Differential Equations for Dyson Processes

Joint work with Harold Widom

I. Overview

We call **Dyson process** any invariant process on ensembles of matrices in which the entries undergo diffusion.

Dyson Brownian motion: Start with $n \times n$ GUE matrix, let entries independently undergo Ornstein-Uhlenbeck diffusion. Eigenvalues describe n curves: **Hermite Process**.

Let $n \to \infty$, scale near the top. Infinitely many curves, **Airy process**. Top curve $A(\tau)$. From work of **Johansson** and **Prähoffer & Spohn**, the Airy process is now believed to underly a large class of growth processes. ($1+1$ KPZ Universality Class)

Scale the Hermite process in the bulk $\longrightarrow$ **sine process**.

Evolution of singular values of complex matrices leads to **Laguerre process**; scaling this at bottom edge gives **Bessel process**.

II. Dyson BM: GUE initial conditions and independent matrix elements independently undergo OU diffusion (BM with a linear drift term):

$$\tau \to H(\tau)$$

Transition density is

$$p(H, H'; \tau_2 - \tau_1) := \exp\left(-\frac{\mathrm{tr}(H - qH')^2}{1 - q^2}\right)$$

where $q = e^{\tau_1 - \tau_2} < 1$. As $\tau_2 \to \infty$, measure approaches GUE measure. Each eigenvalue, viewed as a particle, feels an external electric field

$$E(x_i) = \sum_{i \neq j} \frac{1}{x_i - x_j} - x_i$$

with equilibrium Coulomb measure (at $\beta = 2$).

For many times $\tau_1 < \tau_2 < \cdots < \tau_m$ the density for $H(\tau_k)$ to be infinitesimal neighborhood of H_k is

$$e^{-\mathrm{tr}(H_1^2)} \prod_{j=2}^{m} p(H_j, H_{j-1}, \tau_j - \tau_{j-1})$$

Use HCIZ integral to integrate out unitary parts to obtain **determinantal measure** on eigenvalues $x_j(\tau)$

This leads to **extended kernels** (Eynard and Mehta, Johansson, Prähoffer & Spohn) and by scaling to

extended Hermite kernel, **extended Airy kernel**, **extended sine kernel**,

...

III. Airy Process is defined by its finite dimensional distribution functions

$$\mathbf{Pr}(\mathbf{A}(\tau_\mathbf{1}) \le \xi_\mathbf{1}, \ldots, \mathbf{A}(\tau_\mathbf{m}) \le \xi_\mathbf{m})$$

This probability given as Fredholm determinant of *extended Airy kernel*, an $m \times m$ matrix kernel. Entries $L_{ij}(x, y)$ given by

$$\int_0^\infty e^{-z\,(\tau_i - \tau_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz, \quad i \ge j,$$

$$-\int_{-\infty}^0 e^{-z\,(\tau_i - \tau_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz, \; i < j$$

$$K_{ij}(x, y) = L_{ij}(x, y)\,\chi_{(\xi_j, \infty)}(y).$$

Probability equals $\det(I - K)$.

Remarks

1. For $m = 1$ extended kernel reduces to **Airy kernel**—an integrable kernel in the sense of **A. Its** et al.

2. For $m = 1$ Fredholm determinant is a τ-function for **Painlevé II**, ψ.

3. Relationship between the two is

$$\psi(\xi) = \left(I - K_{\mathsf{Airy}}\right)^{-1} \mathrm{Ai}(x)|_{x=\xi}$$

4. Are there integrable differential equations for $m > 1$? Answered affirmatively by **Adler** and **van Moerbeke** and **TW**. Equations derived by Adler and van Moerbeke are of a different form than those of TW. Might be a multidimensional version of "sigma representations" versus Painlevé representations.

Set $R = K\,(I-K)^{-1}$, then

$$\partial_{\xi_k} \text{log det}\,(I-K) = R_{kk}(\xi_k,\,\xi_k)$$

Kernel not 'integrable'. To find equations take derivatives. New quantities arise. Take derivatives of these. Continue and hope at some point new quantities can be expressed in terms of old.

