APPLICATIONS of RANDOM MATRIX THEORY to COMBINATORICS and GROWTH PROCESSES

Craig A. Tracy

UC Davis

February, 2000

- Random Matrix Models

- Fredholm Det. Repr. of Distribution Fns

- Edge Scaling Limit

- Largest Eigenvalue Distributions

- Applications

  - Random Permutations

  - Random Words

  - Growth Processes

  - Arctic Circle Theorem

  - KPZ Universality Class

Random Matrix Models (RMM)

RMM=Probability space $(\Omega, \mathcal{P}, \mathcal{F})$, Ω=set of matrices, $\mathcal{P}$ = probability measure

- Gaussian Orthogonal Ensemble ($\beta = 1$)
  - $\Omega = N \times N$ real symmetric matrices
  - $\mathcal{P}$ = "unique" measure that is invariant under orthogonal transformations & matrix elements are iid rv's.
- Gaussian Unitary Ensemble ($\beta = 2$)
  - $\Omega = N \times N$ hermitian matrices
  - $\mathcal{P}$= "unique" measure that is invariant under unitary transformations & real and imaginary matrix elements are iid rv's.
- Gaussian Symplectic Ensemble ($\beta = 4$)

Expected Values

$f : \Omega \to \mathbf{C}$,

$$E(f) = \int_\Omega f(M)\, d\mathcal{P}(M)$$

Suppose $f(M)$ depends only on the eigenvalues of M.

Finite N Gaussian Ensembles ($\beta = 1, 2, 4$)

$$\begin{aligned} E_{N\beta}(f) &= c_{N\beta} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_N) \times \\ &\qquad |\Delta(x_1, \ldots, x_N)|^\beta \, e^{-\frac{\beta}{2}\sum x_j^2} \, dx_1 \cdots dx_N \end{aligned}$$

where Δ is the Vandermonde determinant.

$E_{N\beta}(0; J) :=$ probability no eigenvalues lie in J

Here $f(x_1, \ldots, x_N) = \chi_J(x_1) \cdots \chi_J(x_N)$.

Example: $J = (t, \infty)$

$$\begin{aligned} F_{N\beta}(t) &:= \text{Prob}\,(\lambda_{\text{max}} \le t) \\ &= E_{N\beta}(0, J) \end{aligned}$$

Gaudin, Dyson, Mehta showed for $\beta = 1, 2, 4$

$$\begin{aligned} E_{N\beta}(0; J)^{1,2} &= \det\big(I - K_{N\beta}\big) \\ K_{N\beta} &= \text{integral operator on } J \\ &\ (\text{scalar kernel, matrix kernel}) \end{aligned}$$

For $\beta = 2$ the kernel of the operator $K_{N\,2}$ is

$$\frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}$$

$$\begin{aligned} \varphi(x) &= c_N e^{-x^2/2} H_N(x) \\ \psi(x) &= c_N e^{-x^2/2} H_{N-1}(x) \end{aligned}$$

Edge Scaling Limit (Limiting Laws)

Density of eigenvalues decays exponentially fast around

$$2\sigma\sqrt{N}$$

Surprisingly (Bai and Yin, . . .),

$$\lim_{N\to\infty} \frac{\lambda_{\max}(N)}{\sqrt{N}} = 2\sigma, \ a.s.$$

Need scale of fluctuations. Define scaled random variable λ

$$\lambda_{\max} = 2\sigma\sqrt{N} + \frac{\sigma\lambda}{N^{1/6}}$$

Famous $N^{1/6}$ known to Wigner. (Bowick & Brézin, Forrester, Widom & CT, . . .)

CLAIM:

$$\text{Prob}\,(\lambda_{\max} \le t) = \text{Prob}\,(\lambda \le s) \to F_\beta(s)$$

where $t = 2\sigma\sqrt{N} + \sigma s/N^{1/6}$, $N \to \infty$, s fixed.

THEOREM:(Widom, CT)

$$\begin{aligned} F_{N,2}(t) &\to F_2(s) \text{ (edge scaling limit)} \\ F_2(s) &= \det\left(I - K_{\mathrm{Airy}}\right) \\ &= \exp\left(-\int_s^\infty (x-s) q^2(x)\, dx\right) \end{aligned}$$

where

$$\begin{aligned} q'' &= sq + 2q^3 \ (P_{II}) \\ q(s) &\sim \mathrm{Ai}(s) \text{ as } s \to \infty \end{aligned}$$

Remarks:

- $F_2(s)$ is the P_{II} τ-function of Okamoto, Jimbo and Miwa.

