

Distribution Functions for Largest Eigenvalues and Their Applications

Craig A. Tracy* and Harold Widom†

Abstract

It is now believed that the limiting distribution function of the largest eigenvalue in the three classic random matrix models GOE, GUE and GSE describe new universal limit laws for a wide variety of processes arising in mathematical physics and interacting particle systems. These distribution functions, expressed in terms of a certain Painlevé II function, are described and their occurrences surveyed.

2000 Mathematics Subject Classification: 82D30, 60K37.

Keywords and Phrases: Random matrix models, limit laws, growth processes, Painlevé functions.

1. Random Matrix Models

A random matrix model is a probability space $(\Omega, \mathcal{P}, \mathcal{F})$ where the sample space Ω is a set of matrices. There are three classic finite N random matrix models (see, e.g. [31] and for early history [37]):

- Gaussian Orthogonal Ensemble ($\beta = 1$)
 - $\Omega = N \times N$ real symmetric matrices
 - $\mathcal{P} =$ “unique” measure that is invariant under orthogonal transformations and the matrix elements are i.i.d. random variables. Explicitly, the density is

$$c_N \exp(-\text{tr}(A^2)) dA, \quad (1.1)$$

where c_N is a normalization constant and $dA = \prod_i dA_{ii} \prod_{i < j} dA_{ij}$, the product Lebesgue measure on the independent matrix elements.

- Gaussian Unitary Ensemble ($\beta = 2$)

*Department of Mathematics, University of California, Davis, CA 95616, USA. E-mail: tracy@math.ucdavis.edu

†Department of Mathematics, University of California, Santa Cruz, CA 95064, USA. E-mail: widom@math.ucsc.edu

- $\Omega = N \times N$ hermitian matrices
- \mathcal{P} = “unique” measure that is invariant under unitary transformations and the (independent) real and imaginary matrix elements are i.i.d. random variables.
- Gaussian Symplectic Ensemble ($\beta = 4$) (see [31] for a definition)

Generally speaking, the interest lies in the $N \rightarrow \infty$ limit of these models. Here we concentrate on one aspect of this limit. In all three models the eigenvalues, which are random variables, are real and with probability one they are distinct. If $\lambda_{\max}(A)$ denotes the largest eigenvalue of the random matrix A , then for each of the three Gaussian ensembles we introduce the corresponding distribution function

$$F_{N,\beta}(t) := P_{\beta}(\lambda_{\max} < t), \beta = 1, 2, 4.$$

The basic limit laws [46, 47, 48] state that¹

$$F_{\beta}(s) := \lim_{N \rightarrow \infty} F_{N,\beta} \left(2\sigma\sqrt{N} + \frac{\sigma s}{N^{1/6}} \right), \beta = 1, 2, 4,$$

exist and are given explicitly by

$$\begin{aligned} F_2(s) &= \det \left(I - K_{\text{Airy}} \right) \\ &= \exp \left(- \int_s^{\infty} (x-s)q^2(x) dx \right) \end{aligned}$$

where

$$K_{\text{Airy}} \doteq \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y}$$

acting on $L^2(s, \infty)$ (Airy kernel)

and q is the unique solution to the Painlevé II equation

$$q'' = sq + 2q^3$$

satisfying the condition

$$q(s) \sim \text{Ai}(s) \text{ as } s \rightarrow \infty.$$

The orthogonal and symplectic distribution functions are

$$\begin{aligned} F_1(s) &= \exp \left(- \frac{1}{2} \int_s^{\infty} q(x) dx \right) (F_2(s))^{1/2}, \\ F_4(s/\sqrt{2}) &= \cosh \left(\frac{1}{2} \int_s^{\infty} q(x) dx \right) (F_2(s))^{1/2}. \end{aligned}$$

Graphs of the densities dF_{β}/ds are in the adjacent figure and some statistics of F_{β} can be found in the table.

