# Universality of the Distribution Functions of Random Matrix Theory

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## Random Matrix Models (RMM)

RMM=**Probability space** $(\Omega, \mathcal{P}, \mathcal{F})$

$\Omega$=set of matrices, $\mathcal{P}$ = probability measure

- Gaussian Orthogonal Ensemble ($\beta = 1$)
  - $\Omega = N \times N$ real symmetric matrices
  - $\mathcal{P}$ = "unique" measure that is invariant under orthogonal transformations & matrix elements are iid rv's.

$$p(A)dA = c_N \exp\left(-\mathrm{tr}(A^2)\right) dA$$

- Gaussian Unitary Ensemble ($\beta = 2$)
  - $\Omega = N \times N$ hermitian matrices
  - $\mathcal{P}$= "unique" measure that is invariant under unitary transformations & real and imaginary matrix elements are iid rv's.
- Gaussian Symplectic Ensemble ($\beta = 4$)

$$F_{N,\beta}(t) := \text{Prob}\,(\lambda_{\text{max}} < t)\,, \beta = 1, 2, 4.$$

**THEOREM (Tracy-Widom)**

$$F_{N,2}(t) \to F_2(s) \text{ (edge scaling limit)}$$

$$\boxed{\begin{aligned} F_2(s) &= \det\left(I - K_{\text{Airy}}\right) \\ &= \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right) \end{aligned}}$$

where

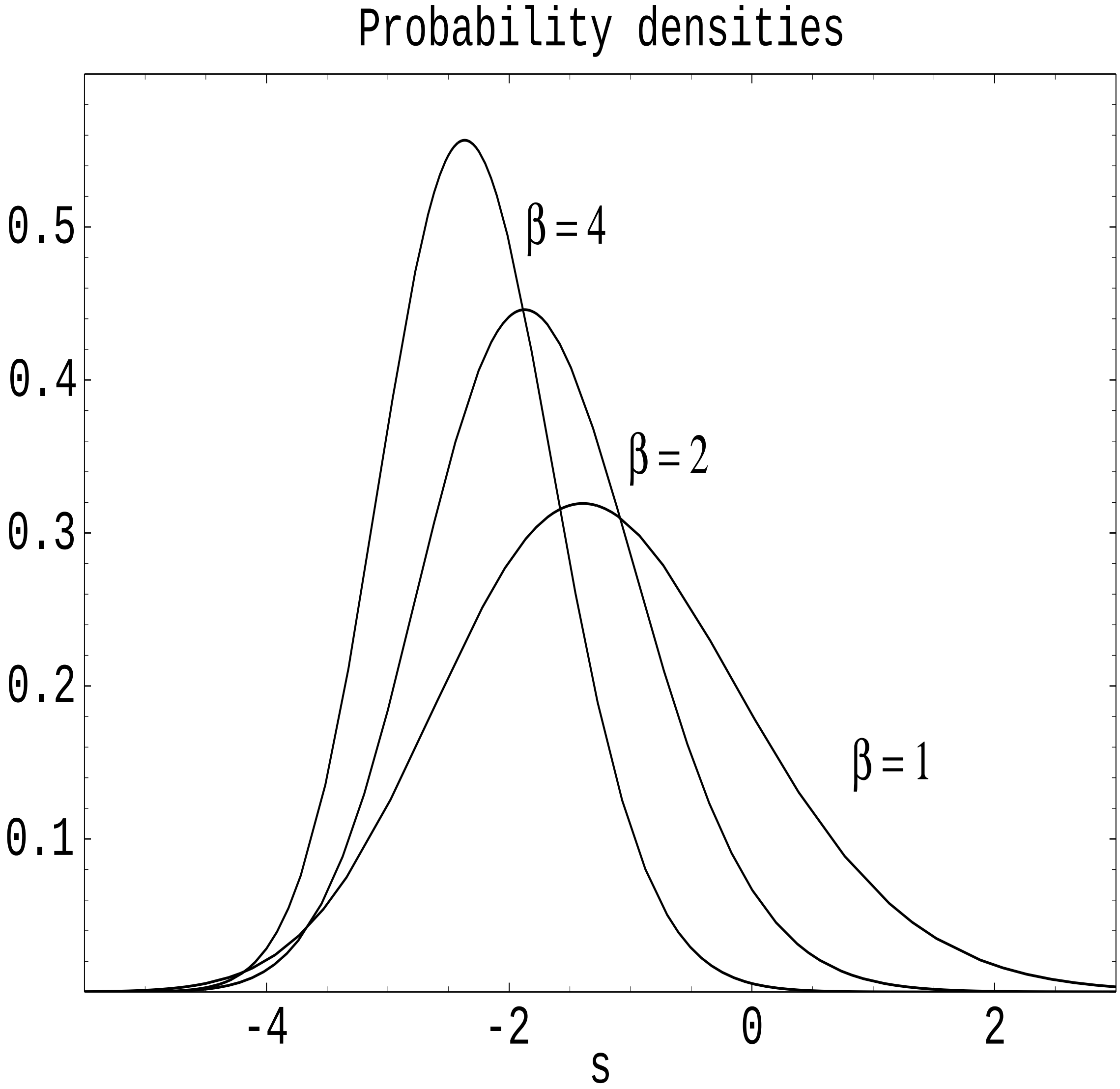
$$K_{\text{Airy}} \doteq \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y}$$

