Differential Equations for Dyson Processes

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Overview

Dyson process: Any process on ensembles of matrices in which the entries undergo diffusion.

Dyson BM: Entries of finite GUE matrix independently undergo Ornstein-Uhlenbeck diffusion. Eigenvalues describe n curves: **Hermite Process**.

Let $n \to \infty$, scale near the top. Infinitely many curves, **Airy process**. Top curve $A(\tau)$. From work of Johansson and Prähoffer & Spohn, the Airy process is now believed to underly a large class of growth processes. ($1 + 1$ KPZ Universality Class)

Scale the Hermite process in the bulk $\longrightarrow$ **sine process**.

Evolution of singular values of complex matrices leads to **Laguerre process**; scaling this at bottom edge gives **Bessel process**.

Aztec Diamond A_n

Elkies, Kuperberg, Larsen, Propp, ...

- A_n: Union of all lattice squares that lie inside $\{|x| + |y| \leq n + 1\}$.
- Tile with 2×1 and 1×2 dominoes.
- Checkerboard lattice. Four types of tiles: N, S, E, W.
- Draw lines on S, E, W dominoes. $X_n(t)$ is top line.
- The Northern Polar Region (NPR) is exactly the part of the domino tiling that lies above $X_n(t)$, and consists only of N-dominoes.

