Integrable Differential Equations in Random Matrix Theory

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Integrable DEs and Determinants

1. Some historical remarks
2. Random Matrix Models (RMM) with unitary symmetry
3. RMM with orthogonal symmetry
4. Wishart distributions
5. RMM and Extended Kernels

Figure 1: E. Ivar Fredholm, 1866–1927.

Figure 2: Paul Painlevé, 1863–1933.

Figure 3: Tai Tsun Wu, 1933–.

Figure 4: Harold Widom, 1932–, with brother Ben, 1927–

§1. Some Historical Remarks

2D Ising Model: First connection between

Toeplitz and Fredholm Dets $\rightsquigarrow$ Painlevé

Wu, McCoy, Tracy, & Barouch (1973–77) [61]:

$$\lim_{\substack{T \to T_c^{\pm}, R^2 = M^2 + N^2 \to \infty \\ r = R/\xi(T) \text{fixed}}} \mathbb{E}\left(\sigma_{00}\sigma_{MN}\right) = \left\{ \begin{array}{c} \sinh \frac{1}{2}\psi(r) \\ \cosh \frac{1}{2}\psi(r) \end{array} \right\} \times$$

$$\exp\left(-\frac{1}{4}\int_r^\infty (\frac{d\psi}{dy})^2 - \sinh^2\psi(y)\, dy\right)$$

where

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = \frac{1}{2}\sinh(2\psi), \;\; \psi(r) \sim \frac{2}{\pi}K_0(r), \;\; x \to \infty.$$

Note: $y(x) = \mathrm{e}^{-\psi(x)}$ is a particular **Painlevé III** transcendent.
(See also Widom [60].)

Next, Sato, Miwa & Jimbo, 1977–1980, (see [32] and references therein) introduced the notion of

τ-functions and holonomic quantum fields,

a class of field theories that include the scaling limit of the Ising model and for which the expression of correlation functions in terms of solutions to holonomic differential equations is a general feature. A **book length** account of these developments can be found in Palmer [45].

These developments led Jimbo-Miwa-Môri-Sato [35] to consider, in 1980, the *Fredholm determinant* and *Fredholm minors* of the operator whose kernel is the familiar **sine kernel**

$$\frac{1}{\pi}\,\frac{\sin \pi(x-y)}{x-y}$$

on the domain

$$\mathbb{J} = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n)$$

Their main interest was the *density matrix of the impenetrable Bose gas*, and only incidentally, random matrices.

For $\mathbb{J} = (0, s)$, the **JMMS result** is

$$\det\left(I - \lambda K_{\text{sine}}\right) = \exp\left(-\int_0^{\pi s} \frac{\sigma(x;\lambda)}{x}\, dx\right)$$

where

$$(x\sigma'')^2 + 4(x\sigma' - \sigma)\left(x\sigma' - \sigma + (\sigma')^2\right) = 0$$

with boundary condition

$$\sigma(x, \lambda) = -\frac{\lambda}{\pi}\, x + \mathrm{O}(x^2), \;\; x \to 0.$$

Remarks:

- σ is expressible in terms of Painlevé V. An example of the *σ-form for Painlevé equations* [34].
- For general $\mathbb{J}$, JMMS [35] obtains a compatible system of nonautonomous Hamiltonian equations generated by Poisson commuting Hamiltonians where the independent variables are the a_j, b_j—i.e. the endpoints of the intervals. (See also Harnad [27].)
- A simplified derivation of the JMMS equations can be found in TW [50]. See Gangardt [26] for recent developments on the impenetrable Bose gas.
- Connections with quantum inverse scattering were developed by Its, Izergin, Korepin, Slavnov and others. (See, e.g., [30, 40].)