acting on $L^2(s,\infty)$ (Airy kernel)

$$q'' = sq + 2q^3 \text{ (Painlevé II)}$$

$$q(s) \sim \text{Ai}(s) \text{ as } s \to \infty$$

For $\beta = 1, 4$, $F_{N,\beta}(t) \to F_\beta(s)$: expressible in terms of $q$.




Probability densities of scaled largest eigenvalue $\beta = 1, 2, 4$

$$f_\beta(s) = \frac{dF_\beta}{ds}$$

# RMT Universality Theorems

**Invariant Measures**

$$\exp\left(-\mathrm{tr}(A^2)\right) \to \exp\left(-\mathrm{tr}(p(A))\right)$$

e.g.

$$p(A) = gA^4 - A^2$$

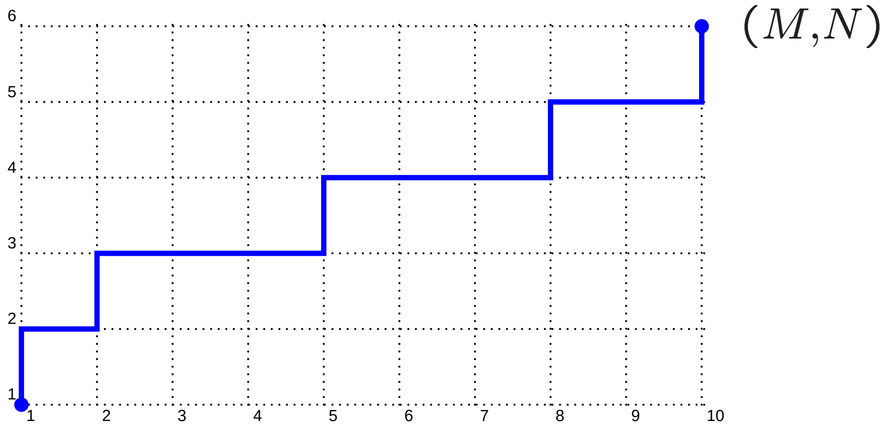
**Bleher-Its** and **Deift**-**Kriecherbauer**- **McLaughlin**-**Venakides**-**Zhou** showed limiting distribution is still $F_2$ for $\beta = 2$ ensembles. Used Riemann-Hilbert methods. For $\beta = 1, 4$ universality theorem known for quartic (**Stojanovic**).

**Noninvariant Measures**: Wigner matrices ($\beta = 1, 2$). **Soshnikov** proved universality in edge scaling limit. This result is very significant since nongaussian Wigner matrices have no Fredholm determinant representation of distribution function. Outside of "integrable" class but limit laws the same.

# RMT & Growth Processes

$$h_t(x) \;=\; \text{height of surface at position } x \text{ at time } t$$

Long history, e.g. **Hammersley-Welsh**, **Aldous-Diaconis**, **Seppäläinen**, but **Johansson** first to find the limiting laws describing the **fluctuations** about a **limiting shape** by building on the important work of **Baik-Deift-Johansson**. This was followed by **Baik-Rains**, **Prähofer-Spohn**, **Gravner-Tracy-Widom**.




$$\Pi_{M,N} \;=\; \text{up/right paths: } (1,1) \to (M,N)$$

$$G(M,N) \;=\; \max_{\pi \in \Pi_{M,N}} \sum_{(i,j)\in\pi} w(i,j)$$

$$\text{Prob}\,(w(i,j) = k) = (1-q)\,q^k$$

**Theorem** (**Johansson**) For each $q \in (0,1)$, $\gamma \geq 1$, and $s \in \mathbf{R}$

$$\lim_{N\to\infty} \text{Prob}\left(\frac{G([\gamma N], N) - c_1\, N}{c_2\, N^{1/3}} \leq s\right) = F_2(s)$$

where $c_1$ and $c_2$ are explicitly given functions of $\gamma$ and $q$.