- q with above boundary condition exists. Existence & asymptotics as $s \to -\infty$ by Hastings, Clarkson, McLeod, Deift, Zhou,. . .

CASES $\beta = 1$ and $\beta = 4$

In these two cases

$$E_\beta(0;J)^2 = \det\left(I - K_{N,\beta}\right)$$

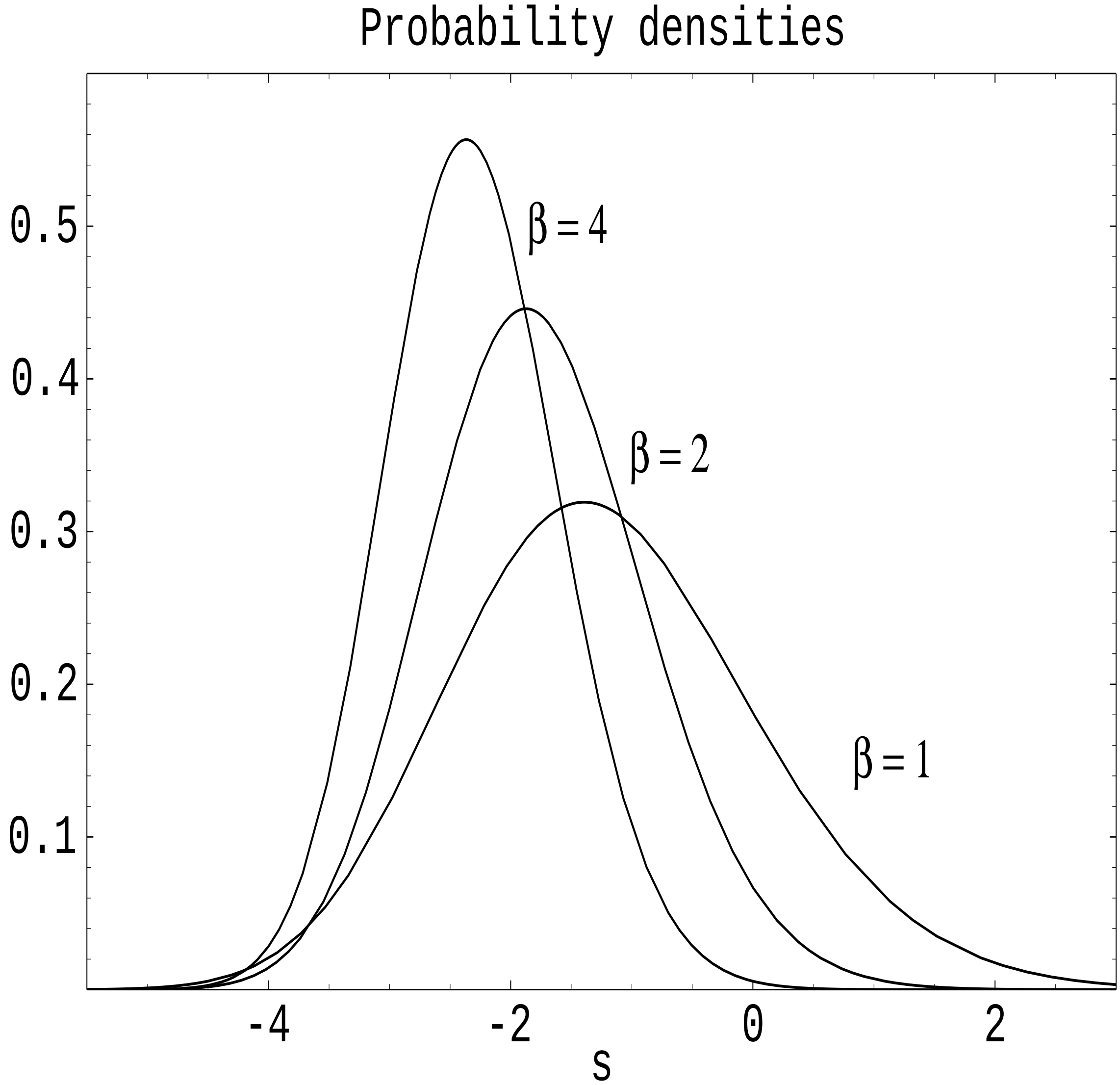
the operator $K_{N,\beta}$ is a 2×2 matrix with operator entries. For Gaussian ensembles and J a union of open intervals—general case worked out by Widom and CT. Specializing to the case