Unknowns: Five **matrix functions** of the ξ_k. First is

$$r_{ij} = R_{ij}(\xi_i,\,\xi_j).$$

To define others, let

$$A = \text{diag}\,(\text{Ai}), \qquad \chi = \text{diag}\,(\chi_{(\xi_k,\infty)}),$$

$$Q = (I-K)^{-1}A, \qquad \tilde{Q} = A\chi(I-K)^{-1}.$$

Other unknowns are

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j),$$

$$q'_{ij} = Q'_{ij}(\xi_i), \quad \tilde{q}'_{ij} = \tilde{Q}'_{ij}(\xi_j).$$

Define r_x and r_y by

$$(r_x)_{ij} = (\partial_x R)_{ij}(\xi_i, \xi_j)$$

$$(r_y)_{ij} = (\partial_y R)_{ij}(x_i, \xi_j).$$

r_x and r_y are **not** unknowns.

Set $\xi = \mathrm{diag}\,(\xi_k)$. Equations are

$$\begin{aligned}
dr &= -r\,d\xi\,r + d\xi\,r_x + r_y\,d\xi, \\
dq &= d\xi\,q' - r\,d\xi\,q, \\
d\tilde{q} &= \tilde{q}'\,d\xi - \tilde{q}\,d\xi\,r, \\
dq' &= d\xi\,\xi\,q - (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q', \\
d\tilde{q}' &= \tilde{q}\,\xi\,d\xi - \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi.
\end{aligned}$$

Have to show diagonal entries of r_x+r_y and off-diagonal entries of r_x and r_y are expressible in terms of the unknowns. Here is where the τ_k enter. Let $\tau = \mathrm{diag}\,(\tau_k)$ and Θ the matrix with all entries equal to one. Commutators

$$[D,\, L] = -A\Theta A + [\tau, L],$$
$$[D^2 - M,\, L] = 0.$$

From these can derive

$$r_x + r_y = -q\,\Theta\,\tilde{q} + r^2 + [\tau,\, r],$$
$$[\tau,\, r_x - r_y] = q'\,\Theta\,\tilde{q} - q\,\Theta\,\tilde{q}' + [r,\, r_x + r_y] + [\xi,\, r].$$

When $m = 1$ these equations reduce ($\tilde{q} = q = \psi,\ \tilde{q}' = q' = d\psi/d\xi + r\psi$) to the single Painlevé II equation

$$\frac{d^2\psi}{d\xi^2} = \xi\psi + 2\psi^3$$

Remarks

Adler and **van Moerbeke** used their DEs to derive $\tau \to \infty$ asymptotics for

$$\frac{\Pr\left(A(0) \le \xi_1, A(\tau) \le \xi_2\right)}{F_2(\xi_1) F_2(\xi_2)} = 1 + \frac{c_2(\xi_1, \xi_2)}{\tau^2} + \frac{c_4(\xi_1, \xi_2)}{\tau^4} + \mathrm{O}(\tau^{-6})$$

and **Widom** derived the same asymptotic expansion directly from the Fredholm determinant representation. The important feature is that c_2 and c_4 are expressible in terms of the Painlevé II function ψ, e.g.

$$c_2(\xi_1, \xi_2) = u(\xi_1) u(\xi_2), u(\xi) = \int_\xi^\infty \psi^2(x)\, dx$$

These same methods, e.g. perturbation expansion of DEs or expansion of Fredholm determinant, show that the **matrix Painlevé function $\mathbf{q}$**

$$q(\xi) = \begin{pmatrix} \psi(\xi_1) & 0 \\ 0 & \psi(\xi_2) \end{pmatrix} +$$

$$\frac{1}{\tau}\begin{pmatrix} 0 & -u(\xi_1)\psi(\xi_2) \\ \psi(\xi_1)u(\xi_2) & 0 \end{pmatrix} + O(\tau^{-2})$$

That is, matrix Painlevé q is decoupling in $\tau \to \infty$ asymptotics to scaler Painlevé II.