Connection with RMT:

$$\text{Prob}\,(G(M,N) \leq t) \;=\; Z^{-1} \sum_{\substack{h\in\mathbf{N}^N \\ \max\{h_i\}\leq t+N-1}} \Delta(h)^2 \prod_{j=1}^{N} w_{N,\text{Mei}}(h_j)$$

$$\text{Prob}\,(\lambda_{\text{max}} \leq t) \;=\; Z^{-1} \int_{(-\infty,t]^N} \Delta(x)^2 \prod_{j=1}^{N} w_{N,\text{Gau}}(x_j)\, d^N x$$

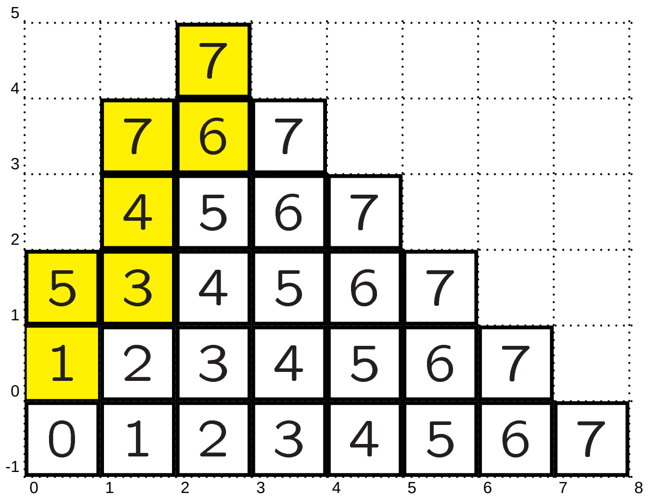
**Johansson**: Restricted geometries lead to $F_1$

## Discrete Time and Space Growth Models
## Oriented Digital Boiling

Introduce a **height function** $h_t(x)$ with rules (**Griffeath, Gravner**)

$$h_{t+1}(x) = \max\left\{h_t(x-1), h_t(x) + \varepsilon_{x,t}\right\}$$

$\varepsilon_{x,t}$ are i.i.d. Bernoulli random variables



## Statement of Results: Gravner-Tracy-Widom

Four limiting regimes

Define

$$\alpha = \frac{x+1}{t-x}, \qquad \alpha_c = \frac{1-p}{p}$$

1. **GUE Universal Regime**: $x \to \infty$, $t \to \infty$ such that $\alpha$ is fixed and $\alpha < \alpha_c$.

$$\mathrm{Prob}\left(\frac{h_t(x) - c_1 t}{c_2 t^{1/3}} < s\right) \to F_2(s)$$

2. **Critical Regime**: $x \to \infty$, $t \to \infty$ such that $\alpha = \alpha_c$. For fixed $\Delta \in \mathbf{Z}_+$,

$$\mathrm{Prob}\left(h_t(x) - (t-x) \le -\Delta\right)$$

converges to a $\Delta \times \Delta$ determinant.

3. **Deterministic Regime**: $x \to \infty$, $t \to \infty$ such that $\alpha > \alpha_c$.

$$\text{Prob}\,(h_t(x) = t - x) \to 1.$$

4. **Finite $x$ GUE Regime**: Fixed $x$ and $t \to \infty$.

