spitzer, 1970.

“default stochastic model for transport phenomena” (**H.-T. Yau**).

Definition of Model

The ASEP is a model for **interacting particles** on a lattice S, say $S = \mathbb{Z}^d$.

1. A **state** η of the system is a map $\eta : S \to \{0,1\}$ such that

$$\eta(x) = \begin{cases} 1 & \text{if site } x \in S \text{ is occupied by a particle,} \\ 0 & \text{if site } x \in S \text{ is vacant.} \end{cases}$$

   States $\Omega = \{0,1\}^S$.

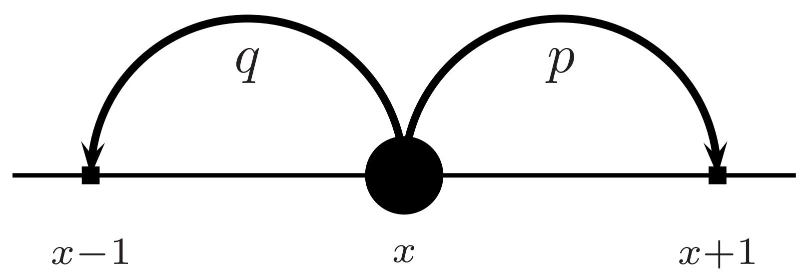
2. Introduce **dynamics**: $t \to \eta_t \in \Omega$:
   (a) Each particle $x \in S$ waits exponential time with parameter 1, independently of all other particles;
   (b) at the end of that time, it chooses a $y \in S$ with probability $p(x,y)$; and
   (c) if y is vacant, it goes to y, while if y is occupied, it stays at x and the clock starts over.

Markov Chain Aspects

Without the "exclusion" of part (c), we would have particles moving on S according to *independent*, continuous time Markov chains on S that have unit exponential holding times. Because time is continuous, two clocks do not go off at the same time.

Now take $S = \mathbb{Z}$ with

$$p(x, y) = \begin{cases} p & \text{if } y = x + 1 \\ q & \text{if } y = x - 1 \\ 0 & \text{otherwise} \end{cases}$$



T(totally)**ASEP**: $p = 1$.

Random matrix connection (**Kurt Johansson** 2000):

Initial configuration $\mathbb{Z}^-$. Probability that at time t the particle initially at $-m$ has moved at least $n\,(\geq m)$ times equals

$$C_{m,n} \int_{[0,t]^m} \prod_{0 \leq i < j < m} (\tau_i - \tau_j)^2 \prod_i (\tau_i^{n-m}\, e^{-\tau_i}) d^m \tau$$

$$= C'_{m,n} \det \left(\int_0^t \tau^{i+j+n-m}\, e^{-\tau}\, d\tau \right).$$

= **distribution of largest eigenvalue** in the unitary Laguerre ensemble.

Thus underlying *determinantal process*

This made possible Johansson's analysis of **current fluctuations**. (Famous 1/3 law for fluctuations.)

Precisely, if $Y(k,t)$ is the number of particles to the right of k at time t, then

Theorem (Johansson, 2000)

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Y([ut],t)-c_1 t}{c_2 t^{1/3}} \le \xi\right) = 1 - F_2(-\xi)$$

where $0 \le u < 1$,

$$\begin{aligned} c_1 &= \frac{1}{4}(1-u)^2 \quad \text{(Rost, 1981)} \\ c_2 &= (1-u)^{2/3}(1+u)^{-1/3} \end{aligned}$$

and F_2 is the scaling function for the distribution of the largest eigenvalue in GUE (TW).

What about the case $p < 1$?

The process is *not obviously*

a determinantal process

so none of the RMT techniques seem to apply.

What we show is the ideas from **integrable systems**; in particular

Bethe Ansatz

will allow us to derive certain exact formulas for the analogous distribution functions. However, **asymptotics** are still out of reach.

The **physics conjecture** is that with appropriate centering and scaling the $1/3$ fluctuations remain the same; and furthermore, the scaling function stays F_2.