¹Here σ is the standard deviation of the Gaussian distribution on the off-diagonal matrix elements. For the normalization we've chosen, $\sigma = 1/\sqrt{2}$; however, for subsequent comparisons, the normalization $\sigma = \sqrt{N}$ is perhaps more natural.

Table 1: The mean (μ_β), standard deviation (σ_β), skewness (S_β) and kurtosis (K_β) of F_β .

β	μ_β	σ_β	S_β	K_β
1	-1.20653	1.2680	0.293	0.165
2	-1.77109	0.9018	0.224	0.093
4	-2.30688	0.7195	0.166	0.050

The Airy kernel is an example of an *integrable integral operator* [19] and a general theory is developed in [49]. A vertex operator approach to these distributions (and many other closely related distribution functions in random matrix theory) was initiated by Adler, Shiota and van Moerbeke [1]. (See the review article [51] for further developments of this latter approach.)

Historically, the discovery of the connection between Painlevé functions (P_{III} in this case) and Toeplitz/Fredholm determinants appears in work of Wu et al. [53] on the spin-spin correlation functions of the two-dimensional Ising model. Painlevé functions first appear in random matrix theory in Jimbo et al. [20] where they prove the Fredholm determinant of the sine kernel is expressible in terms of P_V .

Gaudin [13] (using Mehta's [30] then newly invented method of orthogonal polynomials) was the first to discover the connection between random matrix theory and Fredholm determinants.

1.1. Universality Theorems

A natural question is to ask whether the above limit laws depend upon the underlying Gaussian assumption on the probability measure. To investigate this for unitarily invariant measures ($\beta = 2$) one replaces in (1.1)

$$\exp(-\text{tr}(A^2)) \rightarrow \exp(-\text{tr}(V(A))).$$

Bleher and Its [9] choose

$$V(A) = gA^4 - A^2, g > 0,$$

and subsequently a large class of potentials V was analyzed by Deift et al. [12]. These analyses require proving new Plancherel-Rotach type formulas for nonclassical orthogonal polynomials. The proofs use Riemann-Hilbert methods. It was shown that the generic behavior is GUE; and hence, the limit law for the largest eigenvalue is F_2 . However, by finely tuning the potential new universality classes will emerge at the edge of the spectrum. For $\beta = 1, 4$ a universality theorem was proved by Stojanovic [44] for the quartic potential.

In the case of noninvariant measures, Soshnikov [42] proved that for real symmetric Wigner matrices² (complex hermitian Wigner matrices) the limiting distribution of the largest eigenvalue is F_1 (respectively, F_2). The significance of this result is that nongaussian Wigner measures lie outside the "integrable class" (e.g. there are no Fredholm determinant representations for the distribution functions) yet the limit laws are the same as in the integrable cases.

2. Appearance of F_β in Limit Theorems

In this section we briefly survey the appearances of the limit laws F_β in widely differing areas.

2.1. Combinatorics

A major breakthrough occurred with the work of Baik, Deift and Johansson [3] when they proved that the limiting distribution of the length of the longest increasing subsequence in a random permutation is F_2 . Precisely, if $\ell_N(\sigma)$ is the length of the longest increasing subsequence in the permutation $\sigma \in S_N$, then

$$\mathbb{P}\left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} < s\right) \rightarrow F_2(s)$$

²A symmetric Wigner matrix is a random matrix whose entries on and above the main diagonal are independent and identically distributed random variables with distribution function F . Soshnikov assumes F is even and all moments are finite.

as $N \rightarrow \infty$. Here the probability measure on the permutation group S_N is the uniform measure. Further discussion of this result can be found in Johansson's contribution to these proceedings [26].