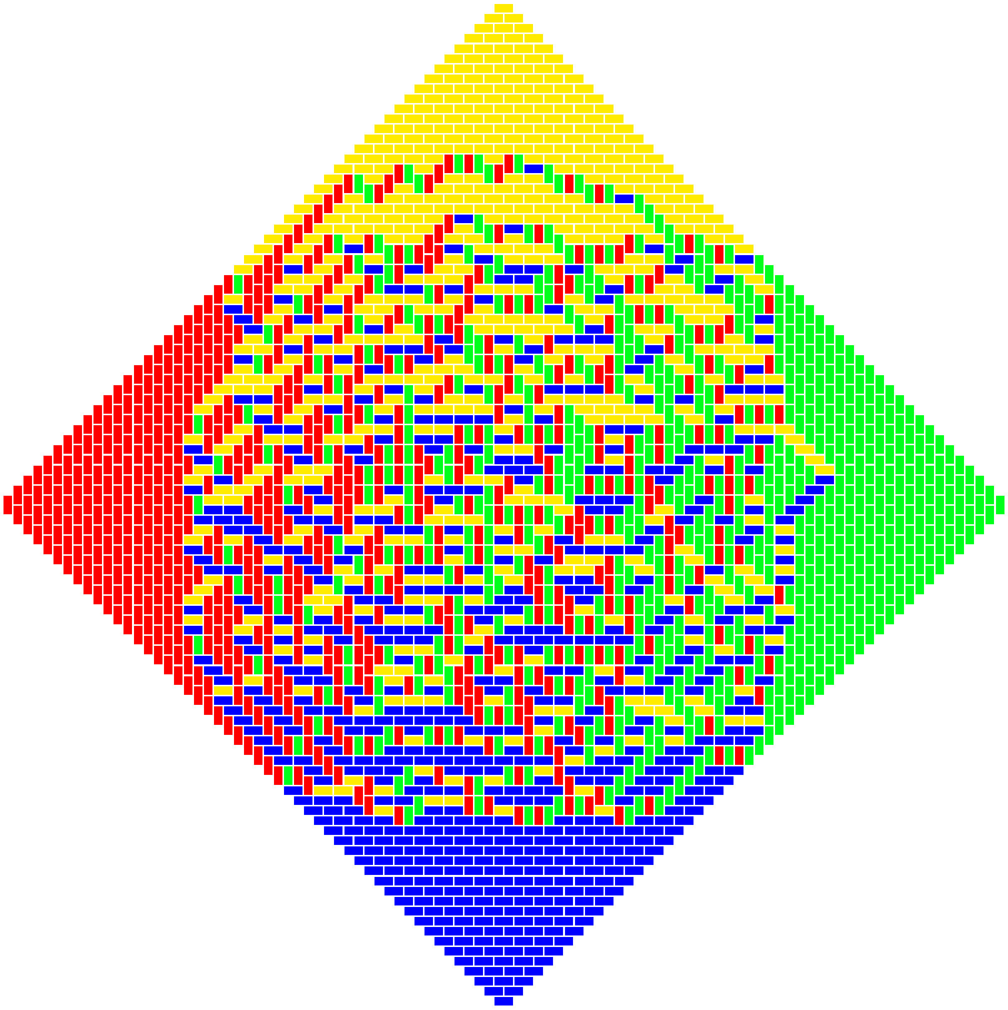
$$\text{Prob}\left(\frac{h_t(x) - p\,t}{(p(1-p)\,t)^{1/2}} < s\right) \to F_{x,2}(s)$$

$F_{x,2}$ = the distr fn for largest eigenvalue in finite $x$ GUE.

# Beyond the Arctic Circle Theorem

Random tilings on **Aztec diamonds**

**Elkies-Kuperberg-Larsen-Propp**, **Jockusch-Propp-Schor**, **Cohn-Elkies-Propp**



**Johansson**: Uses idea (**Randall**) that a domino tiling can be described by a family of

nonintersecting paths

By Lindström-Gessel-Viennot, probabilities on paths are expressed as

determinants

These determinants are related to discrete RMM, Hermite $\to$ Krawtchouk.

Limiting distribution fn. of fluctuations around

circle is

$$F_2$$

**Prähofer-Spohn** have reinterpreted the work of **Aldous-Diaconis, Baik-Deift-Johansson, Baik-Rains** as a growth model ("polynuclear growth") where

$\beta = 2$ is growth from a **single droplet**, GUE $F_2$ fluctuations.

$\beta = 1$ is growth from a **flat substrate**, GOE $F_1$ fluctuations

Third possibility: **stationary growth**, i.e. initial conditions are random with fixed density. **Baik-Rains** computed this distr fn.: call it $F_0$—expressible in terms of same Painlevé function $q$.

1+1 KPZ Equation (**Kardar-Parisi-Zhang**)

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \lambda\left(\frac{\partial h}{\partial x}\right)^2 + \sigma W(t,x)$$

$$\begin{aligned} h &= \text{height function} \\ W &= \text{space-time white noise} \end{aligned}$$

Conjecture (**Prähofer-Spohn**): The large $x$ and large $t$ limiting distribution of an appropriately centered and normalized $h$ equals $F_1$ for growth from a flat substrate, $F_2$ for growth from a single droplet, $F_0$ for growth from a stationary substrate.

Experiments:

**J. Timonen**, **M. Myllys**, **J. Maunuksela**

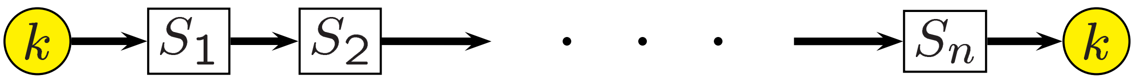
Slow, flameless burning process in a random medium (paper sheets)—$1+1$ KPZ universality class

# Connections with Queueing Theory

Series of $n$ single-server queues each with unlimited waiting space with first in-first out service.