N-particle ASEP: Possible configuration

$$X = \{x_1, \ldots, x_N\}, \quad x_1 < \cdots < x_N, \quad x_i \in \mathbb{Z}.$$

Initial config.: $Y = \{y_1, \ldots, y_N\}$. We obtain integral formulas for

$\mathbb{P}_Y(X; t) =$ prob. that at time t the system is in configuration X.

$\mathbb{P}(x_m(t) = x) =$ prob. that at time t the mth particle from the left is at x.

TASEP: $(p = 1)$

- Schütz (1997): $\mathbb{P}_Y(X; t) = N \times N$ determinant.
- Rákos-Schütz (2005): Schütz $\Rightarrow$ Johansson (2000).
- Particles only influenced by those to the right.

Differential Equation

New notation $u(X;t)$ or $u(X)$ for $\mathbb{P}_Y(X;t)$.

Initial condition: $u(X;0) = \delta_Y(X)$.

$\mathbf{N = 1}$: With one particle no problem with exclusion.

First solve using **probabilistic arguments**. If $N(t)$ is a Poisson process, then

$$\mathbb{P}(N(t) = n) = \frac{t^n}{n!}\, e^{-t}$$

Let X_j denote independent Bernoulli random variables, $\mathbb{P}(X_j = +1) = p$, $\mathbb{P}(X_j = -1) = q$. The position of the walker at time t, starting at 0, is then

$$x_t = X_1 + X_2 + \cdots + X_{N(t)}$$

$$
\begin{aligned}
\mathbb{P}(x_t = x) &= \sum_{n\geq 0} \mathbb{P}(x_t = x | N(t) = n)\, \mathbb{P}(N(t) = n) \\
&= \sum_{n\geq 0} \binom{n}{\frac{n+x}{2}} p^{(n+x)/2} q^{(n-x)/2} \, \frac{t^n}{n!} \, e^{-t} \\
&= \left(\frac{p}{q}\right)^{x/2} e^{-t} I_x \left(2\sqrt{pq}\, t\right), \; I_\nu = \text{Bessel function.}
\end{aligned}
$$

Differential equation solution:

Let $p(x, y) =$ the transition probability for simple random walk and $\mathbb{P}_t^\eta$ the distribution of ASEP at time t with initial configuration η defined by $Y = \{y_1, \ldots, y_N\}$. Set

$$
P_Y(X; t) = \mathbb{P}_t^\eta \{\eta : \eta(x_1) = 1, \ldots, \eta(x_N) = 1, \eta(z) = 0, z \neq x_j\}
$$

where $X = \{x_1, \ldots, x_N\}$.

Then for $N = 1$

$$\frac{d}{dt} P_{y_1}(x_1; t) = p P_{y_1}(x_1 - 1; t) + q P_{y_1}(x_1 + 1; t) - P_{y_1}(x_1; t)$$

This is the **forward equation** or **master equation**.

We solve this DE using Fourier series (with initial condition $P_{y_1}(x_1; 0) = \delta_{x_1, y_1}$): $\hat{P}_1(k, t) = \sum_{x \in \mathbb{Z}} e^{ikx} P_{y_1}(x; t)$. Find

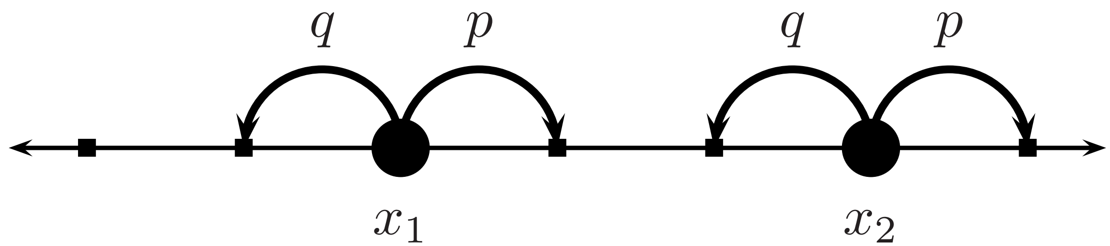
$$P_{y_1}(x_1; t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \xi^{x_1 - y - 1} \, e^{\varepsilon(\xi) t} \, d\xi$$

