# The Universality of the Distribution Functions of Random Matrix Theory

Craig A. Tracy
UC Davis

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# The Central Limit Theorem

Carl F. Gauss was the first to use the normal law (or Gaussian)

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt$$

as a bona fide distribution function. Earlier, A. De Moivre in his "Doctrine of Chances" (1733) had shown the normal law is a good *approximation* to the binomial distribution. However, it was Pierre S. Laplace (1820) who gave the first CLT.

In modern notation: $X_1, \ldots, X_n$ are discrete-valued independent and identically distributed random variables. Set $\mu = E(X_i)$ and $\sigma^2 = \mathrm{Var}(X_i)$, then

$$\Pr\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) \to \Phi(x), \;\; n \to \infty$$

where $S_n = X_1 + \cdots + X_n$.

Important features of Laplace's proof

- Introduced the characteristic function $E(\mathrm{e}^{itS_n})$ and used Laplace's method for approximating integrals
- He made the important observation that the limit law depended only upon $\mu$ and $\sigma$ of the underlying distribution. (Universality).

Laplace's ideas were further developed by Poisson, Dirichlet, Cauchy and others. The St. Petersburg School of probability (Chebyshev, Markov, Lyapunov, ...) relaxed the conditions of identically distributed as well as the important assumption of independence. This includes the important "method of moments"

The final form is attributed to Lindeberg, Feller and Levy where necessary and sufficient conditions are given for convergence to the normal law.

# Random Matrix Models

Probability Space: $(\Omega, \Pr, \mathcal{F})$:

- Gaussian Orthogonal Ensemble (GOE, $\beta = 1$):
  - $\Omega = N \times N$ real symmetric matrices
  - $\Pr =$ "unique" measure that is invariant under orthogonal transformations and matrix elements are iid random variables. Explicitly,

$$\Pr(A \in \mathcal{B}) = \int_{\mathcal{B}} \mathrm{e}^{-\mathrm{tr}(A^2)}\, dA$$

- Gaussian Unitary Ensemble (GUE, $\beta = 2$)
  - $\Omega = N \times N$ (complex) hermitian matrices
  - $\Pr =$ "unique" measure that is invariant under unitary transformations and the independent real and imaginary matrix elements are iid random variables
- Gaussian Symplectic Ensemble (GSE, $\beta = 4$)

# Limit Laws: $N \to \infty$

**Eigenvalues**, which are random variables, are real and with probability one they are distinct.

If $\lambda_{\mathbf{max}}(\mathbf{A})$ denotes the **largest eigenvalue** of the random matrix $A$, then for each of the **three Gaussian ensembles** we introduce the corresponding distribution function

$$F_{N,\beta}(t) := \mathrm{Pr}_\beta\left(\lambda_{\mathrm{max}} < t\right), \beta = 1, 2, 4.$$

The basic limit laws (Tracy-Widom) state that[a]

$$F_\beta(s) := \lim_{N\to\infty} F_{N,\beta}\left(2\sigma\sqrt{N} + \frac{\sigma s}{N^{1/6}}\right), \beta = 1, 2, 4,$$

exist and are given explicitly by

[a] Here $\sigma$ is the standard deviation of the Gaussian distribution on the off-diagonal matrix elements.

$$
\begin{aligned}
F_2(s) &= \det\left(I - K_{\text{Airy}}\right) \\
&= \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right)
\end{aligned}
$$

where

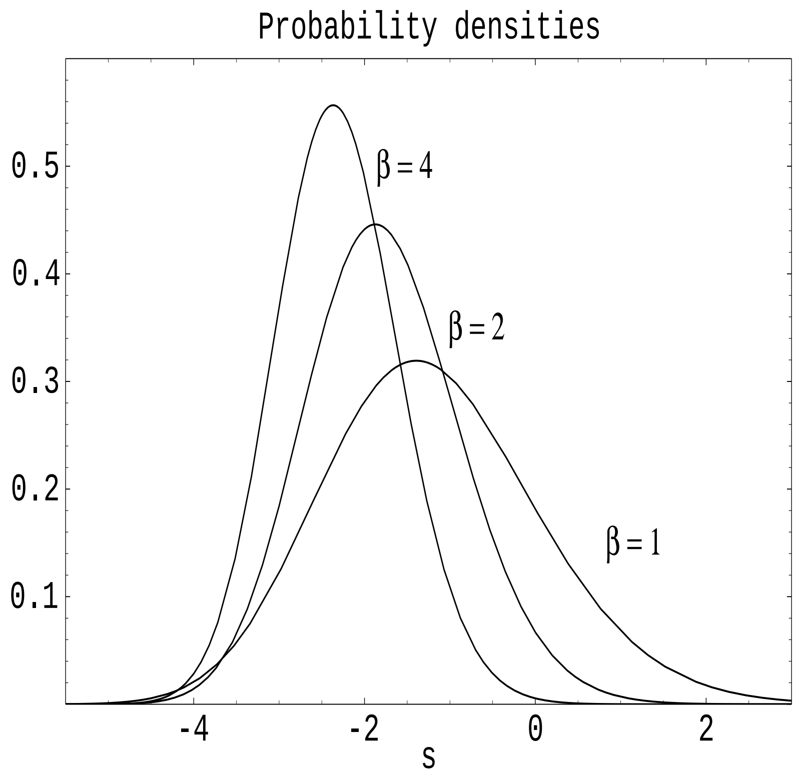
$$
K_{\text{Airy}} \doteq \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}
$$