Service times are i.i.d. with mean 1 and variance $\sigma^2$.

$k$ customers placed in first queue.




$$D(k,n) :=$$

Departure time of customer $k$ from queue $n$

**V(i,j)**:=Service time for customer $i$ at queue $j$

**Basic Recursion Relation:**

$$D(k,n) = \max\{D(k-1,n), D(k,n-1)\} + V(k,n)$$

This implies

$$D(k,n) = \max_{\pi \in \Pi} \left\{ \sum_{(i,j) \in \pi} V(i,j) \right\}$$

$\Pi =$ increasing paths $(1,1) \to (k,n)$

# Glynn-Whitt Functional Central Limit Theorem

Fix $k$, number of customers,

$$\frac{D(k,n)-n}{\sigma\sqrt{n}} \xrightarrow{d} \hat{D}_k, \quad n \to \infty$$

$\hat{D}_k$ is a functional of $k$-dimensional Brownian motion. $\hat{D}_k$ is **independent** of the service-time distribution $V$. $f = (f_1, f_2, \ldots, f_k) : [0,1] \to \mathbf{R}^k$, continuous, $f(0) = 0$,

$$\begin{aligned} F(f) \;:=\; & \max\{f_1(t_1) + f_2(t_2) - f_2(t_1) + \cdots + \\ & f_k(t_k) - f(t_{k-1}) : 0 \le t_1 \le \cdots \le t_k = 1\} \end{aligned}$$

$$B = (B_1, \ldots, B_k), k \text{ dim BM}$$

$$\hat{D}_k := F(B)$$

From **GTW** and **Baryshnikov** this equals the distribution of the largest eigenvalue in the $k \times k$ GUE, $F_{k,2}(s)$.

From **Johansson's** work we know for $V$ exponentially distributed the behavior of the queue for large $n$ and large $k$. Let

$$\gamma(x) = (1+\sqrt{x})^2, \quad \rho(x) = x^{-1/6}(1+\sqrt{x})^{4/3}$$

Then as $n \to \infty$

$$\text{Prob}\left(\frac{D(\lfloor xn \rfloor, n) - \gamma(x)\, n}{\rho(x) n^{1/3}} \leq s\right)$$

$$\to F_2(s)$$

# Distribution of the Largest Principal Component

**Iain Johnstone**

Let $X$ be an $n$ by $p$ data matrix. ($n$=number of observations of a $p$ dimensional row vector) Assume

$$X_{jk} \sim \mathrm{N}(0,1)$$

Denote the sample eigenvalues of the $p \times p$ Wishart matrix $X^t X$ by $\ell_1 > \cdots > \ell_p$. Define center and scaling constants

$$\begin{aligned}
\mu_{np} &= \left(\sqrt{n-1} + \sqrt{p}\right)^2 \\
\sigma_{np} &= \left(\sqrt{n-1} + \sqrt{p}\right)\left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{p}}\right)^{1/3}
\end{aligned}$$

then as $n, p \to \infty$, $n/p \to \gamma \geq 1$

$$\frac{\ell_1 - \mu_{np}}{\sigma_{np}} \xrightarrow{\mathcal{D}} W_1 \sim F_1$$

Remarks

- Reversing roles of $n$ and $p$ gives $\gamma \leq 1$.

- Proof adapts the TW proof for the GOE edge scaling.

- Soshnikov has removed the gaussian assumption but must assume $n - p = O(p^{1/3})$.

# Gap Fluctuations in the Superconductor Proximity Effect

Normal metal grain or semiconductor quantum dot in proximity to a superconductor. **Vavilov**, **Brouwer**, **Ambegaokar**, **Beenakker** predict that the mesoscopic fluctuations of the lowest excited state fluctuates from sample to sample with a **universal** distribution function