Baik and Rains [5, 6] showed by restricting the set of permutations (and these restrictions have natural symmetry interpretations) F_1 and F_4 also appear. Even the distributions F_1^2 and F_2^2 [50] arise. By the Robinson-Schensted-Knuth correspondence, the Baik-Deift-Johansson result is equivalent to the limiting distribution on the number of boxes in the first row of random standard Young tableaux. (The measure is the push-forward of the uniform measure on S_N .) These same authors conjectured that the limiting distributions of the number of boxes in the second, third, etc. rows were the same as the limiting distributions of the next-largest, next-next-largest, etc. eigenvalues in GUE. Since these eigenvalue distributions were also found in [47], they were able to compare the then unpublished numerical work of Odlyzko and Rains [34] with the predicted results of random matrix theory. Subsequently, Baik, Deift and Johansson [4] proved the conjecture for the second row. The full conjecture was proved by Okounkov [33] using topological methods and by Johansson [23] and by Borodin, Okounkov and Olshanski [10] using analytical methods. For an interpretation of the Baik-Deift-Johansson result in terms of the card game *patience sorting*, see the very readable review paper by Aldous and Diaconis [2].

2.2. Growth Processes

Growth processes have an extensive history both in the probability literature and the physics literature (see, e.g. [15, 29, 41] and references therein), but it was only recently that Johansson [22, 26] proved that the *fluctuations* about the limiting shape in a certain growth model (Corner Growth Model) are F_2 . Johansson further pointed out that certain symmetry constraints (inspired from the Baik-Rains work [5, 6]), lead to F_1 fluctuations. This growth model is in Johansson's contribution to these proceedings [26] where the close analogy to largest eigenvalue distributions is explained.

Subsequently, Baik and Rains [7] and Gravner, Tracy and Widom [16] have shown the same distribution functions appearing in closely related lattice growth models. Prähofer and Spohn [38, 39] reinterpreted the work of [3] in terms of the physicists' polynuclear growth model (PNG) thereby clarifying the role of the symmetry parameter β . For example, $\beta = 2$ describes growth from a single droplet where as $\beta = 1$ describes growth from a flat substrate. They also related the distributions functions F_β to fluctuations of the height function in the KPZ equation [28, 29]. (The connection with the KPZ equation is heuristic.) Thus one expects on physical grounds that the fluctuations of any growth process falling into the $1 + 1$ KPZ universality class will be described by the distribution functions F_β or one of the generalizations by Baik and Rains [7]. Such a physical conjecture can be tested experimentally; and indeed, Timonen and his colleagues [45] have taken up this challenge. Earlier Timonen et al. [32] established experimentally that a slow, flameless burning process in a random medium (paper!) is in the $1 + 1$ KPZ universality class. This sequence of events is a rare instance in which new results in mathematics inspires

new experiments in physics.

In the context of the PNG model, Prähofer and Spohn have given a process interpretation, the *Airy process*, of F_2 . Further work in this direction can be found in Johansson [25].

There is an extension of the growth model in [16] to growth in a *random environment*. In [17] the following model of interface growth in two dimensions is considered by introducing a height function on the sites of a one-dimensional integer lattice with the following update rule: the height above the site x increases to the height above $x - 1$, if the latter height is larger; otherwise the height above x increases by one with probability p_x . It is assumed that the p_x are chosen independently at random with a common distribution function F , and that the initial state is such that the origin is far above the other sites. In the *pure regime* Gravner-Tracy-Widom identify an asymptotic shape and prove that the fluctuations about that shape, normalized by the square root of the time, are asymptotically normal. This contrasts with the quenched version: conditioned on the environment and normalized by the cube root of time, the fluctuations almost surely approach the distribution function F_2 . We mention that these same authors in [18] find, under some conditions on F at the right edge, a *composite regime* where now the interface fluctuations are governed by the extremal statistics of p_x in the annealed case while the fluctuations are asymptotically normal in the quenched case.