acting on $L^2(s,\infty)$ (**Airy kernel**)

and $q$ is the unique solution to the **Painlevé II equation**

$$
q'' = sq + 2q^3, \quad q(s) \sim \mathrm{Ai}(s) \text{ as } s \to \infty.
$$

(Called the Hastings-McLeod solution.)




$$F_1(s) \quad = \quad \exp\left(-\frac{1}{2}\int_s^\infty q(x)\,dx\right)\ (F_2(s))^{1/2}\,,$$

$$F_4(s/\sqrt{2}) \quad = \quad \cosh\left(\frac{1}{2}\int_s^\infty q(x)\,dx\right)\ (F_2(s))^{1/2}\ .$$

# RMT Universality Theorems

Do limit laws depend upon the underlying Gaussian assumption on the probability measure?

To investigate this for unitarily invariant measures ($\beta = 2$):

$$\exp\left(-\mathrm{tr}(A^2)\right) \to \exp\left(-\mathrm{tr}(V(A))\right).$$

Bleher & Its chose

$$V(A) = gA^4 - A^2, g > 0,$$

and subsequently a large class of potentials $V$ was analyzed by Deift/Kriecherbauer/McLaughlin/Venakides/Zhou.

Requires proving new **Plancherel-Rotac**h type formulas for **nonclassical** orthogonal polynomials. The proofs use **Riemann-Hilbert methods**. Generic behavior is GUE. However, by tuning $V$ new universality classes will emerge.

Universality theorems for **orthogonal & symplectic** invariant measures:

- Stojanovic analyzed the quartic potential.
- Deift & Gioev considered a class of polynomial potentials whose equilibrium measure is supported on a single interval. Their starting point is Widom's representation of the correlation kernels for the beta=1,4 cases in terms of the unitary (beta=2) correlation kernel plus a correction.

All these results can be summarized by

**Generic edge behavior is described by Airy kernel**

# Noninvariant RMT Measures

Soshnikov proved that for real symmetric Wigner matrices[a] (complex hermitian Wigner matrices) the limiting distribution of the largest eigenvalue is $F_1$ (respectively, $F_2$). The **significance** of this result is that nongaussian Wigner measures lie **outside** the **"integrable class"** (e.g. there are no Fredholm determinant representations for the distribution functions) yet the limit laws are the same as in the integrable cases.

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[a] A symmetric Wigner matrix is a random matrix whose entries on and above the main diagonal are independent and identically distributed random variables with distribution function $F$. Soshnikov assumes $F$ is even and all moments are finite.

# Appearance of Limit Laws Outside of RMT

**Major breakthrough** when Baik, Deift, Johansson proved that the limiting distribution of the **length of the longest increasing subsequence** in a random permutation is $F_2$.

Random permutation of $\{1, 2, \ldots, 10\}$:

$$\sigma = \{\mathbf{3}, 7, 10, \mathbf{5}, 9, \mathbf{6}, \mathbf{8}, 1, 4, 2\}, \;\; \ell_{10}(\sigma) = 4$$

Patience Sorting Algorithm (Aldous, Diaconis)

$$\begin{array}{cccc} & 2 & & \\ & 4 & 6 & \\ 1 & 5 & 9 & \\ 3 & 7 & 10 & 8 \end{array}$$

**BDJ Theorem:**

$$\lim_{n\to\infty} \Pr\left(\frac{\ell_n - 2\sqrt{n}}{n^{1/6}} \le x\right) = F_2(x)$$

and with convergence of moments, e.g.

$$\begin{aligned} E(\ell_n) &= 2\sqrt{n} + \int_{-\infty}^{\infty} x f_2(x)\,dx\; n^{1/6} + \mathrm{o}(n^{1/6}) \\ &= 2\sqrt{n} - 1.7710868074\, n^{1/6} + \mathrm{o}(n^{1/6}) \end{aligned}$$

A simulation with 100 trials for $n = 10^5$ gives an average number of piles per trial

**621.96**

which should be compared with the asymptotic expected value

**620.389**

The $2\sqrt{n}$ term alone gives **632.456**.