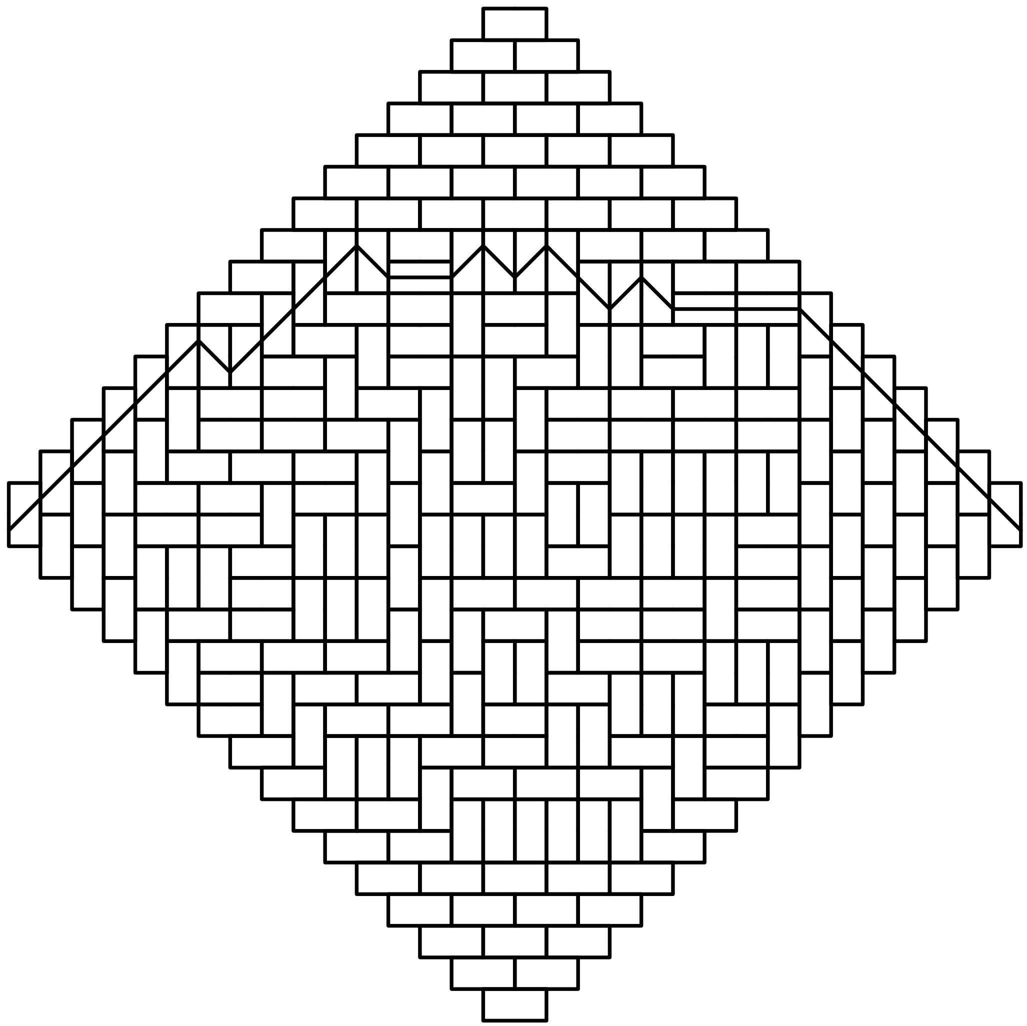
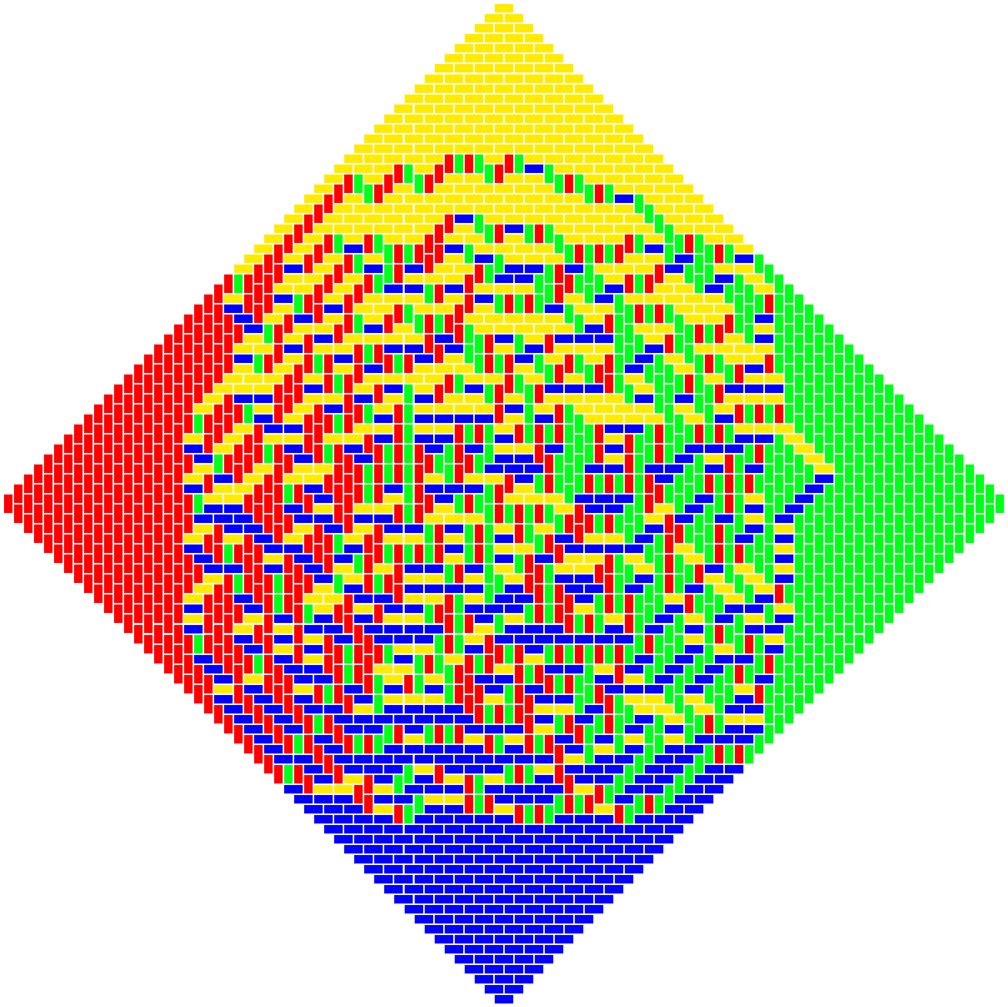


Figure 1: Top Curve $X_n(t)$ [Johansson]



Random Tilings Research Group

Theorem (Johansson) Let $X_n(t)$ be the NPR-boundary process and $A(\tau)$ the Airy process, then

$$\frac{X_n(2^{-1/6}n^{2/3}t) - n/\sqrt{2}}{2^{-5/6}n^{1/3}} \to A(t) - t^2,$$

as $n \to \infty$, in the sense of convergence of finite-dimensional distributions.