$$J = (t,\infty)$$

and in the edge scaling limit

$$\begin{aligned} F_{N,\beta}(t) &\to F_\beta(s),\ \beta = 1,4, \\ F_1(s)^2 &= \exp\left(-\int_s^\infty q(x)\,dx\right) F_2(s) \\ F_4(s/\sqrt{2})^2 &= \cosh\left(\frac{1}{2}\int_s^\infty q(x)\,dx\right) F_2(s) \end{aligned}$$

where q is the P_{II} function and $F_2(s)$ the limiting distribution function for $\beta = 2$.



Probability densities of scaled largest eigenvalue $\beta = 1, 2, 4$

$$f_\beta(s) = \frac{dF_\beta}{ds}$$

PATIENCE SORTING

D. Aldous and P. Diaconis

Shuffle a deck of cards $\{1, 2, \ldots, N\}$

Rules of the game

- Turn over first card.
- Turn over second card. If second card is of higher rank, start new pile to the right of the first card. Otherwise place second card on top of first card.
- Turn over third card. If third card is of higher rank than either first or second card, start a new pile to the right of the second card. Otherwise place third card on top of card of higher rank. If both first and second are of higher rank, place third card on the smaller ranked card.
- Continue playing game until no more cards.

- Object of game is to get a minimum number of piles.

A shuffled deck of cards

$$\{i_1, i_2, \ldots, i_N\}$$

is a permutation σ of $\{1, 2, \ldots, N\}$. Let

$$\ell_N(\sigma)$$

equal the number of piles at the end of the game.

CLAIM: $\ell_N(\sigma)$ is the length of the longest increasing subsequence in the permutation σ.

Example: $N = 12$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 3 & 2 & 4 & 1 & 10 & 12 & 7 & 6 & 9 & 8 & 11 \end{pmatrix}$$

$$\begin{array}{lllll} 1 & & & & \\ 2 & & 6 & 8 & \\ 3 & & 7 & 9 & \\ 5 & 4 & 10 & 12 & 11 \end{array}$$

CONNECTION WITH RSK and RMT

For permutations RSK is a bijection

$$\sigma \leftrightarrow (P, Q)$$

P and Q are SYTs of same shape $\lambda \vdash N$. For previous example

$$\sigma = (5\ 3\ 2\ 4\ 1\ 10\ 12\ 7\ 6\ 9\ 8\ 11)$$

$\lambda = (5, 3, 3, 1) \vdash 12$ and

$$P = \begin{array}{lllll} 1 & 4 & 6 & 8 & 11 \\ 2 & 7 & 9 & & \\ 3 & 10 & 12 & & \\ 5 & & & & \end{array} \qquad Q = \begin{array}{lllll} 1 & 4 & 6 & 7 & 12 \\ 2 & 8 & 10 & & \\ 3 & 9 & 11 & & \\ 5 & & & & \end{array}$$

Furthermore,

$$\ell_N(\sigma) = \#\ \text{boxes in first row}$$

$\boxed{\text{G}}$ = Gessel, $\boxed{\text{BO}}$ = Borodin & Okounkov

$\boxed{\text{BDJ}}$ = Baik, Deift & Johansson

$\boxed{\text{J}}$ = Johansson

$$e^{-t}\sum_{N=0}^{\infty} \text{Prob}\,(\ell_N(\sigma) \le n)\,\frac{t^N}{N!}$$

$$= e^{-t}\det\,(T_n(\varphi))\,,\ \varphi(z) = e^{\sqrt{t}(z+1/z)}\ \boxed{\text{G}}$$

$$= \det\,(I - K_B)\,,\ B = \text{Discrete Bessel}\ \boxed{\text{BO}}$$

$$\to \det\left(I - K_{\text{Airy}}\right) = F_2(s)\ \boxed{\text{BDJ, BO, J}}$$

Arrow means set

$$n = [2\sqrt{t} + s\,t^{1/6}]$$

and let $t \to \infty$ such that s is fixed.