The BDJ Theorem resulted in a burst of activity relating the distribution functions of RMT to problems in combinatorics, representation theory of the symmetric group, growth processes and determinantal random fields

**Cast of Players**

M. Adler, D. Aldous, J. Baik, P. Bleher, T. Bodineau, A. Borodin, P. Deift, P. Diaconis, P. Ferrari, P. Forrester, J. Gravner, T. Imamura, A. Its, K. Johannson, J. Martin, K. McLaughlin, N. O'Connell, A. Okounkov, G. Olshanski, M. Prähoffer, E. Rains, N. Reshetikhin, T. Sasamoto, A.Soshnikov, H. Spohn, C. Tracy, P. van Moerbeke, H. Widom, ...

# From Brownian Motion to the Airy Process

$$t \longrightarrow B_t$$

is a Gaussian process: Fix $t_1 < t_2 < \cdots < t_m$,

$$(B_{t_1}, B_{t_2}, \ldots, B_{t_m})$$

is a multivariate Gaussian, e.g.

$$\Pr\left(B_t \leq x\right) = \Phi(x)$$

The **Airy Process** (Prähoffer, Spohn, Johansson)

$$t \longrightarrow A_t$$

is the process underlying $F_2$, e.g.

$$\Pr\left(A_t \leq x\right) = F_2(x)$$

# Dyson BM

GUE initial conditions and independent matrix elements of a Hermitian matrix $H$ independently undergo Ornstein-Uhlenbeck diffusion

$$t \to H_t.$$

Transition density

$$p(H, H'; t_2 - t_1) := \exp\left(-\frac{\mathrm{tr}(H - qH')^2}{1 - q^2}\right) / Z$$

$q = \mathrm{e}^{t_1 - t_2} < 1.$

As $t_2 \to \infty$, measure approaches GUE measure.

Each eigenvalue feels an electric field

$$E(x_i) = \sum_{i \neq j} \frac{1}{x_i - x_j} - x_i$$

**Many times**: $t_1 < t_2 < \cdots < t_m$

With GUE initial conditions the density for $H_t$ in neighborhood of $H_k$ at time $t = t_k$ is

$$\mathrm{e}^{-\mathrm{tr}(H_1^2)} \prod_{j=2}^{m} p(H_j, H_{j-1}, t_j - t_{j-1})$$

Use HCIZ integral to integrate out unitary parts to obtain determinantal measure on eigenvalues $x_j(t)$

Focus on the **largest eigenvalue**

$$t \to x_{\max}(t)$$

In **edge scaling limit** obtain

$$t \to A_t$$

# Airy Process

Defined by the distribution functions

$$\Pr(A_{t_1} \le \xi_1, \ldots, A_{t_m} \le \xi_m)$$

Probability expressed as a Fredholm determinant of extended Airy kernel, an $m \times m$ matrix kernel. Entries $L_{ij}(x,y)$ given by

$$\int_0^{\infty} \mathrm{e}^{-z\,(t_i - t_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz, \quad i \ge j,$$

$$-\int_{-\infty}^{0} \mathrm{e}^{-z\,(t_i - t_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz, \; i < j$$

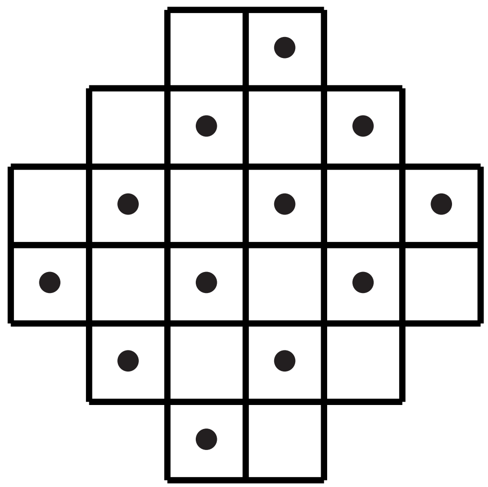
$$K_{ij}(x,y) = L_{ij}(x,y)\,\chi_{(\xi_j,\infty)}(y).$$

Probability equals $\det\,(I - K)$.