2.3. Random Tilings

The Aztec diamond of order n is a tiling by dominoes of the lattice squares $[m, m + 1] \times [\ell, \ell + 1]$, $m, n \in \mathbb{Z}$, that lie inside the region $\{(x, y) : |x| + |y| \leq n + 1\}$. A *domino* is a closed 1×2 or 2×1 rectangle in \mathbb{R}^2 with corners in \mathbb{Z}^2 . A typical tiling is shown in the accompanying figure. One observes that near the center the tiling appears random, called the *temperate zone*, whereas near the edges the tiling is frozen, called the *polar zones*. It is a result of Jockush, Propp and Shor [21] (see also [11]) that as $n \rightarrow \infty$ the boundary between the temperate zone and the polar zones (appropriately scaled) converges to a circle (Arctic Circle Theorem). Johansson [24] proved that the fluctuations about this limiting circle are F_2 .

2.4. Statistics

Johnstone [27] considers the largest principal component of the covariance matrix $X^t X$ where X is an $n \times p$ data matrix all of whose entries are independent standard Gaussian variables and proves that for appropriate centering and scaling, the limiting distribution equals F_1 in the limit $n, p \rightarrow \infty$ with $n/p \rightarrow \gamma \in \mathbb{R}^+$. Soshnikov [43] has removed the Gaussian assumption but requires that $n - p = O(p^{1/3})$. Thus we can anticipate applications of the distributions F_β (and particularly F_1) to the statistical analysis of large data sets.

2.5. Queuing Theory

Glynn and Whitt [14] consider a series of n single-server queues each with unlimited waiting space with a first-in and first-out service. Service times are i.i.d. with mean one and variance σ^2 with distribution V . The quantity of interest is $D(k, n)$, the departure time of customer k (the last customer to be served) from the last queue n . For a fixed number of customers, k , they prove that

$$\frac{D(k, n) - n}{\sigma\sqrt{n}}$$

converges in distribution to a certain functional \hat{D}_k of k -dimensional Brownian motion. They show that \hat{D}_k is independent of the service time distribution V . It was shown in [8, 16] that \hat{D}_k is equal in distribution to the largest eigenvalue of a $k \times k$ GUE random matrix. This fascinating connection has been greatly clarified in recent work of O'Connell and Yor [35] (see also [36]).

From Johansson [22] it follows for V Poisson that

$$\mathrm{P}\left(\frac{D(\lfloor xn \rfloor, n) - c_1 n}{c_2 n^{1/3}} < s\right) \rightarrow F_2(s)$$

as $n \rightarrow \infty$ for some explicitly known constants c_1 and c_2 (depending upon x).

2.6. Superconductors

Vavilov et al. [52] have conjectured (based upon certain physical assumptions supported by numerical work) that the fluctuation of the excitation gap in a metal grain or quantum dot induced by the proximity to a superconductor is described by F_1 for zero magnetic field and by F_2 for nonzero magnetic field. They conclude their paper with the remark:

The universality of our prediction should offer ample opportunities for experimental observation.

References

- [1] M. Adler, T. Shiota and P. van Moerbeke, Random matrices, vertex operators and the Virasoro algebra, *Phys. Letts. A* **208** (1995), 67–78.
- [2] D. Aldous and P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theory, *Bull. Amer. Math. Soc.* **36** (1999), 413–432.
- [3] J. Baik, P. Deift and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, *J. Amer. Math. Soc.* **12** (1999), 1119–1178.
- [4] J. Baik, P. Deift and K. Johansson, On the distribution of the length of the second row of a Young diagram under Plancherel measure, *Geom. Funct. Anal.* **10** (2000), 702–731.
- [5] J. Baik and E. M. Rains, The asymptotics of monotone subsequences of involutions, *Duke Math. J.* **109** (2001), 205–281.
- [6] J. Baik and E. M. Rains, Symmetrized random permutations, in *Random Matrix Models and their Applications*, eds. P. Bleher and A. Its, Math. Sci. Res. Inst. Publications **40**, Cambridge Univ. Press, 2001, pgs. 1–19.
- [7] J. Baik and E. M. Rains, Limiting distributions for a polynuclear growth model, *J. Stat. Phys.* **100** (2000), 523–541.
- [8] Yu. Baryshnikov, GUEs and queues, *Probab. Th. Rel. Fields* **119** (2001), 256–274.
- [9] P. Bleher and A. Its, Semiclassical asymptotics of orthogonormal polynomials, Riemann-Hilbert problem, and universality in the matrix model, *Ann. Math.* **150** (1999), 185–266.
- [10] A. Borodin, A. Okounkov and G. Olshanski, Asymptotics of Plancherel measures for symmetric groups, *J. Amer. Math. Soc.* **13** (2000), 481–515.
- [11] H. Cohn, N. Elkies and J. Propp, Local statistics for random domino tilings of the Aztec diamond, *Duke Math. J.* **85** (1996), 117–166.
- [12] P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weight and applications to universality questions in random matrix theory, *Commun. Pure Appl. Math.* **52** (1999), 1335–1425.