This implies (dePoissonization)

Theorem(Baik-Deift-Johansson)

$$\lim_{N\to\infty} \text{Prob}\left(\frac{\ell_N(\sigma) - 2\sqrt{N}}{N^{1/6}} \le s\right) = F_2(s)$$

SYMMETRIZED RANDOM PERMUTATIONS

Look at random permutations that are involutions and fixed point free. Ask for limiting laws of random variables for longest increasing subsequences and longest decreasing subsequences.

Baik and Rains show limiting laws are now $F_4(s)$ and $F_1(s)$, respectively.

RANDOM WORDS

Homogeneous: Widom & CT, Johansson,

Inhomogeneous: Its, Widom & CT.

Words of length N are formed from letters from an alphabet of fixed size k.

w = b c a a d c c a a , N=9, k=4

Letters occur independently but with possibly different probabilities p_i. Interested in length $\ell_N(w)$ of longest weakly increasing subsequence in word w.

Version of RSK for words leads to measure

$$s_\lambda(p_1, p_2, \ldots) f^\lambda$$

on partitions. Here s_λ are the Schur functions and f^λ is the number of SYTs of shape λ.

Stanley was led to this same measure for generalized riffle shuffles generalizing work of Bayer and Diaconis.

Order the p_i

$$p_1 \geq p_2 \geq \cdots \geq p_k$$

Decompose alphabet into subsets $\mathcal{A}_1$,$\mathcal{A}_2$,... such that $p_i = p_j$ if and only if i and j belong to the same $\mathcal{A}_\alpha$. Set $k_\alpha = |\mathcal{A}_\alpha|$.

Theorem: The limiting distribution as $N \to \infty$ for the appropriately centered and normalized random variable ℓ_N is related to the distribution function for the eigenvalues ξ_i in the direct sum of mutually independent $k_\alpha \times k_\alpha$ GUEs conditioned on the eigenvalues satisfying

$$\sum_i \sqrt{p_i}\,\xi_i = 0$$

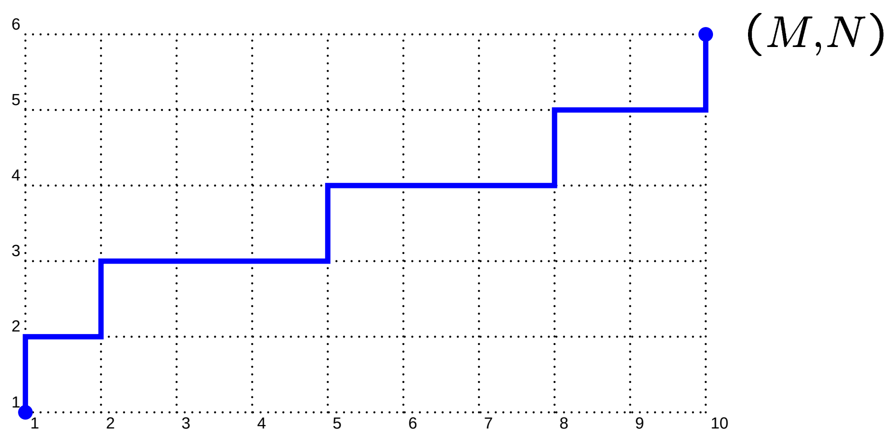
For all p_i distinct and $N \to \infty$

$$E(\ell_N) = Np_1 + \sum_{j>1} \frac{p_j}{p_1 - p_j} + O(\frac{1}{\sqrt{N}})$$

GROWTH PROCESSES

Long history, but Johansson first to find the limiting laws involving distributions $F_\beta(s)$. Since then work by Prähofer & Spohn; Gravner, Widom, & CT and possibly other works in progress.

$$\text{Prob}\,(w(i,j) = k) = (1-q)\,q^k$$



$$\begin{aligned} \Pi_{M,N} &= \text{up/right paths from } (1,1) \text{ to } (M,N) \\ G(M,N) &= \max_{\pi\in\Pi_{M,N}} \sum_{(i,j)\in\pi} w(i,j) \end{aligned}$$

Theorem(Johansson) For each $q \in (0,1)$, $\gamma \geq 1$, and $s \in \mathbf{R}$

$$\lim_{N\to\infty} \text{Prob}\left(\frac{G([\gamma N],N) - N\omega(\gamma,q)}{\sigma(\gamma,q)N^{1/3}} \leq s\right) = F_2(s)$$

where ω and σ are explicitly given functions of γ and q.