# Aztec Diamond $A_n$

Elkies, Kuperberg, Larsen, Propp, . . .

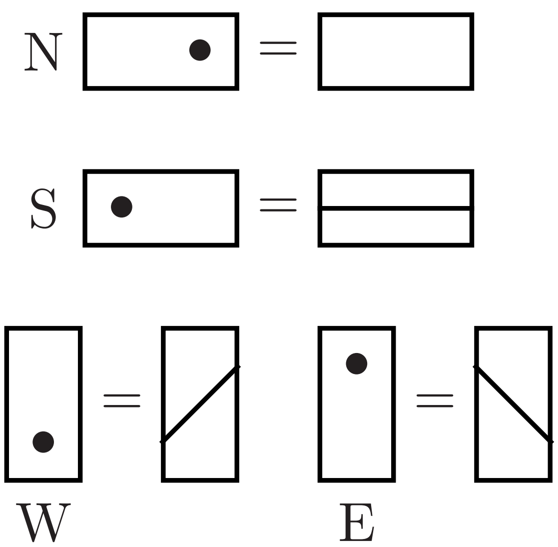
$A_n$: Union of all lattice squares that lie inside $\{|x| + |y| \leq n + 1\}$.



$A_3$ with checkerboarding

‡ Tile with $2 \times 1$ and $1 \times 2$ dominoes.

‡ Checkerboard lattice. Four types of tiles: N, S, E, W.




‡ $X_n(t)$ is top line.

‡ The Northern Polar Region (NPR) is exactly the part of the domino tiling that lies above $X_n(t)$, and consists only of N-dominoes.