§2. RMM with Unitary Symmetry

Many RMM with unitary symmetry come down to the evaluation of Fredholm determinants $\det(I - \lambda K)$ where K has kernel of the form

$$\frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}\, \chi_{\mathbb{J}}(y)$$

where

$$\mathbb{J} = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n).$$

Examples:

- Sine kernel: $\varphi(x) = \sin \pi x$, $\psi(x) = \cos \pi x$.
- Airy kernel: $\varphi(x) = \mathrm{Ai}(x)$, $\psi(x) = \mathrm{Ai}'(x)$.
- Bessel kernel: $\varphi(x) = J_\alpha(\sqrt{x})$, $\psi(x) = x\varphi'(x)$.
- Hermite kernel: $\varphi(x) = (\frac{N}{2})^{1/4}\varphi_N(x), \psi(x) = (\frac{N}{2})^{1/4}\varphi_{N-1}(x)$ where $\varphi_k(x) = \frac{1}{\sqrt{2^k\, k!\pi^{1/2}}}\, \mathrm{e}^{-x^2/2}\, H_k(x)$.

A **general theory** of such Fredholm determinants was developed in TW [52] under the additional hypothesis that

$$m(x)\frac{d}{dx}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\begin{pmatrix}A(x) & B(x)\\ -C(x) & -A(x)\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

where m, A, B and C are polynomials. For example, for the Airy kernel

$$m(x)=1,\ A(x)=0,\ B(x)=1,\ C(x)=-x.$$

The **basic objects** of the theory are

$$Q_j(x;\mathbb{J})=(I-K)^{-1}\,x^j\varphi(x),\ \ P_j(x;\mathbb{J})=(I-K)^{-1}\,x^j\psi(x),$$

and

$$u_j=(Q_j,\varphi)\,,v_j=(P_j,\varphi)\,,\tilde{v}_j=(Q_j,\psi)\,,w_j=(P_j,\psi)$$

where $(\cdot,\cdot)$ denotes the inner product. The **independent variables** are the endpoints a_j and b_j making up $\mathbb{J}$.

There are **two types of differential equations**:

- Universal equations, i.e. equations that hold *independently* of the differential equations for φ and ψ.
- Equations that depend upon m, A, B and C.

For $K = K_{\mathrm{Airy}}$ with $\mathbb{J} = (s, \infty)$, $q(s) := Q_0(s, \mathbb{J})$, $p(s) := P_0(s, \mathbb{J})$, $u = u_0$, $v = v_0$, the general theory reduces to the differential equations

$$\begin{aligned} \frac{dq}{ds} &= p - qu, \quad \frac{dp}{ds} = sq - 2qv + pu, \\ \frac{du}{ds} &= -q^2, \qquad \frac{dv}{ds} = -pq, \end{aligned}$$

together with

$$\frac{d}{ds}R(s,s) = -q^2, \quad \frac{d}{ds}\log\det(I - K) = -R(s,s),$$

where $R(x, y)$ is the resolvent kernel of K.

Using the first integral $u^2 - 2v = q^2$, one easily derives that q satisfies the **Painlevé II equation**

$$\frac{d^2q}{ds^2} = sq + 2q^3.$$

- **Key features** of the proof are simple expressions for $(m(x) \equiv 1)$

$$[D, (I-K)^{-1}] \ \text{ and } \ [M^k, (I-K)^{-1}]$$

  where D is differentiation with respect to the independent variable and M is multiplication by the independent variable. For example, when $K = K_{\text{Airy}}$ with $\mathbb{J} = (s, \infty)$ we have

$$[D, (I-K)^{-1}] \doteq -Q(x)Q(y) + R(x,s)\rho(s,y)$$

- Palmer [44] and Harnad & Its [28] have given an **isomondromic deformation** approach to these type of kernels.

- ADLER, SHIOTA, & VAN MOERBEKE'S [1] Virasoro algebra approach gives directly equations for the resolvent kernel $R(s, s)$. The connection between these two approaches has been clarified by HARNAD [27].

- After one has the differential equations, e.g., Painlevé II, one is faced with the **asymptotic analysis of the solutions**. This generally involves finding **connection formulae**.

Integral Equation Approach: Using, e.g., **inverse scattering methods**, one associates to an integrable DE, e.g. Painlevé II, a **linear integral equation**. An asymptotics analysis of these linear equations then leads to **nonlinear connection formulae**. Some of the early work here is [10, 29, 41]. These methods are now generally recognized as less powerful than Riemann-Hilbert methods. However, they still, on occasion, produce some delicate new results [54].