Johansson: Restricted geometries lead to $F_1(s)$

Prähofer & Spohn using the symmetrized random permutations work of Baik & Rains have shown in a certain polynuclear growth model that the universality class $\beta = 2$ is associated with growth from a single droplet and $\beta = 1$ class is associated with growth from a flat substrate, i.e. limiting distribution is $F_1(s)$.

Discrete Time and Space Growth Models

Oriented Digital Boiling

Gravner, Widom, CT

Introduce a height function $h_t(x)$ with rules

(i) $h_t(x) \le h_{t+1}(x)$

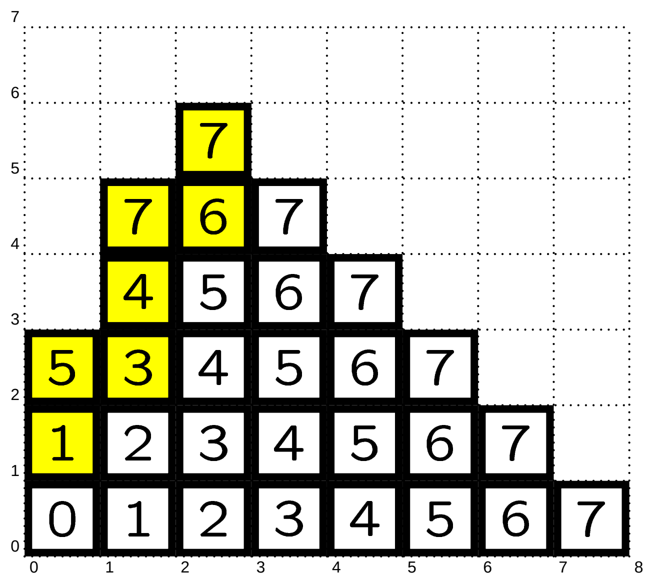
(ii) If $h_t(x-1) > h_t(x)$,

then $h_{t+1}(x) = h_t(x-1)$

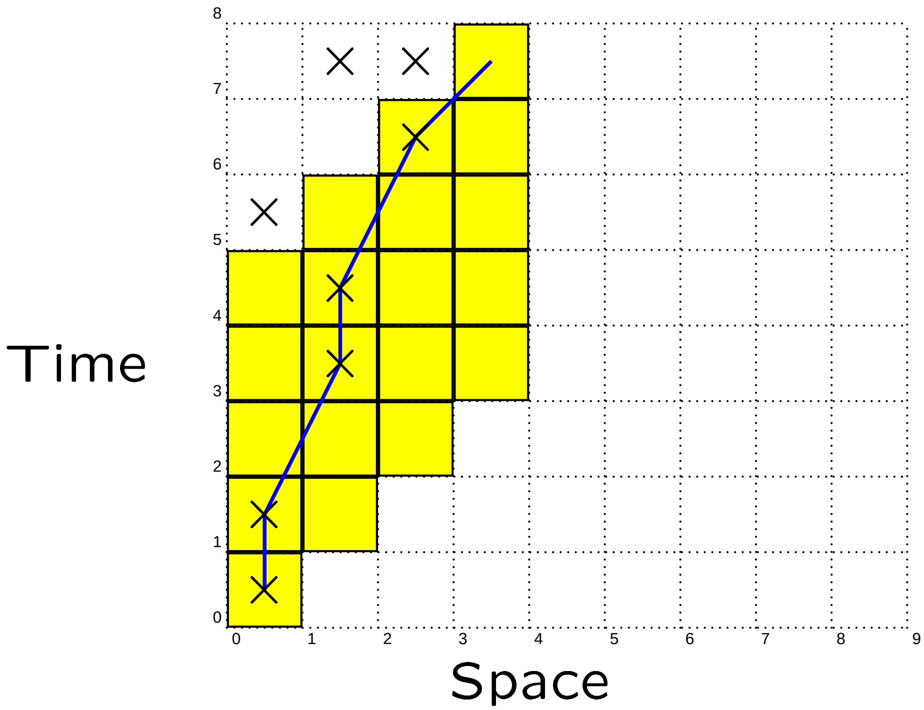
(iii) Otherwise, then independently of other sites and other times,

$h_{t+1}(x) = h_t(x) + 1$ with prob p

$h_{t+1}(x) = h_t(x)$ with prob $1-p$.