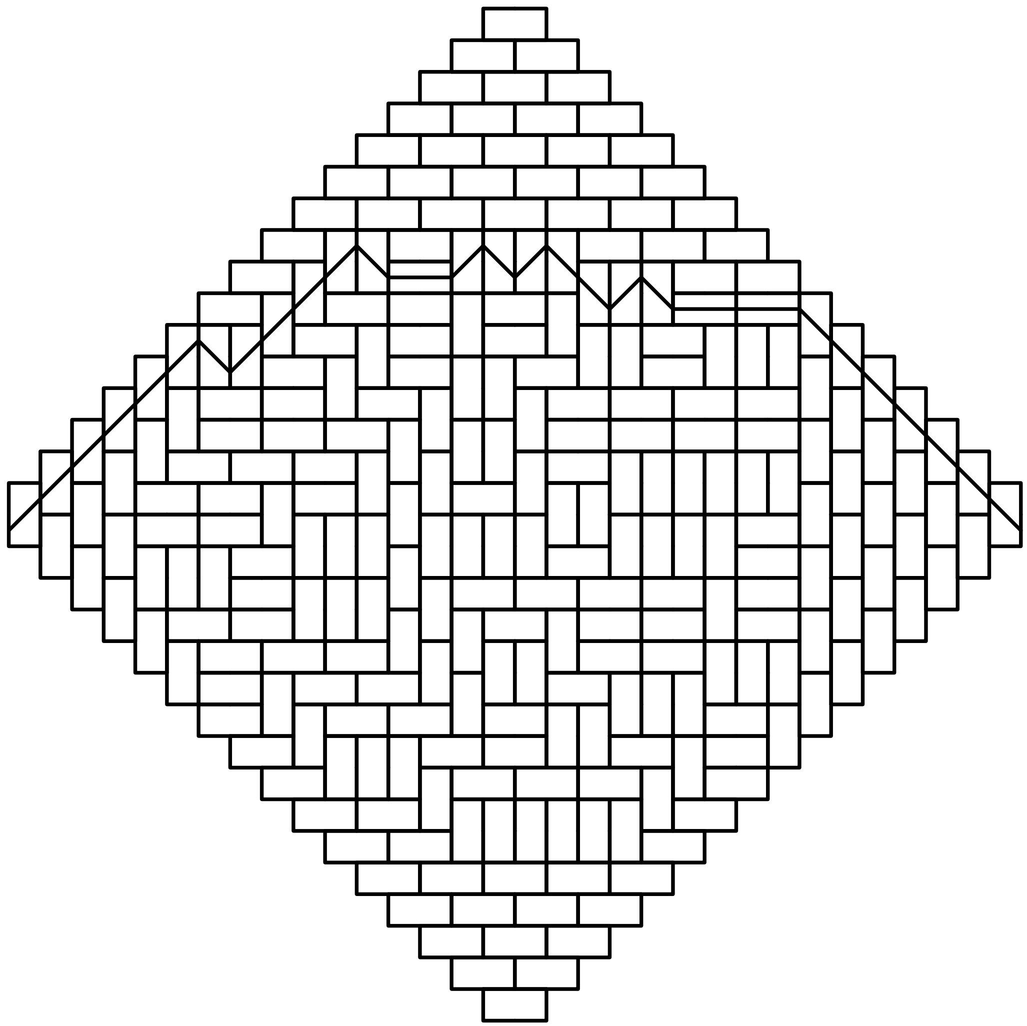
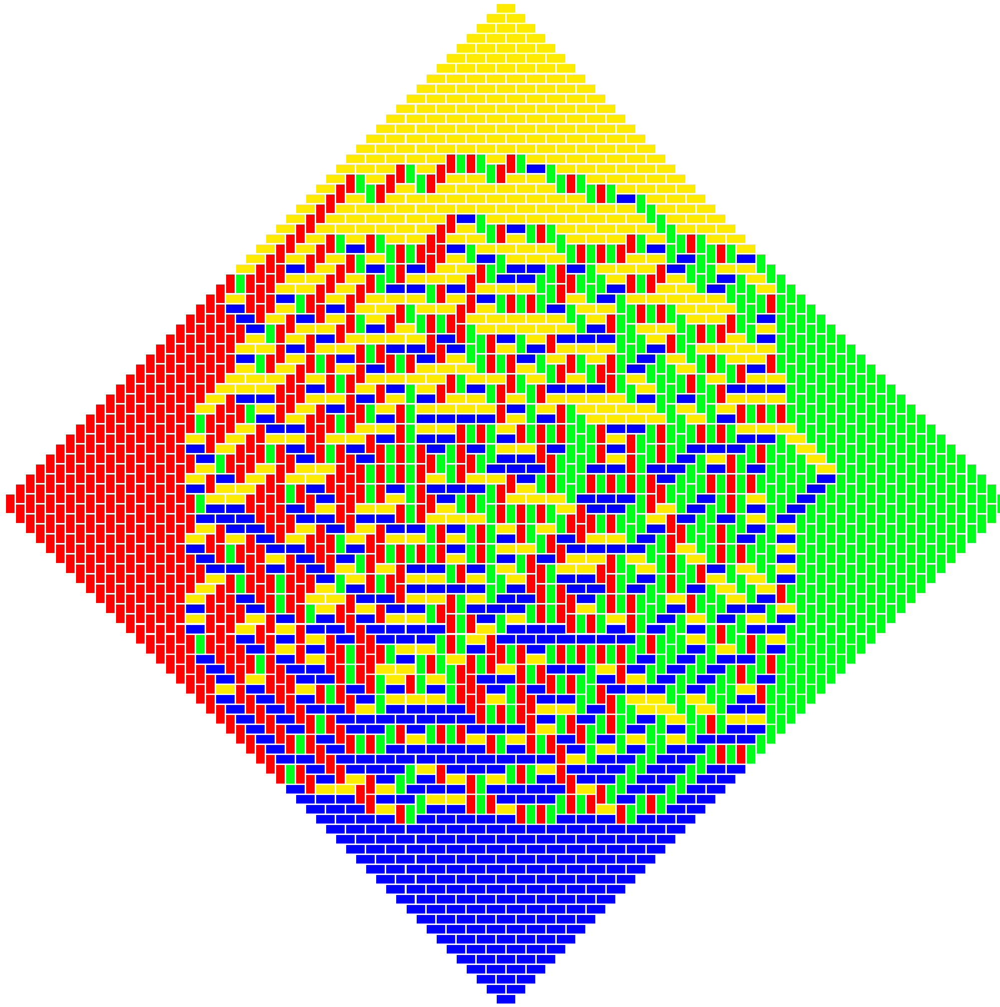


Figure 1: Top Curve $X_n(t)$ [Johansson]



Random Tilings Research Group

**Theorem** (Johansson) Let $X_n(t)$ be the NPR-boundary process and $A_t$ the Airy process, then

$$\frac{X_n(2^{-1/6}n^{2/3}t) - n/\sqrt{2}}{2^{-5/6}n^{1/3}} \to A_t - t^2,$$

as $n \to \infty$, in the sense of convergence of finite-dimensional distributions.