Riemann-Hilbert Approach: The modern Riemann-Hilbert approach, which has its origins in the isomondromy deformation method of Flaschka & Newell [23] and Jimbo, Miwa & Ueno [33] in 1980s, began with the work of Deift & Zhou [15] when they proposed a nonlinear version of the classical steepest descent method for oscillatory Riemann-Hilbert problems. The **Deift-Zhou approach** has the advantage of not using any prior information about the solutions of the Painlevé equations. For Painlevé II consult [15] and [31].

- A recent achievement [14] of the RH approach is a proof that as $s \to -\infty$,

$$\log\det(I - K_{\mathrm{Airy}}) = -\frac{s^3}{12} - \frac{1}{8}\log s + \kappa + \mathrm{O}(s^{-3/2})$$

where

$$\kappa = \frac{1}{24}\log 2 + \zeta'(-1)$$

and $\zeta(s)$ is the Riemann zeta function.

Remark: The first two terms follows from the HASTINGS-MCLEOD [29] solution of Painlevé II as was shown in TW [51]. The constant κ was conjectured [51] to be as above, but a proof only came recently in the work of DEIFT, ITS & KRASOVSKY [14] using RH methods.

- CHOUP [9] has given explicit Painlevé representations for **corrections to edge scaling** for both finite n GUE and LUE.

$$\begin{aligned}\mathbb{P}^{\mathrm{GUE,LUE}}(\lambda_{\max} < t) &= F_2(s)\Big\{1 + c_1^{G,L} E_1(s) n^{-1/3} + \\ &\qquad c_2^{G,L} E_2(s) n^{-2/3} + \mathrm{O}(n^{-1})\Big\}\end{aligned}$$

where t and s are related by explicit norming and centering constants.

§3. RMM with Orthogonal Symmetry

The added difficulty with RMM with orthogonal symmetry is that the kernels are **matrix kernels** [20, 42, 53, 55, 59]. For example, for finite N GOE the operator is

$$K_1 = \chi \begin{pmatrix} K_2 + \psi \otimes \varepsilon\varphi & K_2 D - \psi \otimes \varphi \\ \varepsilon K_2 - \varepsilon + \varepsilon\psi \otimes \varepsilon\varphi & K_2 + \varepsilon\varphi \otimes \psi \end{pmatrix} \chi$$

where

$$K_2 \doteq \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y),$$

ε is the operator with kernel $\frac{1}{2}\mathrm{sgn}(x-y)$, D is the differentiation operator, and χ is the indicator function for the domain $\mathbb{J}$. Notation: $A \otimes B \doteq A(x)B(y)$.

The idea of the proof in TW [53] is to factor out the GUE part

$$(I - K_2\chi)$$

and through various determinant manipulations show that the **remaining part is a finite rank perturbation**. Thus one ends up with formulas like

$$\det(I - K_1) = \det(I - K_2\chi)\ \det\left(I - \sum_{j=1}^{k} \alpha_j \otimes \beta_j\right)$$

For the case $\mathbb{J} = (s, \infty)$, an asymptotic analysis shows that as $N \to \infty$ the distribution of the scaled largest eigenvalue in GOE is expressible in terms of the **same Painlevé II function** appearing in GUE. (Similar remarks for the symplectic ensemble.)

- The **edge scaling limit** is more **subtle for GOE** than for GUE or GSE. For GUE and GSE we have **convergence in trace norm** to limiting operators $K_{2,\text{Airy}}$ and $K_{4,\text{Airy}}$, but for GOE the **convergence is to a regularized determinant**, i.e. $\det_2$. This extra subtleness is due to the presence of the ε. This lack of trace norm convergence is basically explains why the limit $N \to \infty$ was taken at the end in [53]. The pointwise limit of finite N K_1 was worked out by Ferrari [21] and by Forrester, Nagao & Honner [24]. The convergence at the operator level is in TW [57].