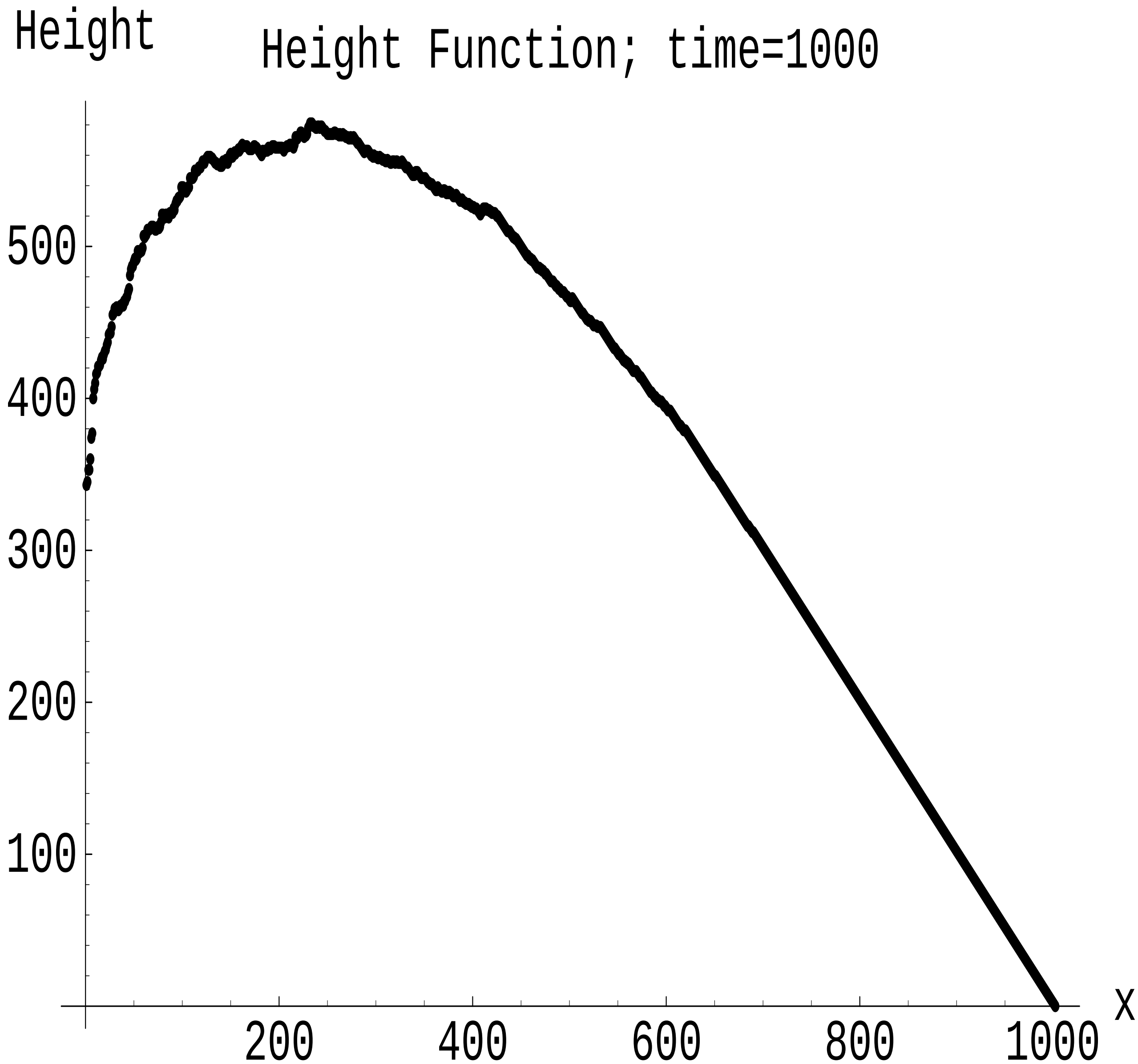


Space-Time and Path Description

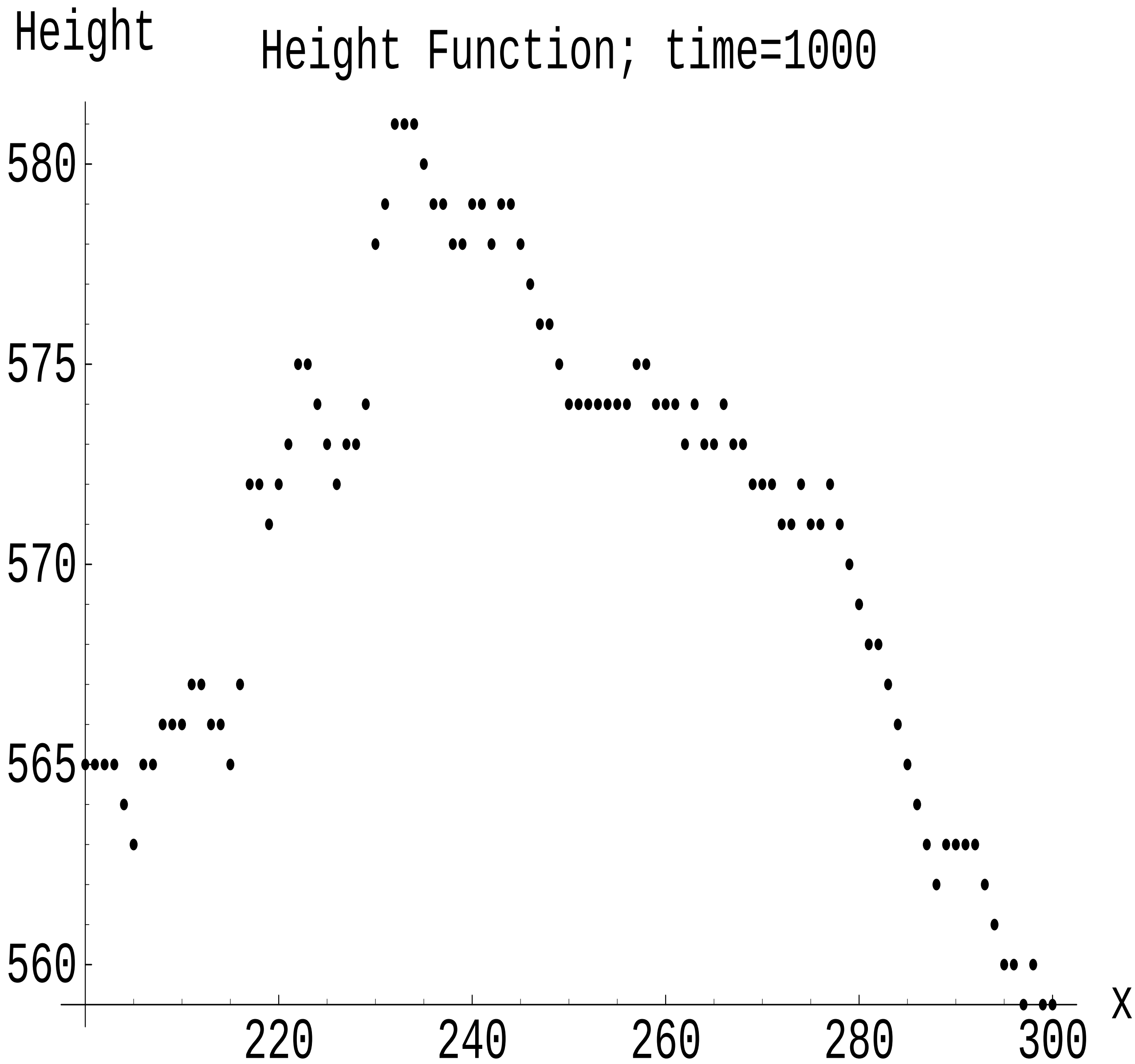


Yellow is the (discrete) backwards light cone of the point $(x, t) = (3, 7)$. The length of the longest increasing path (weakly increasing in x coordinate and strongly increasing in t coordinate) through the nucleation events starting at the initial box and ending at the box with coordinates (x, t) is the height of the surface at (x, t), i.e. $h_t(x)$. The Length of the path is defined as the number of nucleation events the path passes through counting the initial nucleation event at $t = 0$.

Transform backwards light cone into a random $(0,1)$-matrix of size $(t-x)\times(x+1)$. Relate via dual RSK to Young tableaux. (At this point equivalent to a first passage model of Seppäläinen as shown by Johansson.)