where

$$\varepsilon(\xi) = \frac{p}{\xi} + q\,\xi - 1$$

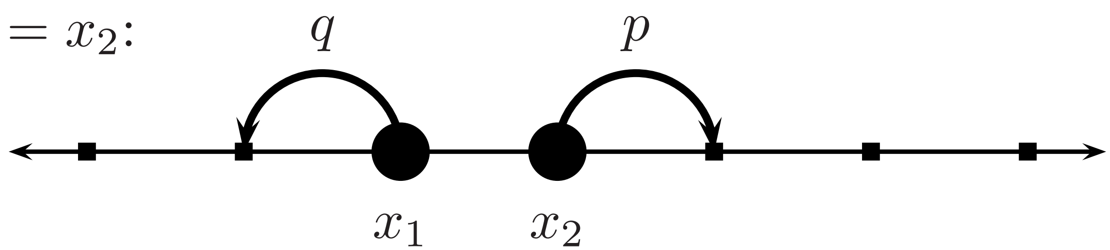
The latter is an integral representation for I-Bessel functions.

$\mathbf{N=2}$ and $x_1 + 1 < x_2$:



$$\begin{aligned}\frac{d}{dt}u(x_1,x_2) \quad &= \quad p\,u(x_1-1,x_2)+q\,u(x_1+1,x_2) \\ &\quad\quad +p\,u(x_1,x_2-1)+q\,u(x_1,x_2+1)-2\,u(x_1,x_2).\end{aligned}$$

For $x_1 + 1 = x_2$:



$$\frac{d}{dt}u(x_1,x_2) = p\,u(x_1-1,x_2)+q\,u(x_1,x_2+1)-u(x_1,x_2).$$

Formally subtract when $x_2 = x_1 + 1$ to obtain.

$$0 = p\,u(x_1,x_1)+q\,u(x_1+1,x_1+1)-u(x_1,x_1+1).$$

Treat this as a **boundary condition**. Consider $(x_1, x_2) \in \mathbb{Z}^2$, then

First Equation + Boundary Condition $\Rightarrow$ **Second Equation**.

Find a solution in $\mathbb{Z}^2$ satisfying first DE, BC and the initial condition in region $x_1 < x_2 \Longrightarrow$ found the probability $\mathbb{P}_Y(X;t)$.

General N: $X = \{x_1, \ldots, x_N\} \in \mathbb{Z}^N$.

Differential equation (master equation):

$$\frac{d}{dt} u(X;t) = \sum_i \left[p\, u(\ldots, x_i - 1, \ldots) + q\, u(\ldots, x_i + 1, \ldots) - u(X)\right].$$

Boundary conditions ($i = 1, \ldots, N-1$) for **2-particle neighbors**:

$$\begin{aligned} &u(\ldots, x_i, x_i + 1, \ldots) \\ &= p\, u(\ldots, x_i, x_i, \ldots) + q\, u(\ldots, x_i + 1, x_i + 1, \ldots). \end{aligned}$$

Important Point:

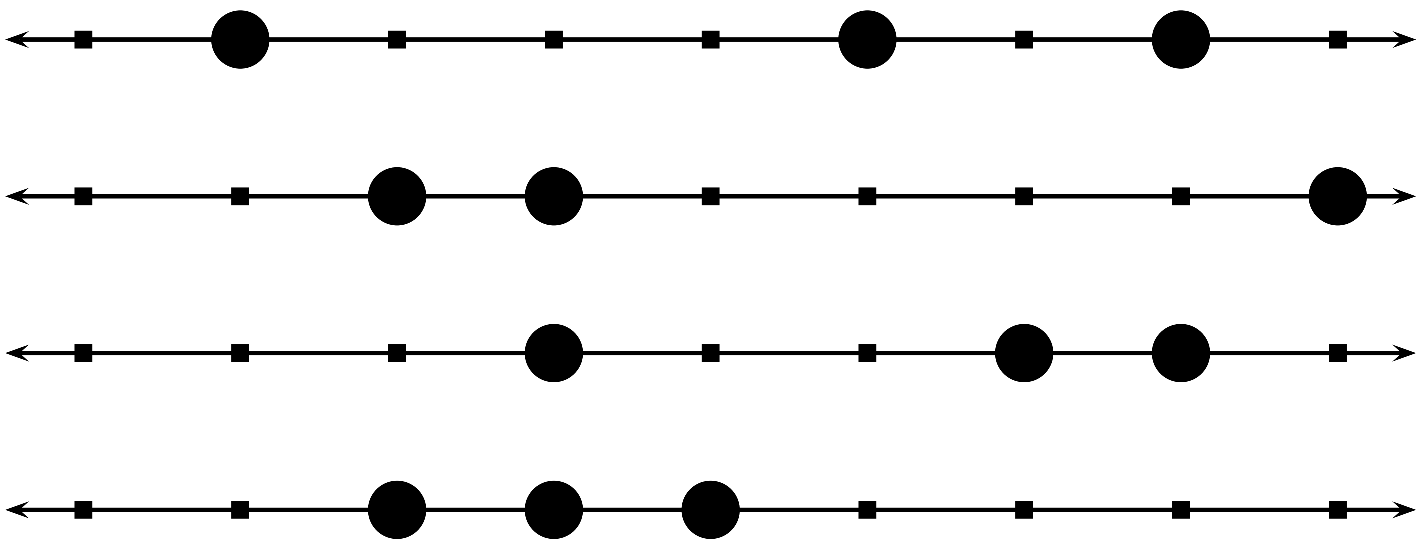
Additional boundary conditions arise when, say,

3 particles are adjacent

4 particles are adjacent

etc.

For example, for $N = 3$ we have four cases



These "new conditions" are automatically satisfied by the boundary conditions arising when **2 particles are adjacent**.

The underlying algebraic reason is that the S-matrix satisifes the

Yang-Baxter equations

See the book

Beautiful Models:
70 Years of Exactly Solved Quantum Many-Body Problems

by **Bill Sutherland** for a nice description of integrability and the Yang-Baxter equations

Initial condition

$$u(X;0) = \delta_Y(X) \text{ when } x_1 < \cdots < x_N.$$

Equation + boundary conditions + initial condition $\Rightarrow$

$u(X;t) = P_Y(X;t)$ when $x_1 < \cdots < x_N$.

Bethe Ansatz Solution

$\varepsilon(\xi) := p\,\xi^{-1} + q\,\xi - 1$. For any $\xi_1, \ldots, \xi_N \in \mathbb{C}\backslash\{0\}$ a solution of the DE is

$$\prod_j \left(\xi_j^{x_j}\, e^{\varepsilon(\xi_j)\,t} \right).$$

For any $\sigma \in \mathcal{S}_N$, another solution is

$$\prod_j \xi_{\sigma(j)}^{x_i} \prod_j e^{\varepsilon(\xi_j)\,t}$$

or any linear combination of these, or any integral of a linear combination.

Bethe Ansatz:

$$u(X;t) = \int \sum_{\sigma \in \mathcal{S}_N} F_\sigma(\xi) \prod_j \xi_{\sigma(j)}^{x_j} \prod_j e^{\varepsilon(\xi_j)\,t}\, d^N\xi.$$

Want the boundary conditions to be satisfied.