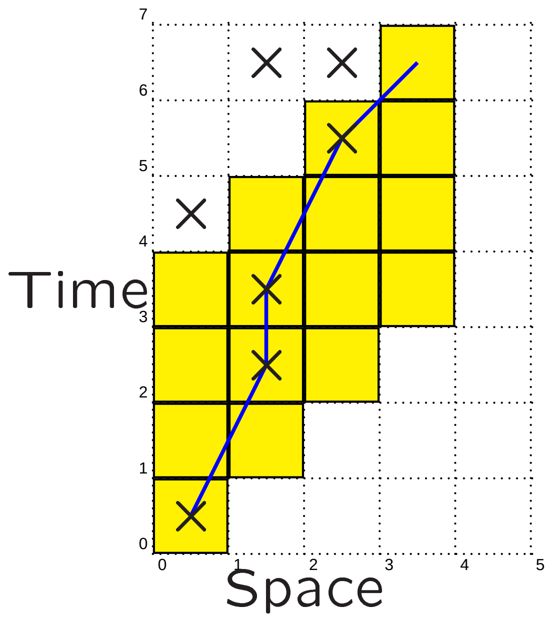
$$\begin{aligned} x &= (\varepsilon_1 - E_g)/\Delta_g \\ E_g &\sim N\delta \\ \Delta_g &\sim N^{1/3}\delta \\ N &= \#\text{ modes at Fermi surface} \gg 1 \\ \delta &= \text{mean level spacing} \end{aligned}$$

Fluctuations described by

$$\begin{array}{rl} F_1 & \text{zero magnetic field} \\ F_2 & \text{nonzero magnetic field} \end{array}$$

# ODB: Ideas in Proof

## Space-Time and Path Description




The backwards lightcone of the point $(x, t) = (3, 7)$ for the process shown earlier. The $\times$'s denote the marked points and polyogonal line gives a longest 1-increasing path. The length of this path is equal to the number of $\times$'s in the path. **This length equals $h_t(x)$.**

Transform backwards light cone into a random (0, 1)-matrix of size $(t-x) \times (x+1)$. Relate via **dual RSK** to **Young tableaux**. The length of the longest 1-increasing path becomes the number of boxes in the first row of $P$ in RSK.

$$\begin{aligned}\text{Prob}\,(h_t(x) \le h) &= (1-p)^{mn} \sum_{k=0}^{mn} r^k \\ &\quad \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \le h}} d_\lambda(m) d_{\lambda'}(n)\end{aligned}$$

where

$$\begin{aligned} d_\lambda(M) &= \#\text{ of SSYTs of shape } \lambda \\ &\quad \text{formed from } \{1, 2, \ldots, M\} \\ r &= p/(1-p) \\ m &= t-x \\ n &= x+1 \end{aligned}$$

# Toeplitz Determinant Representation

**Gessel's theorem** is

$$\sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le h}} r^{|\lambda|} s_\lambda(x) s_\lambda(y) = D_h(\varphi)$$

where $s_\lambda$ are the Schur functions and $D_h(\varphi)$ is the $h \times h$ Toeplitz determinant with symbol

$$\varphi(z) = \prod_{j=1}^{\infty} (1 - x_j z)^{-1} \prod_{j=1}^{\infty} (1 - y_j r z^{-1})^{-1}.$$

If we apply to both sides of this identity the automorphism $\omega$ $\omega(s_\lambda) = s_{\lambda'}$, to the symmetric functions in the $x$-variables we obtain

$$\sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le h}} r^{|\lambda|} s_{\lambda'}(x) s_\lambda(y) = D_h(\varphi)$$

where now the symbol is

$$\varphi(z) = \prod_{j=1}^{\infty} (1 + x_j z) \prod_{j=1}^{\infty} (1 - y_j r z^{-1})^{-1}.$$

Recalling the specialization $\mathrm{ps}_n^1$, we apply $\mathrm{ps}_n^1$ to the $x$-variables and $\mathrm{ps}_m^1$ to the $y$-variables in Gessel's identity and observe* that the resulting LHS is precisely what we want. Since the specialization $\mathrm{ps}_n^1$ is a ring homomorphism, we may apply it directly to the symbol. Doing so we obtain

$$\mathrm{Prob}\,(h_t(x) \leq h) = (1-p)^{mn} D_h(\varphi)$$

where

$$\varphi(z) = (1+z)^n (1 - r/z)^{-m}.$$

*Note that $\mathrm{ps}_n^1\, s_\lambda = d_\lambda(n)$ which follows from the combinatorial definition of the Schur function.

# Borodin-Okounkov Identity

$$D_h(\varphi) = Z \, \det(I - K_h)$$

where $K_h$ acts on $\ell^2(\{0,1,\ldots\})$ and is given by

$$K_h(j,k) = \sum_{\ell=1}^{\infty} \left(\varphi_-/\varphi_+\right)_{h+j+\ell+1} \left(\varphi_+/\varphi_-\right)_{-h-k-\ell-1}$$

and

$$\varphi = \varphi_+ \, \varphi_-$$

is the Wiener-Hopf factorization of $\varphi$, and $Z$ is an overall normalization constant.