The expectation is that the Airy process governs a large class of growth processes which physicists call the KPZ Universality Class. (KPZ=Kardar, Parisi, Zhang).

Dyson BM

GUE initial conditions and independent matrix elements independently undergo OU diffusion:

$$\tau \to H(\tau)$$

Transition density

$$p(H, H'; \tau_2 - \tau_1) := \exp\left(-\frac{\mathrm{tr}(H - qH')^2}{1 - q^2}\right) / Z$$

$q = e^{\tau_1 - \tau_2} < 1.$

As $\tau_2 \to \infty$, measure approaches GUE measure.

Each eigenvalue feels an electric field

$$E(x_i) = \sum_{i \neq j} \frac{1}{x_i - x_j} - x_i$$

Many times: $\tau_1 < \tau_2 < \cdots < \tau_m$

With GUE initial conditions the density for $H(\tau_k)$ in neighborhood of H_k is

$$e^{-\mathrm{tr}(H_1^2)} \prod_{j=2}^{m} p(H_j, H_{j-1}, \tau_j - \tau_{j-1})$$

Use HCIZ integral to integrate out unitary parts to obtain determinantal measure on eigenvalues $x_j(\tau)$

Leads to **extended kernels** (Eynard & Mehta, Johansson, Prähoffer & Spohn) and by scaling to

extended Airy kernel, extended sine kernel, extended Bessel kernel. . .

Airy Process

Defined by the distribution functions

$$\Pr(A(\tau_1) \le \xi_1, \ldots, A(\tau_m) \le \xi_m)$$

Probability expressed as a Fredholm determinant of extended Airy kernel, an $m \times m$ matrix kernel. Entries $L_{ij}(x, y)$ given by

$$\int_0^\infty e^{-z\,(\tau_i - \tau_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz, \quad i \ge j,$$

$$-\int_{-\infty}^0 e^{-z\,(\tau_i - \tau_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz, \; i < j$$

$$K_{ij}(x, y) = L_{ij}(x, y)\,\chi_{(\xi_j, \infty)}(y).$$

Probability equals $\det\,(I - K)$.

Remarks

1. For $m = 1$ extended kernel reduces to Airy kernel—an integrable kernel in the sense of A. Its et al.. Not 'integrable' for $m > 1$.

2. For $m = 1$ Fredholm determinant is a τ-function for Painlevé II, ψ.

3. Relationship between the two is

$$\psi(\xi) = (I - K_{\mathrm{Airy}})^{-1} \, \mathrm{Ai}(x)|_{x=\xi}$$

4. Integrable differential equations for $m > 1$? Answered affirmatively by Adler and van Moerbeke and TW.

Set $R = K\,(I-K)^{-1}$, then

$$\partial_{\xi_k} \log \det (I-K) = R_{kk}(\xi_k,\, \xi_k)$$

Unknowns: Five **matrix functions** of the ξ_k. First is

$$r_{ij} = R_{ij}(\xi_i,\, \xi_j).$$

To define others, let

$$A = \mathrm{diag}\,(\mathrm{Ai}), \quad \chi = \mathrm{diag}\,(\chi_{(\xi_k,\infty)}),$$

$$Q = (I-K)^{-1} A, \quad \tilde{Q} = A\chi(I-K)^{-1}.$$

Other unknowns are

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j),$$

$$q'_{ij} = Q'_{ij}(\xi_i), \quad \tilde{q}'_{ij} = \tilde{Q}'_{ij}(\xi_j).$$

Define r_x and r_y by

$$(r_x)_{ij} = (\partial_x R)_{ij}(\xi_i, \xi_j)$$

$$(r_y)_{ij} = (\partial_y R)_{ij}(x_i, \xi_j).$$

r_x and r_y are **not** unknowns.