- Recently Ferrari & Spohn [22] gave a different determinantal expression for edge scaling in GOE. It would be interesting to explore further their approach.

Universality Theorems

Though not a part of this survey proper, it's important to mention that these *same* distribution functions (and hence integrable DEs) arise for a much wider class of models than the Gaussian cases discussed here.

Invariant Measures: $\mathrm{e}^{-\mathrm{Tr}(A^2)} \longrightarrow \mathrm{e}^{-N\mathrm{Tr}(V(A))}$

Unitary:

- Bleher & Its [8], $V(x) = \frac{1}{2}tx^2 + \frac{1}{4}gx^4$, $g > 0, t < 0$.
- Deift, Kriecherbauer, McLaughlin, Venakides, Zhou [12, 13], V real analytic and $V/\log|x| \to +\infty$ as $|x| \to \infty$.

Orthogonal & Symplectic: Deift & Gioev [11], poly. V

Noninvariant Measures:

- Soshnikov [48], Real symmetric and complex Hermitian Wigner matrices.

Next Largest, Next-Next Largest, ... Eigenvalue Distributions

$$\mathbf{D}_\beta(\mathbf{s},\lambda) := \mathbf{det}\,(\mathbf{I} - \lambda \mathbf{K}_{\beta,\mathrm{Airy}}),\ \ \beta = 1, 2, 4,\ 0 \le \lambda \le 1.$$

($\det_2$ for $\beta = 1$.) One needs

$$\left.\frac{\partial^j D_\beta(s,\lambda)}{\partial \lambda^j}\right|_{\lambda=1}$$

for next largest, next-next largest eigenvalue, etc. distributions.

For $\beta = 2, 4$ there is a **simple answer**: Let

$$q(x) \longrightarrow q(x,\lambda)$$

in $\lambda = 1$ distributions where now q satisifies **same Painlevé II** equation but with boundary condition

$$q(x,\lambda) \sim \sqrt{\lambda}\,\mathrm{Ai}(x), \quad x \to \infty.$$

Not So for Orthogonal Symmetry!

Dieng [18] proved (see also [19])

$$D_1(s,\lambda) = D_2(s,\tilde{\lambda})\,\frac{\lambda - 1 - \cosh\mu(s,\tilde{\lambda}) + \sqrt{\tilde{\lambda}}\sinh\mu(s,\lambda)}{\lambda - 2}$$

with

$$\mu(s,\lambda) = \int_s^\infty q(x,\lambda)\,dx \quad \text{and} \quad \tilde{\lambda} := 2\lambda - \lambda^2.$$

Note evaluation at $\tilde{\lambda}$ in above. For $\lambda = 1$ this reduces to TW [53].

From this follows distribution functions for next-largest, next-next largest, etc. for GOE universality class in terms of Painlevé II function q and derivatives