- [13] M. Gaudin, Sur la dlo limite de l'espacement des valeurs propres d'une matrice aléatoire, *Nucl. Phys.* **25** (1961), 447–458; reprinted in [37].
- [14] P. W. Glynn and W. Whitt, Departure times from many queues, *Ann. Appl. Probab.* **1** (1991), 546–572.
- [15] J. Gravner and D. Griffeath, Cellular automaton growth on \mathbb{Z}^2 : Theorems, examples and problems, *Adv. Appl. Math.* **21** (1998), 241–304.
- [16] J. Gravner, C. A. Tracy and H. Widom, Limit theorems for height fluctuations in a class of discrete space and time growth models, *J. Stat. Phys.* **102** (2001), 1085–1132.
- [17] J. Gravner, C. A. Tracy and H. Widom, A growth model in a random environment, *Ann. Probab.* **30** (2002), 1340–1368.
- [18] J. Gravner, C. A. Tracy and H. Widom, Fluctuations in the composite regime of a disordered growth model, *Commun. Math. Phys.* **229** (2002), 433–458.
- [19] A. R. Its, A. G. Izergin, V. E. Korepin and N. A. Slavnov, Differential equations for quantum correlations, *J. Mod. Phys. B* **4** (1990), 1003–1037.
- [20] M. Jimbo, T. Miwa, Y. Môri and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, *Physica D* **1** (1980), 80–158.
- [21] W. Jockush, J. Propp and P. Shor, Random domino tilings and the arctic circle theorem, preprint.
- [22] K. Johansson, Shape fluctuations and random matrices, *Commun. Math. Phys.* **209** (2000), 437–476.
- [23] K. Johansson, Discrete orthogonal polynomial ensembles and the Plancherel measure, *Ann. Math.* **153** (2001), 259–296.
- [24] K. Johansson, Non-intersecting paths, random tilings and random matrices, *Probab. Th. Rel. Fields 123* (2002), 225–280.
- [25] K. Johansson, Discrete polynuclear growth and determinantal processes, preprint (arXiv: math.PR/0206208).
- [26] K. Johansson, Toeplitz determinants, random growth and determinantal processes, ICM 2002, Vol. III, 53–62.
- [27] I. Johnstone, On the distribution of the largest principal component, *Ann. Statistics* **29** (2001), 295–327.
- [28] M. Kardar, G. Parisi and Y-C Zhang, Dynamical scaling of growing interfaces, *Phys. Rev. Letts.* **56** (1986), 889–892.
- [29] P. Meakin, *Fractals, Scaling and Growth Far from Equilibrium*, Cambridge Univ. Press, 1998.
- [30] M. L. Mehta, On the statistical properties of the level-spacings in nuclear spectra, *J. Math. Phys.* **18** (1960), 395–419; reprinted in [37].
- [31] M. L. Mehta, *Random Matrices*, second ed., Academic Press, 1991.
- [32] M. Myllys, J. Maunuksela, M. Alava, T. Ala-Nissila, J. Merikoski and J. Timonen, Kinetic roughening in slow combustion of paper, *Phys. Rev. E* **64**, 036101-1–036101-12.
- [33] A. Okounkov, Random matrices and random permutations, *Internat. Math. Res. Notices* no. 20 (2000), 1043–1095.
- [34] A. M. Odlyzko and E. M. Rains, On the longest increasing subsequences in random permutations, in *Analysis, Geometry, Number Theory: The Mathematics*