Set $\xi = \operatorname{diag}(\xi_k)$. Equations are

$$\begin{aligned}
dr &= -r\, d\xi\, r + d\xi\, r_x + r_y\, d\xi, \\
dq &= d\xi\, q' - r\, d\xi\, q, \\
d\tilde{q} &= \tilde{q}'\, d\xi - \tilde{q}\, d\xi\, r, \\
dq' &= d\xi\, \xi\, q - (r_x\, d\xi + d\xi\, r_y)\, q + d\xi\, r\, q', \\
d\tilde{q}' &= \tilde{q}\, \xi\, d\xi - \tilde{q}\, (d\xi\, r_y + r_x\, d\xi) + \tilde{q}'\, r\, d\xi.
\end{aligned}$$

Diagonal entries of $r_x + r_y$ and off-diagonal entries of r_x and r_y are expressible in terms of the unknowns. Here is where the τ_k enter. Let $\tau = \operatorname{diag}(\tau_k)$ and Θ the matrix with all entries equal to one.

$$r_x + r_y = -q\,\Theta\,\tilde{q} + r^2 + [\tau,\, r],$$
$$[\tau,\, r_x - r_y] = q'\,\Theta\,\tilde{q} - q\,\Theta\,\tilde{q}' + [r,\, r_x + r_y] + [\xi,\, r].$$

To prove these we used the Airy commutators

$$\begin{aligned} [D,\, L] &= -A\Theta A + [\tau,\, r] \\ \left[D^2 - M\,,L\right] &= 0 \end{aligned}$$

When $m = 1$ these equations reduce ($\tilde{q} = q = \psi$, $\tilde{q}' = q' = d\psi/d\xi + r\psi$) to the single Painlevé II equation

$$\frac{d^2\psi}{d\xi^2} = \xi\psi + 2\psi^3$$

Remarks

Adler & van Moerbeke used their DEs to derive $\tau \to \infty$ asymptotics for

$$\frac{\Pr\left(A(0) \le \xi_1, A(\tau) \le \xi_2\right)}{F_2(\xi_1) F_2(\xi_2)} = 1 + \frac{c_2(\xi_1, \xi_2)}{\tau^2} + \frac{c_4(\xi_1, \xi_2)}{\tau^4} + \mathrm{O}(\tau^{-6})$$

and Widom derived the same asymptotic expansion directly from the Fredholm determinant representation. The important feature is that c_2 and c_4 are expressible in terms of the Painlevé II function ψ, e.g.

$$c_2(\xi_1, \xi_2) = u(\xi_1) u(\xi_2), \;\; u(\xi) = \int_\xi^\infty \psi^2(x)\, dx$$

These same methods, e.g. perturbation expansion of DEs or expansion of Fredholm determinant, show that the **matrix Painlevé function q**

$$q(\xi) = \begin{pmatrix} \psi(\xi_1) & 0 \\ 0 & \psi(\xi_2) \end{pmatrix} +$$

$$\frac{1}{\tau} \begin{pmatrix} 0 & -u(\xi_1)\psi(\xi_2) \\ \psi(\xi_1)u(\xi_2) & 0 \end{pmatrix} + \mathrm{O}(\tau^{-2})$$

That is, matrix Painlevé q is decoupling in $\tau \to \infty$ asymptotics to scalar Painlevé II.

Open Problems for Airy System

1. Are equations deformation equations for some isomondromy problem and is Fredholm determinant the associated τ-function in sense of Jimbo-Miwa-Ueno?

2. We proved compatibility for small m using Maple. Give general conceptual proof. Difficulty lies with the conditions determining r_x and r_y.

3. Systemize large τ asymptotics. Find small τ expansions.

We have systems of PDEs that determine the Fredholm determinant of

- Extended Hermite kernel
- Extended Sine kernel
- Extended Bessel kernel

They are more complicated than the extended Airy system. Each requires a special trick. Adler & van Moerbeke also have system of DEs for extended Hermite kernel.

Higher Universality Classes

1. Airy kernel arises as a fold singularity: coalescence of two saddle points

2. Pearcey kernel (Brézin & Hikami, Okounkov & Reshetikhin, Bleher & Kuijlaars) arises as a cusp singularity: coalescence of three saddle points.

3. General Problem:

$$\text{Singularity} \longrightarrow \text{Diffraction Integral} \longrightarrow \text{Kernel} \longrightarrow \text{Process}$$