$$\left.\frac{\partial^k q(x,\lambda)}{\partial\lambda^k}\right|_{\lambda=1}$$

Simulations

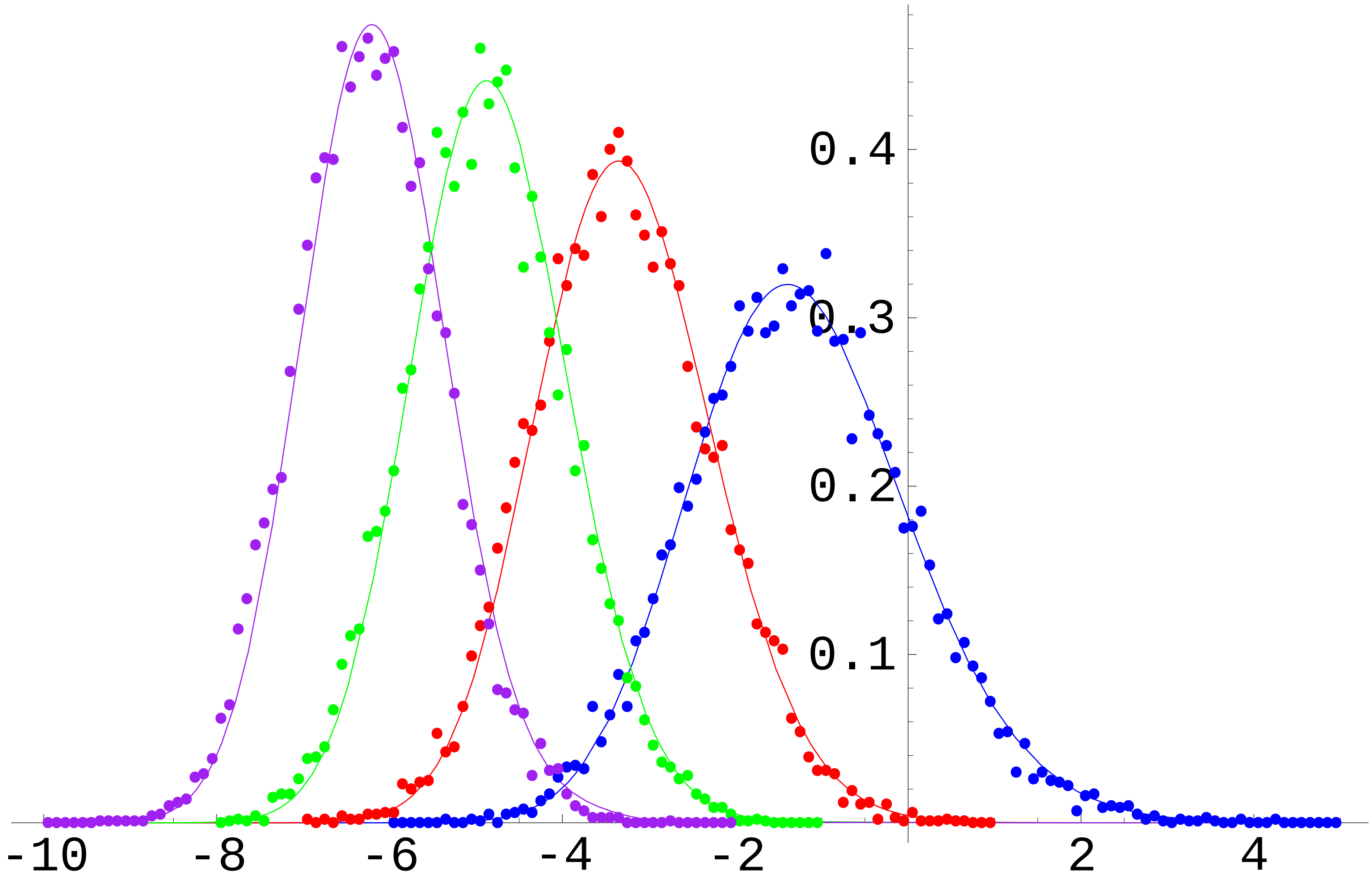

Figure 5: 10^4 realizations of $10^3 \times 10^3$ GOE matrices

§4. Wishart Distributions

If

$$A = X^T X$$

where the $n \times p$ matrix X is $N_p(0, I_n \otimes \Sigma)$, $\Sigma > 0$, then A is said to have **Wishart distribution** with n degrees of freedom and covariance Σ. The Wishart distribution is the multivariate generalization of the χ^2-distribution. We will say A is $W_p(n, \Sigma)$.

The quantity $\frac{1}{n}A$ is also called the **sample covariance matrix**.

The classic references are Anderson [4] and Muirhead [43] (see also [19]).

Eigenvalues of a Wishart Matrix

Theorem: If A is $W_p(n, \Sigma)$, $n \geq p$, the **joint density function** for the eigenvalues $\ell_1, \ldots, \ell_p$ of A is

$$c_{p,n,\Sigma} \prod_{j=1}^{p} \ell_j^{(n-p-1)/2} \prod_{j<k} |\ell_j - \ell_k| \times$$

$$\int_{\mathcal{O}(p)} \mathrm{e}^{-\frac{1}{2}\,\mathrm{tr}(\Sigma^{-1} Q L Q^T)}\, dQ,$$

where $L = \mathrm{diag}(\ell_1, \ldots, \ell_p)$ and dQ is normalized Haar measure.