The asymptotic analysis ($h \to \infty$) of the Fredholm determinant is easier than the asymptotic analysis of the Toeplitz determinant. (Note that $Z$ is determined by the strong Szegö limit theorem.)

## Saddle Point Method

$(\varphi_+/\varphi_-)_{-n}$ and $(\varphi_-/\varphi_+)_n$ are Fourier coeffs—$1D$ integrals—apply **steepest descent method**.

### GUE Universal Regime

- Two saddle pts: Nontrivial limit only when pts coincide. Coincidence determines the **time constant**.
- Corrections about saddle point are **cubic**. Coefficient of cubic determines **variance** normalization.
- Fourier coeff. goes to an **Airy function**.
- **Airy kernel**:
$$K_{\mathrm{Airy}}(x,y) = \int_0^\infty \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)\,dt$$
- $K_h \to K_{\mathrm{Airy}}$
- $\det(I - K_h) \to \det(I - K_{\mathrm{Airy}}) = F_2$

# Sketch of Proof of $P_{II}$ Representation of $F_2$

$$\frac{d}{ds}\,\mathrm{log\,det}\,(I-K) = -\mathrm{tr}\left((I-K)^{-1}\,\frac{\partial K}{\partial s}\right)$$

where

$$(Kf)\,(x) = \int_s^\infty K(x,y)f(y)\,dy$$

$\Longrightarrow$

$$\frac{\partial K}{\partial s} \doteq -K(x,s)\delta(y-s)$$

Substitute

$$\frac{d}{ds}\,\mathrm{log\,det}\,(I-K) = -R(s,s)$$

where

$$R = (I-K)^{-1}\,K$$

Thus we need the resolvent kernel at

$$\mathbf{R(s,s)}$$

# Derive Expression for $\mathrm{R}(\mathrm{x}, \mathrm{y})$

Let **M** denote the multiplication operator;

$$(Mf)(x) = xf(x)$$

$$\begin{aligned} [M, K] &\doteq xK(x,y) - K(x,y)y \\ &= (x-y)K(x,y) \\ &= \varphi(x)\psi(y) - \psi(x)\varphi(y) \end{aligned}$$

or

$$[M, K] = \varphi \otimes \psi - \psi \otimes \varphi$$

Since

$$\left[M, (I-K)^{-1}\right] = (I-K)^{-1}\,[M,K]\,(I-K)^{-1}$$

We define

$$\begin{aligned} Q(x;s) &= (I-K)^{-1}\,\varphi(x) \\ P(x;s) &= (I-K)^{-1}\,\psi(x) \end{aligned}$$

$\Rightarrow$

$$\left[\mathbf{M}, (\mathbf{I}-\mathbf{K})^{-1}\right] = \mathbf{Q} \otimes \mathbf{P} - \mathbf{P} \otimes \mathbf{Q}$$

Since $(I-K)^{-1} \doteq \rho(x,y) = \delta(x-y) + R(x,y)$ we compute LHS to find

$$\left[M,(I-K)^{-1}\right] = (x-y)\rho(x,y) = (x-y)R(x,y)$$

$\Rightarrow$

$$R(x,y) = \frac{Q(x)P(y) - P(x)Q(y)}{x-y}$$

Observe this only depends upon the form of $K(x,y)$ (**Its, Izergin, Korepin, Slavnov**), i.e.

$$K(x,y) = \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x-y}$$

Thus

$$R(x,x) = Q'(x;s)P(x;s) - P'(x;s)Q(x;s)$$

Defining

$$\mathbf{q}(\mathrm{s}) = \mathbf{Q}(\mathrm{s};\mathrm{s}) \text{ and } \mathbf{p}(\mathrm{s}) = \mathbf{P}(\mathrm{s};\mathrm{s})$$

we have

$$\mathbf{R}(\mathrm{s},\mathrm{s}) = \mathbf{Q}'(\mathrm{s};\mathrm{s})\mathbf{p}(\mathrm{s}) - \mathbf{P}'(\mathrm{s},\mathrm{s})\mathbf{q}(\mathrm{s})$$

# Compute $\mathrm{Q}'$ and $\mathrm{P}'$

Let $\mathbf{D}$ denote the differentiation operator. Recall $\varphi$ and $\psi$ for Airy kernel satisfy

$$D\varphi = \psi, \;\; D\psi = x\varphi$$

$$\begin{aligned} Q'(x) &= D\,(I-K)^{-1}\,\varphi \\ &= (I-K)^{-1}\,D\varphi + \left[D, (I-K)^{-1}\right]\varphi \end{aligned}$$