Look for F_σ such that the integrand satisfies the BCs *pointwise*:

$$\sum_{\sigma\in\mathcal{S}_N} F_\sigma\left(p+q\,\xi_{\sigma(i)}\,\xi_{\sigma(i+1)}-\xi_{\sigma(i+1)}\right)\cdot\left(\xi_{\sigma(i)}\xi_{\sigma(i+1)}\right)^{x_i}\prod_{j\neq i,\,i+1}\xi_{\sigma(j)}^{x_j}=0.$$

Definition: $T_i\sigma$ differs from σ by an intechange of the ith and $(i+1)$st entries. If $\sigma=(2\ 3\ 1\ 4)$ then $T_2\sigma=(2\ 1\ 3\ 4)$.

Replace σ by $T_i\sigma$ and add. Sufficient conditions

$$\frac{F_{T_i\sigma}}{F_\sigma}=-\frac{p+q\xi_{\sigma(i)}\xi_{\sigma(i+1)}-\xi_{\sigma(i+1)}}{p+q\xi_{\sigma(i)}\xi_{\sigma(i+1)}-\xi_{\sigma(i)}}.$$

The Yang-Yang S-matrix in the XXZ spin Hamiltonian is

$$S_{\alpha\beta}=-\frac{p+q\xi_\alpha\xi_\beta-\xi_\alpha}{p+q\xi_\alpha\xi_\beta-\xi_\beta}$$

Upshot: If we define

$$A_\sigma = \operatorname{sgn}\sigma \, \frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i\xi_j - \xi_i)}$$

then

$$u(X;t) = \sum_\sigma \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} \, d^N\xi \tag{1}$$

satisfies the master equation + boundary conditions.

The $\sigma = id$ summand satisfies initial condition.

TASEP

$$A_\sigma = \operatorname{sgn}\sigma \prod (1-\xi_{\sigma(i)})^{\sigma(i)-i},$$

$$u(X;t) = \det\left(\int (1-\xi)^{j-i}\,\xi^{x_i-y_j-1} e^{(\xi^{-1}-1)t}\,d\xi\right).$$

Schütz (1997): If all contours $\mathcal{C}_r$ with $r<1$ the initial conditions are satisfied. Therefore

$$P_Y(X;t) = \det\left(\int_{\mathcal{C}_r} (1-\xi)^{j-i}\,\xi^{x_i-y_j-1} e^{(\xi^{-1}-1)t}\,d\xi\right).$$

ASEP (TW, 2007)

Want contours so that when $x_1 < \cdots < x_N$

$$\sum_{\sigma \neq id} \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \xi_i^{-y_i - 1} \, d^N \xi = 0.$$

$N = 2$: If both contours are small enough get Schütz's formula.
$N = 3$: Take all contours small. Three integrals are zero, other two are negatives.

So for $N = 2, 3$, $\mathbb{P}_Y(X; t)$ equals (1) with integrals over $\mathcal{C}_r$, $r \ll 1$.

General N:

$$I(\sigma) = \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \xi_i^{-y_i - 1} \, d^N \xi.$$

Some $I(\sigma) =\ 0$, others come in pairs (σ, σ') such that
$I(\sigma) + I(\sigma') = 0$.

Theorem (TW): If $p \neq 0$ and r is small enough then

$$\mathbb{P}_Y(X;t) = \sum_{\sigma \in \mathcal{S}_N} \int_{\mathcal{C}_r^N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i - 1} \, e^{\varepsilon(\xi_i)\, t} \right) d^N \xi.$$

where

$$A_\sigma = \operatorname{sgn} \sigma \, \frac{\prod_{i<j} (p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j} (p + q\xi_i \xi_j - \xi_i)}$$

and satisfies

$$\mathbb{P}_Y(X;0) = \delta_Y(X).$$

Remarks:

- There is no Ansatz in our work!
- Usual Bethe Ansatz calculates the spectrum of the operator. This leads to transcendental equations for the eigenvalues and issues of completeness of the eigenfunctions.
- We compute the semigroup directly. No spectral theory.