Height
Height Function; time=1000
500
400
300
200
100
200
400
600
800
1000
X


Height
Height Function; time=1000
580
575
570
565
560
220
240
260
280
300
X

Summary of Results: (Work in Progress)

(1) Fix x and let $t \to \infty$. Will get "random word" type behavior. Simplest case is $x = 0$ where it is gaussian—simple random walk.

(2) Let $x \to \infty$ and $t \to \infty$ such that

$$p_c := x/t < 1$$

is fixed. Then we have three possibilities:

(a) For $p < p_c$ the fluctuations are described by $F_2(s)$, i.e. for an explicit constants $b = b(p)$, $c = c(p)$

$$\text{Prob}\left(\frac{h_t(x) - b\,t}{c\,t^{1/3}} \leq s\right) \to F_2(s)$$

(b) For $p = p_c$ the limit

$\text{Prob}\,(h_t(x) - t \leq -n) \to p_c(n)$, $n = -1, 0, 1, 2, \ldots$

exists and is nonzero for all $n \geq -1$.

(c) For $p > p_c$ there are no fluctuations, i.e.

$$\text{Prob}\,(h_t(x) = t + 1) \to 1$$

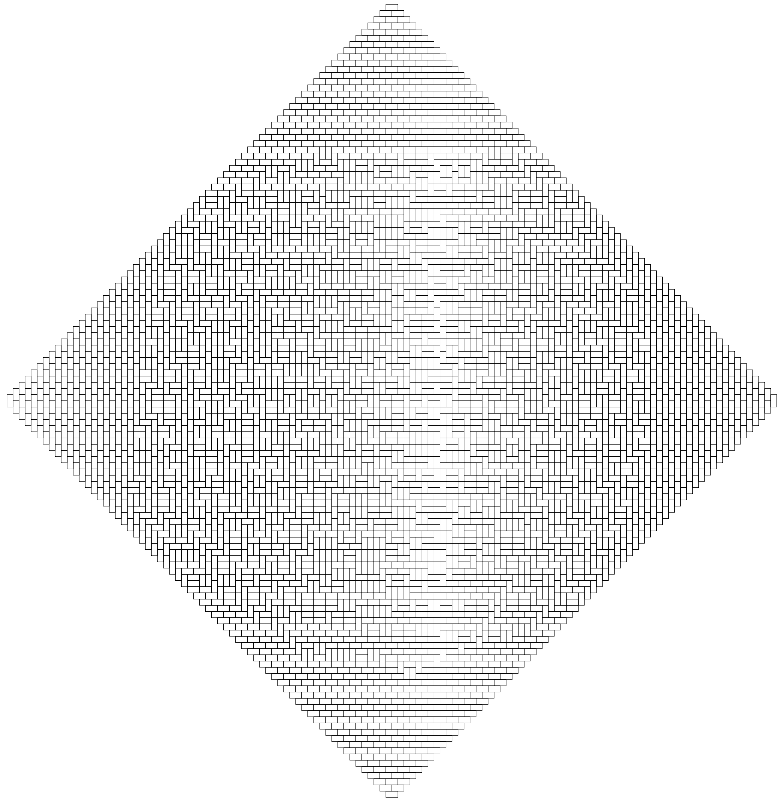


FIGURE 1. A random domino tiling of an Aztec diamond of order 64.

Arctic Circle Theorem - Cohn, Elkies, Propp

Johansson showed fluctuations around limiting circle are F_2.

Relationship to the KPZ Universality Class

KPZ=Kardar, Parisi, Zhang

Let $h_t(x)$ = the position of the growing interface at time t and position x

$$\frac{\partial h_t}{\partial t} = \nu \Delta h_t + \frac{\lambda}{2}(\nabla h_t)^2 + \eta(x,t)$$

Remarks:

(i) First term: relaxation of the interface by a surface tension ν.

(ii) Second term: nonlinear growth

(iii) Noise $\eta(x,t)$ has a gaussian distribution.

Conjecture: (Prähofer, Spohn, among others) For all $D = 1$ growth models in the KPZ Universality class we require self-similar macroscopic shape. If the limiting ray has nonzero curvature, then the conjectured fluctuations are F_2. If the limiting ray has zero curvature, then the conjectured fluctuations are F_1.