Compute commutator

$$\left[D, (I-K)^{-1}\right] = (I-K)^{-1}\,[D,K] \times (I-K)^{-1}$$

Differentiation and integration by parts $\Rightarrow$

$$\begin{aligned}[D, K] \;&\doteq\; \left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\right) \\ &\quad + K(x,s)\delta(y-s)\end{aligned}$$

Now use explicit form of $\varphi$ and $\psi$ to get

$$\left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\right) = -\varphi(x)\varphi(y)$$

Thus

$$\left[D, (I-K)^{-1}\right] \doteq -Q(x)Q(y) + R(x,s)\rho(s,y)$$

Now ready to compute:

$$\begin{aligned}Q'(x;s) &= (I-K)^{-1} D\varphi + \left[D, (I-K)^{-1}\right]\varphi \\ &= (I-K)^{-1}\psi - Q(x)\left(Q,\varphi\right) + R(x,s)q(s) \\ &= P(x) - Q(x)u(s) + R(x,s)q(s)\end{aligned}$$

So

$$Q'(s;s) = p - qu + R(s,s)q$$

Similarly,

$$\begin{aligned} P'(x;s) &= xQ(x) - 2Q(x)v(s) + P(x)u(s) \\ &\quad +R(x,s)p(s) \\ P'(s;s) &= sq - 2qv + pu + R(s,s)p \end{aligned}$$

Where

$$u = (\varphi, Q)\,,\ \ v = (\varphi, P)$$

Put these into

$$\begin{aligned} R(s,s) &= Q'(s;s)p - P'(s;s)q \\ &= p^2 - sq^2 + 2q^2v - 2pqu \end{aligned}$$

Now derive DE equations for $q$, $p$, $u$ and $v$

$$\frac{dq}{ds} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial s}\right) Q(x;s)|_{x=s}$$

$$
\begin{aligned}
Q(x;s) &= (I-K)^{-1}\,\varphi \\
\frac{\partial Q(x;s)}{\partial s} &= (I-K)^{-1}\,\frac{\partial K}{\partial s}\,(I-K)^{-1} \\
&= -R(x,s)q
\end{aligned}
$$

To see this recall

$$\frac{\partial K}{\partial s} \doteq -K(x,s)\delta(y-s)$$

Together with expression for $Q'(x;s) \Rightarrow$

$$\frac{dq}{ds} = p - qu$$

Similarly,

$$\frac{dp}{ds} = sq - 2qv + pu$$

Differentiating inside inner products gives

$$\frac{du}{ds} = -q^2, \;\; \frac{dv}{ds} = -pq$$

Using first integral $u^2 - 2v = q^2$, we eliminate $p$ in $q$ equation to get $P_{II}$ for $q$. Simplify $R(s,s)$ to get theorem.

**THEOREM:**(**Tracy-Widom**) Let

$$J = \bigcup_{j=1}^{m} (a_{2j-1} - a_{2j})$$

be a union of open intervals. Define

$$\tau(a) = \det(I - K)$$

where $K$ is an integral operator acting on $L^2(J)$ whose kernel is of the form

$$\frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}$$

where $\varphi$ and $\psi$ are assumed to satisfy

$$\frac{d}{dx}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \Omega(x)\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

with $\Omega(x)$ a $2 \times 2$ matrix, trace zero, with rational entries in $x$, then

$$\frac{\partial}{\partial a_j} \log\det(I - K)$$

are expressible polynomially in terms of solutions to a total system of partial differential equations ($a_j$ are the independent variables).

# Cases $\beta = 1$ and $\beta = 4$

In these two cases

$$E_\beta(0; J)^2 = \det\left(I - K_{N,\beta}\right)$$

the operator $K_{N,\beta}$ is a $2 \times 2$ matrix with operator entries. For Gaussian ensembles and $J$ a union of open intervals—general case worked out by TW. Specializing to the case

$$J = (t, \infty)$$

and in the **edge scaling limit**

$$\begin{aligned} F_{N,\beta}(t) &\to F_\beta(s), \;\; \beta = 1, 4, \\ F_1(s)^2 &= \exp\left(-\int_s^\infty q(x)\, dx\right) F_2(s) \\ F_4(s)^2 &= \cosh\left(\frac{1}{2}\int_s^\infty q(x)\, dx\right) F_2(s) \end{aligned}$$

where $\mathbf{q}$ is the same $P_{II}$ function and $F_2(s)$ the limiting distribution function for $\beta = 2$.