- of *Leon Ehrenpreis*, eds. E. L. Grinberg, S. Berhanu, M. Knopp, G. Mendoza and E. T. Quinto, Amer. Math. Soc. 2000, pgs. 439–451.
- [35] N. O’Connell and M. Yor, A representation for non-colliding random walks, *Elect. Commun. in Probab.* **7** (2002), 1–12.
- [36] N. O’Connell, Random matrices, non-colliding processes and queues, preprint (arXiv: math/PR/0203176).
- [37] C. E. Porter, *Statistical Theories of Spectra: Fluctuations*, Academic Press, 1965.
- [38] M. Prähofer and H. Spohn, Statistical self-similarity of one-dimensional growth processes, *Physica A* **279** (2000), 342–352.
- [39] M. Prähofer and H. Spohn, Universal distributions for growth processes in $1+1$ dimensions and random matrices, *Phys. Rev. Letts.* **84** (2000), 4882–4885.
- [40] M. Prähofer and H. Spohn, Scale invariance of the PNG droplet and the Airy process, preprint (arXiv: math.PR/0105240).
- [41] T. Seppäläinen, Exact limiting shape for a simplified model of first-passage percolation in the plane, *Ann. Prob.* **26** (1998), 1232–1250.
- [42] A. Soshnikov, Universality at the edge of the spectrum in Wigner random matrices, *Commun. Math. Phys.* **207** (1999), 697–733.
- [43] A. Soshnikov, A note on universality of the distribution of the largest eigenvalue in certain classes of sample covariance matrices, preprint (arXiv: math.PR/0104113).
- [44] A. Stojanovic, Universality in orthogonal and symplectic invariant matrix models with quartic potential, *Math. Phys., Anal. and Geom.* **3** (2000), 339–373.
- [45] J. Timonen, private communication.
- [46] C. A. Tracy and H. Widom, Level-spacing distribution and the Airy kernel, *Phys. Letts. B* **305** (1993), 115–118.
- [47] C. A. Tracy and H. Widom, Level-spacing distribution and the Airy kernel, *Commun. Math. Phys.* **159** (1994), 151–174.
- [48] C. A. Tracy and H. Widom, On orthogonal and symplectic matrix ensembles, *Commun. Math. Phys.* **177** (1996), 727–754.
- [49] C. A. Tracy and H. Widom, Fredholm determinants, differential equations and matrix models, *Commun. Math. Phys.* **163** (1994), 151–174.
- [50] C. A. Tracy and H. Widom, Random unitary matrices, permutations and Painlevé, *Commun. Math. Phys.* **207** (1999), 665–685.
- [51] P. van Moerbeke, Integrable lattices: Random matrices and random permutations, in *Random Matrix Models and their Applications*, eds. P. Bleher and A. Its, Math. Sci. Res. Inst. Publications **40**, Cambridge Univ. Press, 2001, pgs. 321–406.
- [52] M. G. Vavilov, P. W. Brouwer, V. Ambegaokar and C. W. J. Beenakker, Universal gap fluctuations in the superconductor proximity effect, *Phys. Rev. Letts.* **86** (2001), 874–877.
- [53] T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch, Spin-spin correlation functions of the two-dimensional Ising model: Exact theory in the scaling region, *Phys. Rev. B* **13** (1976), 316–374.