Left-Most Particle: Formulas for $\mathbb{P}(\mathbf{x_1(t) = x})$

Since $x_1 < \cdots < x_N$ set

$$x_1 = x,\ x_2 = x + z_1, \ldots, x_N = x + z_1 + z_2 + \cdots z_{N-1}.$$

When $r < 1$ can sum the formula for $\mathbb{P}_Y(X;t)$ over $z_i > 0$. Integrand becomes

$$\frac{\prod_i (\xi_i^{x-y_i-1} e^{\varepsilon(\xi_i)t})}{\prod_{i<j}(p + q\xi_i\xi_j - \xi_i)} \cdot \sum_\sigma \operatorname{sgn}\sigma \left(\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)}) \right.$$

$$\left. \times \frac{\xi_{\sigma(2)}\xi_{\sigma(3)}^2 \cdots \xi_{\sigma(N)}^{N-1}}{(1 - \xi_{\sigma(2)}\xi_{\sigma(3)} \cdots \xi_{\sigma(N)})(1 - \xi_{\sigma(3)} \cdots \xi_{\sigma(N)}) \cdots (1 - \xi_{\sigma(N)})} \right).$$

A miracle! The sum equals

$$p^{N(N-1)/2} \frac{(1 - \xi_1 \cdots \xi_N) \prod_{i<j}(\xi_j - \xi_i)}{\prod_i (1 - \xi_i)}.$$

