Differential Equations for Dyson Processes

Craig Tracy & Harold Widom

- Integrable Differential Equations
- Painlevé Equations in Probability
- Dyson Brownian Motion
- Airy Process
- Differential Equations for Airy Process
- Future Directions & Open Problems

DEs in Probability

The Gaussian distribution is

$$\Phi(x) = \int_{-\infty}^{x} f(t)\,dt \bigg/ \int_{-\infty}^{\infty} f(t)\,dt$$

where f satisfies the ODE

$$\frac{df}{dx} + xf = 0, \quad f(0) = 1\,.$$

This is an integrable differential equation.

What exactly is an integrable DE?

This requires we understand what we mean by integrable.

On that subject books are written ...

Integrable DEs

In Hamiltonian dynamics there is the well-known concept of Arnold-Liouville integrability.

Another class of integrable differential equations arise as integrability conditions for linear total differential equations:

$$dw = \Omega w$$

where Ω is a matrix of one-forms. Use $d^2 = 0$ to conclude

$$d\Omega = \Omega \wedge \Omega,$$

a nonlinear DE the matrix elements of Ω must satisfy.

The classic examples are the six Painlevé equations.

Consider the linear (matrix) Fuchsian DE

$$\frac{dY}{dz} = \sum_{\nu} \frac{A_\nu}{z - a_\nu} Y$$

We require that as the a_ν change that the monodromy matrices of the fundamental solution matrix $Y(z)$ are independent of the a_ν's. This requires $A_\nu = A_\nu(a)$.

One extends the DE to a system

$$dY = \Omega(z, a)Y$$

where d is exterior differentiation with respect to z and the a_ν's.

The Schlesinger equations—nonlinear DEs for $A_\nu = A_\nu(a)$— are then the integrablity conditions on Ω.

For the simplest case of 2×2 matrices with four singularities at 0, 1, t, and ∞, the Schlesinger equations can be reduced to the single Painlevé VI equation.

In fact all of the six Painlevé equations arise from isomondromy considerations but the other five require we consider linear DEs with irregular singular points.

Paul Painlevé (with help from Gambier) came to the equations bearing his name from the classification of nonlinear ODEs

$$\frac{d^2w}{dz^2} = R(\frac{dw}{dz}, w, z)$$

that have the property (now called the Painlevé property) that the only moveable singularities are poles. The classification gives 50 canonical types of which six are essentially new transcendental functions (the irreducibility property). These new six types are now called the Painlevé equations.

They have many nice properties, e.g. one can solve nonlinear connection problems and are closely related to certain Riemann-Hilbert problems.

Painlevé Equations in Probability

- 2-point scaling functions of the 2D Ising model are expressed in terms of P_{III} (1976: Wu, McCoy, Tracy, Barouch).
- The Gaudin distribution in GUE (bulk scaling) is expressed in terms of P_V (1980: Jimbo, Miwa, Môri, Sato) .
- The distribution function for the largest eigenvalue (edge scaling) in the three ensembles GOE, GUE and GSE are expressed in terms of P_{II} (1994–96: Tracy, Widom), e.g.

$$F_2(t) = \exp\left(- \int_t^\infty (x-t)\psi(x)^2 \, dx \right)$$

where

$$\psi'' = x\,\psi + 2\psi^3, \;\; \psi(x) \sim \mathrm{Ai}(x), x \to \infty.$$

Note: Above distributions are all expressible in terms of Fredholm determinants, e.g. $F_2 = \det(I - K_{\mathrm{Airy}})$.

Dyson BM

GUE initial conditions and independent matrix elements independently undergo Ornstein-Uhlenbeck diffusion:

$$\tau \to H(\tau).$$

Transition density

$$p(H, H'; \tau_2 - \tau_1) := \exp\left(-\frac{\mathrm{tr}(H - qH')^2}{1 - q^2}\right) /Z$$

$q = e^{\tau_1 - \tau_2} < 1.$

As $\tau_2 \to \infty$, measure approaches GUE measure.

Each eigenvalue feels an electric field

$$E(x_i) = \sum_{i \neq j} \frac{1}{x_i - x_j} - x_i$$

Many times: $\tau_1 < \tau_2 < \cdots < \tau_m$

With GUE initial conditions the density for $H(\tau_k)$ in neighborhood of H_k is

$$e^{-\mathrm{tr}(H_1^2)} \prod_{j=2}^{m} p(H_j, H_{j-1}, \tau_j - \tau_{j-1})$$

Use HCIZ integral to integrate out unitary parts to obtain determinantal measure on eigenvalues $x_j(\tau)$

Leads to **extended kernels** (Eynard & Mehta, Johansson, Prähoffer & Spohn) and by scaling to

extended Airy kernel, extended sine kernel, extended Bessel kernel. . .

Airy Process

Defined by the distribution functions

$$\Pr(A(\tau_1) \le \xi_1, \ldots, A(\tau_m) \le \xi_m)$$

Probability expressed as a Fredholm determinant of extended Airy kernel, an $m \times m$ matrix kernel. Entries $L_{ij}(x,y)$ given by

$$\int_0^\infty e^{-z\,(\tau_i - \tau_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz, \quad i \ge j,$$

$$-\int_{-\infty}^0 e^{-z\,(\tau_i - \tau_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz, \; i < j$$

$$K_{ij}(x,y) = L_{ij}(x,y)\,\chi_{(\xi_j,\infty)}(y).$$

Probability equals $\det\,(I - K)$.

Remarks

1. For $m = 1$ extended kernel reduces to Airy kernel—an integrable kernel in the sense of A. Its et al.. Not 'integrable' for $m > 1$.