Open Problem for Extended Airy System of DEs

1. Are equations deformation equations for some **isomondromy** problem and is Fredholm determinant the associated τ-function in sense of **Jimbo-Miwa-Ueno**?

2. We proved **compatibility** for small m using Maple. Give general conceptual proof. Difficulty lies with the conditions determing r_x and r_y.

3. Systemize large τ asymptotics. Find small τ expansions. Both might be useful in applications. (Numerics is easy when expressed in terms of Painlevé II ψ.)

We have systems of PDEs that determine the Fredholm determinant of

- Extended Hermite kernel
- Extended Sine kernel
- Extended Bessel kernel

They are more complicated than the extended Airy system. Each requires a special trick. **Adler** and **van Moerbeke** also have system of DEs for extended Hermite kernel.

Extended Hermite kernel (Johansson, Eynard-Mehta) has entries $L_{ij}(x,y)$:

$$\sum_{k=0}^{n-1} e^{(k-n)\,(\tau_i-\tau_j)}\,\varphi_k(x)\,\varphi_k(y) \qquad \text{if } i \geq j,$$

$$-\sum_{k=n}^{\infty} e^{(k-n)\,(\tau_i-\tau_j)}\,\varphi_k(x)\,\varphi_k(y) \quad \text{if } i < j.$$

Set

$$\varphi = (2n)^{1/4}\,\varphi_n, \quad \psi = (2n)^{1/4}\,\varphi_{n-1},$$

and define

$$Q = (I-K)^{-1}\,\varphi, \quad P = (I-K)^{-1}\,\psi,$$

$$\tilde{Q} = \varphi\chi\,(I-K)^{-1}, \quad \tilde{P} = \psi\chi\,(I-K)^{-1}.$$

Unknowns $r_{ij} = R_{ij}(\xi_i, \xi_j)$ and $q,\ \tilde{q},\ p,\ \tilde{p}$ given by

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j),$$

$$p_{ij} = P_{ij}(\xi_i), \quad \tilde{p}_{ij} = \tilde{P}_{ij}(\xi_j),$$

$$q'_{ij} = Q'_{ij}(\xi_i), \quad \tilde{q}'_{ij} = \tilde{Q}_{ij}(\xi_j),$$

$$p'_{ij} = P'_{ij}(\xi_i), \quad \tilde{p}'_{ij} = \tilde{P}'_{ij}(\xi_j).$$

Equations

$$
\begin{aligned}
dr &= -r\,d\xi\,r + d\xi\,r_x + r_y\,d\xi, \\
dq &= d\xi\,q' - r\,d\xi\,q, \\
d\tilde{q} &= \tilde{q}'\,d\xi - \tilde{q}\,d\xi\,r, \\
dq' &= d\xi\,(\xi^2 - 2n - 1)\,q - \\
&\quad (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q', \\
d\tilde{q}' &= \tilde{q}\,(\xi^2 - 2n - 1)\,dx - \\
&\quad \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi, \\
dp &= d\xi\,p' - r\,d\xi\,p, \\
d\tilde{p} &= \tilde{p}'\,d\xi - \tilde{p}\,d\xi\,r, \\
dp' &= d\xi\,(\xi^2 - 2n + 1)\,p - \\
&\quad (r_x\,d\xi + d\xi\,r_y)\,p + d\xi\,r\,p', \\
d\tilde{p}' &= \tilde{p}\,(\xi^2 - 2n + 1)\,d\xi - \\
&\quad \tilde{p}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{p}'\,r\,d\xi.
\end{aligned}
$$

Commutators with $e^{\tau}(D-M)$ and $e^{-\tau}(D+M)$.

Case $m=1$. Can eliminate q and p, arrive at

$$\frac{d^3 r}{d\xi^3} = 4(\xi^2 - 2n)\frac{dr}{d\xi} - 4\xi r - 6\left(\frac{dr}{d\xi}\right)^2.$$

Integrates to Painlevé IV.