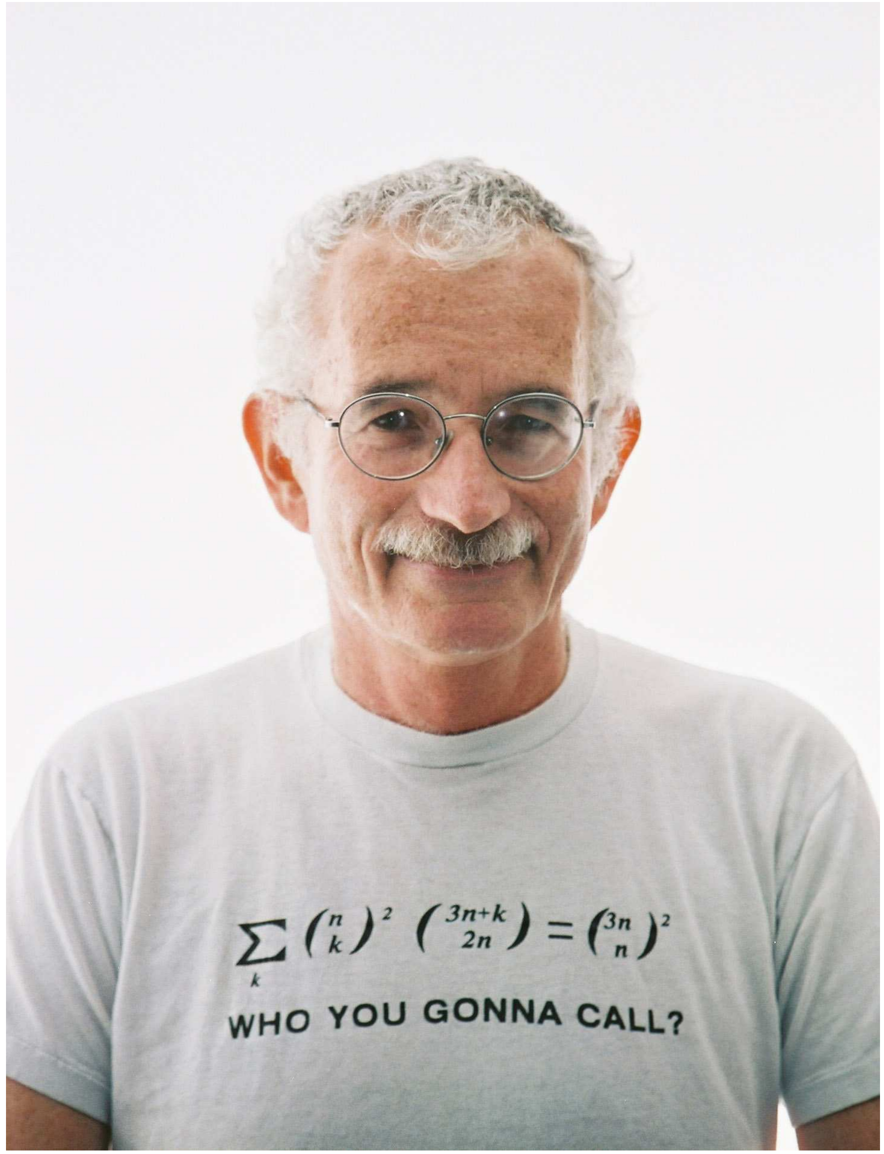


Figure 2: Doron Zeilberger, 2005.

Define

$$I(x,Y,\xi)=\prod_{i<j}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}\;\frac{1-\xi_1\cdots\xi_N}{\prod_i(1-\xi_i)}\;\prod_i\left(\xi_i^{x-y_i-1}e^{\varepsilon(\xi_i)t}\right).$$

Theorem (TW): When $p\neq 0$ and r is small enough

$$\mathbb{P}(x_1(t)=x)=p^{N(N-1)/2}\int_{\mathcal{C}_r^N}I(x,Y,\xi)\,d^N\xi.$$

Remark: To show that $\sum_{x=-\infty}^{\infty}\mathbb{P}(x_1(t)=x)=1$, to compute $\mathbb{E}(x_1(t))$ or to let $N\to\infty$, need integrals over $\mathcal{C}_R$ with R large.

Expand contours: Another miracle—only poles at $\xi_i=1$ contribute.

For $S \subset \{1, \ldots, N\}$, $I(x, Y_S, \xi)$ is $I(x, Y, \xi)$ but only indices $i \in S$.

Theorem (TW): When $q \neq 0$ and R is large enough

$$\mathbb{P}(x_1(t) = x) = \sum_S c_S \int_{\mathcal{C}_R^{|S|}} I(x, Y_S, \xi)\, d^{|S|}\xi,$$

where

$$c_S = \frac{p^{\,\sigma(S)-|S|}}{q^{\,\sigma(S)-|S|(|S|+1)/2}}, \quad \sigma(S) = \sum_{i \in S} i.$$

For initial configuration

$$y_1 < y_2 < \cdots \to +\infty$$

sum over all finite subsets of $\mathbb{Z}^+$.

Formulas for $\mathbb{P}(\mathbf{x_m(t) = x})$

$$x_1 = x - v_{m-1} - \cdots - v_1, \cdots, x_{m-2} = x - v_2 - v_1,\ x_{m-1} = x - v_1,$$

$$x_m = x,\ x_{m+1} = x + z_1, \ldots,\ x_N = x + z_1 + \cdots + z_{N-m}.$$

Sum over $z_i > 0$, $v_i > 0$, so take integrals over $\mathcal{C}_r$ and $\mathcal{C}_R$.

Take $S \subset \{1, \ldots, N\}$ with $|S| = m$ and sum over all σ such that

$$\sigma(\{1, \ldots, m\}) \subset S, \quad \sigma(\{m+1, \ldots, N\}) \subset S^c.$$

Use the miraculous formula twice and another one,

$$\sum_{|S|=m} \prod_{i\in S,\, j\in S^c} \frac{p + q\xi_i\xi_j - \xi_i}{\xi_j - \xi_i} \cdot (1 - \prod_{j\in S^c} \xi_j)$$

$$= C_{N,m}\, (1 - \prod_{j=1}^{N} \xi_j).$$

$$[N] = \frac{p^N - q^N}{p - q}, \qquad [N]! = [N]\,[N-1]\,\cdots\,[1],$$

$$\begin{bmatrix} N \\ m \end{bmatrix} = \frac{[N]!}{[m]!\,[N-m]!}, \qquad (q - \text{binomial coefficient}),$$

$$C_{N,m} = q^m \begin{bmatrix} N-1 \\ m \end{bmatrix}.$$

Theorem (TW): When $p \neq 0$

$$\mathbb{P}(x_m(t) = x) = \sum_{|S^c|<m} c_{N,m,S} \int_{\mathcal{C}_r^{|S|}} I(x, Y_S, \xi)\, d^{|S|}\xi.$$