2. For $m = 1$ Fredholm determinant is a τ-function for Painlevé II, ψ.

3. Relationship between the two is

$$\psi(\xi) = (I - K_{\text{Airy}})^{-1} \operatorname{Ai}(x)|_{x=\xi}$$

4. Integrable differential equations for $m > 1$? Answered affirmatively by Adler and van Moerbeke and TW.

Set $R = K\,(I-K)^{-1}$, then

$$\partial_{\xi_k} \log \det (I-K) = R_{kk}(\xi_k,\, \xi_k)$$

Unknowns: Five **matrix functions** of the ξ_k. First is

$$r_{ij} = R_{ij}(\xi_i,\, \xi_j).$$

To define others, let

$$A = \operatorname{diag}(\mathrm{Ai}), \quad \chi = \operatorname{diag}(\chi_{(\xi_k,\infty)}),$$

$$Q = (I-K)^{-1}A, \quad \tilde{Q} = A\chi(I-K)^{-1}.$$

Other unknowns are

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j),$$

$$q'_{ij} = Q'_{ij}(\xi_i), \quad \tilde{q}'_{ij} = \tilde{Q}'_{ij}(\xi_j).$$

Define r_x and r_y by

$$(r_x)_{ij} = (\partial_x R)_{ij}(\xi_i,\, \xi_j)$$

$$(r_y)_{ij} = (\partial_y R)_{ij}(x_i,\, \xi_j).$$

r_x and r_y are **not** unknowns.

Set $\xi = \mathrm{diag}\,(\xi_k)$. Equations are

$$\begin{aligned}
dr &= -r\,d\xi\,r + d\xi\,r_x + r_y\,d\xi, \\
dq &= d\xi\,q' - r\,d\xi\,q, \\
d\tilde{q} &= \tilde{q}'\,d\xi - \tilde{q}\,d\xi\,r, \\
dq' &= d\xi\,\xi\,q - (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q', \\
d\tilde{q}' &= \tilde{q}\,\xi\,d\xi - \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi.
\end{aligned}$$

Diagonal entries of $r_x + r_y$ and off-diagonal entries of r_x and r_y are expressible in terms of the unknowns. Here is where the τ_k enter. Let $\tau = \text{diag}\,(\tau_k)$ and Θ the matrix with all entries equal to one.

$$r_x + r_y = -q\,\Theta\,\tilde{q} + r^2 + [\tau,\, r],$$
$$[\tau,\, r_x - r_y] = q'\,\Theta\,\tilde{q} - q\,\Theta\,\tilde{q}' + [r,\, r_x + r_y] + [\xi,\, r].$$

To prove these we used the Airy commutators

$$\begin{aligned} [D,\, L] &= -A\Theta A + [\tau,\, r] \\ \left[D^2 - M\,,\, L\right] &= 0 \end{aligned}$$

When $m = 1$ these equations reduce ($\tilde{q} = q = \psi$, $\tilde{q}' = q' = d\psi/d\xi + r\psi$) to the single Painlevé II equation

$$\frac{d^2\psi}{d\xi^2} = \xi\psi + 2\psi^3$$

Remarks

Adler & van Moerbeke used their DEs to derive $\tau \to \infty$ asymptotics for

$$\frac{\Pr\left(A(0) \le \xi_1, A(\tau) \le \xi_2\right)}{F_2(\xi_1) F_2(\xi_2)} = 1 + \frac{c_2(\xi_1, \xi_2)}{\tau^2} + \frac{c_4(\xi_1, \xi_2)}{\tau^4} + \mathrm{O}(\tau^{-6})$$

and Widom derived the same asymptotic expansion directly from the Fredholm determinant representation. The important feature is that c_2 and c_4 are expressible in terms of the Painlevé II function ψ, e.g.

$$c_2(\xi_1, \xi_2) = u(\xi_1) u(\xi_2), \;\; u(\xi) = \int_\xi^\infty \psi^2(x)\, dx$$

These same methods, e.g. perturbation expansion of DEs or expansion of Fredholm determinant, show that the **matrix Painlevé function q**

$$q(\xi) = \begin{pmatrix} \psi(\xi_1) & 0 \\ 0 & \psi(\xi_2) \end{pmatrix} + \frac{1}{\tau} \begin{pmatrix} 0 & -u(\xi_1)\psi(\xi_2) \\ \psi(\xi_1)u(\xi_2) & 0 \end{pmatrix} + \mathrm{O}(\tau^{-2})$$

That is, matrix Painlevé q is decoupling in $\tau \to \infty$ asymptotics to scalar Painlevé II.

Open Problems for Airy System

1. Are equations deformation equations for some isomondromy problem and is Fredholm determinant the associated τ-function in sense of Jimbo-Miwa-Ueno?

2. We proved compatibility for small m using Maple. Give general conceptual proof. Difficulty lies with the conditions determining r_x and r_y.

3. Systematize large τ asymptotics. Find small τ expansions.

We have systems of PDEs that determine the Fredholm determinant of

- Extended Hermite kernel
- Extended Sine kernel
- Extended Bessel kernel

They are more complicated than the extended Airy system. Each requires a special trick. Adler & van Moerbeke also have system of DEs for extended Hermite kernel.

Higher Universality Classes

1. Airy kernel arises as a fold singularity: coalescence of two saddle points

2. Pearcey kernel (Brézin & Hikami, Okounkov & Reshetikhin, Bleher & Kuijlaars) arises as a cusp singularity: coalescence of three saddle points.

3. General Problem:

$$\text{Singularity} \longrightarrow \text{Diffraction Integral} \longrightarrow \text{Kernel} \longrightarrow \text{Process}$$