Corollary: If A is $W_p(n, I_p)$, then the integral over the orthogonal group in the previous theorem is

$$\mathrm{e}^{-\frac{1}{2}\sum_j \ell_j}\,.$$

- One is interested in **limit laws** as $n, p \to \infty$. For $\Sigma = I_p$, Johnstone [37] proved, using RMT methods, for centering and scaling constants

$$
\begin{aligned}
\mu_{np} &= \left(\sqrt{n-1} + \sqrt{p}\right)^2, \\
\sigma_{np} &= \left(\sqrt{n-1} + \sqrt{p}\right)\left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{p}}\right)^{1/3}
\end{aligned}
$$

that

$$
\frac{\ell_1 - \mu_{np}}{\sigma_{np}}
$$

converges in distribution as $n, p \to \infty$, $n/p \to \gamma < \infty$, to the **GOE largest eigenvalue distribution** TW [53].

- El Karoui [38] has extended the result to $\gamma \leq \infty$. The case $p \gg n$ appears, for example, in microarray data.

- Soshnikov [49] has **removed the Gaussian assumption** under the additional restriction $n - p = \mathrm{O}(p^{1/3})$.

- For $\Sigma \neq I_p$, the difficulty in establishing limit theorms comes from the integral

$$\int_{\mathcal{O}(p)} \mathrm{e}^{-\frac{1}{2}\operatorname{tr}(\Sigma^{-1} Q \Lambda Q^T)}\,(dQ).$$

  Using **zonal polynomials**, infinite series expansions have been derived for this integral, but these expansions are difficult to analyze and converge slowly. See MUIRHEAD [43].

- For **complex Gaussian data matrices** X similar density formulas are known for the eigenvalues of X^*X. Limit theorems for $\Sigma \neq I_p$ are known since the analogous group integral, now over the unitary group, is known explicitly—the HARISH CHANDRA–ITZYKSON–ZUBER integral. See the work of BAIK, BEN AROUS & PÉCHÉ [5, 6] and EL KAROUI [39].

The BBP phase transition.

- These RMT developments have had recent application to the analysis of **genetic data**; in particular, determining if the samples are from a homogeneous population [46]. The use of **integrable DEs** together with **good software** allows Patterson, Price and Reich [46] to write

> The complexity of the TW definition is irrelevant to its application to real data. One computes a statistic, and then looks up a *p*-value in tables or through a computational interface. This is little different from how one uses (say) a conventional chi-squared test.

§5. RMM & Extended Kernels

Airy Process: The Airy process $\mathcal{A}(\tau)$, introduced by Prähoffer & Spohn [47] and Johansson [36], is a continuous stochastic process whose distribution functions are given by

$$\mathbb{P}\left(\mathcal{A}(\tau_1) < a_1, \ldots, \mathcal{A}(\tau_m) < a_m\right) = \det\left(I - K\right)$$

for $\tau_1 < \cdots < \tau_m$. Here K is the operator with $m \times m$ matrix kernel (K_{ij}) where

$$K_{ij}(x,y) = L_{ij}(x,y)\chi_{(a_j,\infty)}(y)$$

$$L_{ij}(x,y) = \begin{cases} \int_0^\infty \mathrm{e}^{-z(\tau_i-\tau_j)}\mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\,dz & i \geq j, \\ -\int_{-\infty}^0 \mathrm{e}^{-z(\tau_i-\tau_j)}\mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\,dx & i < j \end{cases}$$

For $m = 1$ this reduces to the Airy kernel (independent of τ).

Extended kernels are more difficult than “integrable kernels” in unitary ensembles. Nevertheless, it is possible to find (complicated!) systems of integrable differential equations: See ADLER & VAN MOERBEKE [2, 3] and TW [56, 58].

Much analysis remains to be done on these equations

THANK YOU FOR YOUR ATTENTION!

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