When $q \neq 0$

$$\mathbb{P}(x_m(t) = x) = \sum_{|S|\geq m} c_{m,S} \int_{\mathcal{C}_R^{|S|}} I(x, Y_S, \xi)\, d^{|S|}\xi.$$

Second representation holds for $N = \infty$.

$Y = \mathbb{Z}^+$: Define k-dimensional integrand

$$J_k(x, \xi) = \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \, \frac{1}{\prod_i (1 - \xi_i)\,(q\xi_i - p)} \prod_i \left(\xi_i^{x-1} e^{\varepsilon(\xi_i)t} \right).$$

Corollary. When $Y = \mathbb{Z}^+$ and $q \neq 0$

$$\mathbb{P}(x_m(t) \leq x) = \sum_{k \geq m} c_{m,k} \int_{\mathcal{C}_R^k} J_k(x, \xi)\, d^k\xi.$$

When $p = 0$ (left-moving TASEP) only $c_{m,m}$ survives. Get $m \times m$ Toeplitz determinant

$$\mathbb{P}(x_m(t) \leq x) = \det \left(\int_{\mathcal{C}_R} \xi^{i-j+x-1} \, (\xi - 1)^{-m} \, e^{(\xi-1)t} \, d\xi \right).$$

This equals the Rákos-Schütz determinant$\Rightarrow$Johansson's result.

When $p, q \neq 0$,

$$J_k(x,\xi) = (-1)^k (pq)^{-k(k-1)/2} \det\left(\frac{\xi_j e^{\varepsilon(\xi_j)t}}{p + q\xi_i\xi_j - \xi_i}\right)_{1\le i,j\le k}$$

The integral

$$\int_{\mathcal{C}_R^k} J_k(x,\xi)\, d^k\xi.$$

is a coefficient in the expansion of $\det(I - \lambda q K)$ where K acts on $L^2(\mathcal{C}_R)$ and has kernel

$$K(\xi,\xi') = \frac{\xi^x e^{\varepsilon(\xi)t}}{p + q\xi\xi' - \xi}$$

Summing over k in the expression for $\mathbb{P}(x_m(t) \le x)$ gives our new representation.

Let $\tau = p/q < 1$ and set

$$(\lambda;\tau)_m = (1-\lambda)(1-\lambda\tau)\cdots(1-\lambda\tau^{m-1})$$

then with initial condition $Y = \mathbb{Z}^+$,

$$\boxed{\mathbb{P}(x_m(t) \le x) = \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{\det(I-\lambda q K)}{(\lambda;\tau)_m}\,\frac{d\lambda}{\lambda}}$$

Recall:

$$(Kf)(\xi) = \frac{\xi^x e^{\varepsilon(\xi)t}}{2\pi i}\int_{\mathcal{C}_R} \frac{1}{p+q\xi\xi'-\xi}\, f(\xi')\, d\xi'$$

In particular,

$$\mathbb{P}(x_1(t) \le x) = 1 - \det(I - qK)$$

Future Directions

1. **Limit Theorems:** The asymptotic regime of **universal current fluctuations** remains out of reach, but there is more reason for optimism now that we have reduced the problem to Fredholm determinants.

2. **Why miracles?** There were a number of places where certain *miraculous identities* saved the day. It would be helpful to understand this in a more structural way.

3. **Initial Conditions:** We assumed initial conditions Y that satisfy

$$y_1 < y_2 < \cdots < y_n < \cdots \to \infty$$

This restriction does not permit us to study the case of **stationary** ASEP, e.g. initial condition that is a product of Bernoulli measures.

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