Nonintersecting Brownian Excursions

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ICMP 2006

August 2006

Outline of Talk

1. Brownian Excursion and the Airy Distribution
2. Nonintersecting Path Models and Extended Kernels
3. Extended Kernel for n-Brownian Excursion
4. Distribution of Bottom Curve at One Fixed Time: P_V
5. Scaling of Bottom Curve to Bessel Process, $\alpha = 1/2$
6. Expected Areas under Bottom and Top Curves

Brownian Excursion

A *Brownian excursion* (BE) $X(\tau)$ is a Brownian path conditioned to remain positive for $0 < \tau < 1$ with boundary conditions $X(0) = X(1) = 0$. BE can be defined by scaling one-dimensional simple random walk conditioned to stay positive and conditioned to start and to end at the origin—a process known as *Bernoulli excursion.*

Let B_τ denote standard Brownian motion (BM), so that

$$\mathbb{P}_x(B_\tau \in dy) = P(x, y, \tau)\, dy = \frac{1}{\sqrt{2\pi\tau}}\, e^{-(x-y)^2/(2\tau)}\, dy, \tau > 0, x, y \in \mathbb{R},$$

Let

$$H_a = \inf\{s : s > 0, B_s = a\} = \text{hitting time of } a.$$

Then

$$\mathbb{P}_x(B_\tau \in dy, H_0 > \tau) = (P(x, y, \tau) - P(x, -y, \tau))\ dy := P_-(x, y, \tau) dy.$$

Airy Distribution

Let $\mathcal{A}$ denote the area under a BE path.

Then

$$F_{\text{Ai}}(a) := \mathbb{P}(\mathcal{A} < a)$$

is called the *Airy distribution*. (Not to be confused with the *Airy process*!) For example,

$$\mathbb{E}(\mathcal{A}) = \sqrt{\frac{\pi}{8}}, \quad \mathbb{E}(\mathcal{A}^2) = \frac{5}{12}, \ \ldots$$

The Airy distribution appears as the limit law for a number of combinatorial problems: path length in trees, total displacement of a random parking sequence, …(Flajolet & Louchard)

Extended Kernels

Given: stationary Markov process with continuous paths, transition probability density $P(x, y, \tau)$, and a family of n nonintersecting paths, $\{X_i(\tau)\}_{i=1}^n$, beginning at $a_1, \ldots, a_n$ at time $\tau = 0$ and ending at $b_1, \ldots, b_n$ at time $\tau = 1$.

An *extended kernel* is a matrix kernel $K(x, y) = (K_{k\ell}(x, y))_{k\, l=1,\ldots,m}$ depending on $0 < \tau_1 < \cdots < \tau_m < 1$ with the following property: Given functions f_k the expected value of

$$\prod_{k=1}^{m} \prod_{i=1}^{n} (1 + f_k(X_i(\tau_k)))$$

is equal to $\det(I + K\, f)$, where f denotes multiplication by $\text{diag}(f_k)$. In the special case where $f_k = -\chi_{J_k}$ this is the probability that for $k = 1, \ldots, m$ no path passes through the set J_k at time τ_k.

Nonintersecting Brownian Bridges and Excursions

Define *Extended Hermite kernel*:

$$K_n^{\mathrm{GUE}}(x,y;\hat\tau) = \begin{cases} \sum_{j=0}^{n-1} e^{j\hat\tau}\varphi_j(x)\varphi_j(y) & \hat\tau \geq 0 \\ -\sum_{j=n}^{\infty} e^{j\hat\tau}\varphi_j(x)\varphi_j(y) & \hat\tau < 0 \end{cases}$$

- Brownian Bridge Extended Kernel:

$$\frac{1}{\sqrt{2(1-\tau_k)\tau_\ell}}\, K_n^{\mathrm{GUE}}\left(X_k, Y_\ell; \hat\tau_k - \hat\tau_\ell\right),$$

$$X_k = \frac{x}{\sqrt{2\tau_k(1-\tau_k)}},\ \ Y_\ell = \frac{y}{\sqrt{\tau_\ell(1-\tau_\ell)}},\ \ \frac{\tau_k}{1-\tau_k} = \hat\tau_k$$

- Brownian Excursion Extended Kernel:

$$\frac{1}{\sqrt{2(1-\tau_k)\tau_\ell}}\ \left\{K_{2n}^{\mathrm{GUE}}\left(X_k, Y_\ell; \hat\tau_k - \hat\tau_\ell\right) - K_{2n}^{\mathrm{GUE}}\left(X_k, -Y_\ell; \hat\tau_k - \hat\tau_\ell\right)\right\}$$

Case of a Single Time: $\mathbf{m = 1}$

Order paths: $X_n(\tau) > \cdots > X_1(\tau) > 0$

$$\mathbb{P}(X_1(\tau) \geq x) = \det\left(I - K\chi_{J_1}\right), \;\; J_1 = (0, s)$$

$$\mathbb{P}\left(X_n(\tau) < x\right) = \det\left(I - K\chi_{J_2}\right) \;\; J_2 = (s, \infty)$$

where

$$K = K_{2n}^{\mathrm{GUE}}(x, y) - K_{2n}^{\mathrm{GUE}}(x, -y), \;\; s = \frac{x}{\sqrt{\tau(1-\tau)}}$$

- Operator K is finite rank $\Longrightarrow$ distributions are $n \times n$ dets.
- Operator K has kernel of the form

$$\frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y} \;\; \text{where}$$

$$\varphi(x) = n^{1/4} x^{1/4} \varphi_{2n}(\sqrt{x}), \;\; \psi(x) = n^{1/4} x^{-1/4} \varphi_{2n-1}(\sqrt{x})$$

Painlevé V Representation

$$\mathbb{P}\left(X_1(\tau) \geq x\right) = \exp\left(-\int_0^{s^2} \frac{r(t)}{t}\, dt\right)$$

where r satisfies the Jimbo-Miwa σ form of Painlevé V,[a] $s = x/\sqrt{\tau(1-\tau)}$ with boundary condition as $s \to 0^+$

$$r(s) = r_0 s^{3/2} + \mathrm{O}(s^{5/2}), \;\; r_0 = \frac{1}{\sqrt{\pi}} \frac{1}{2^{2n}} \binom{2n}{n} \frac{4n(2n+1)}{3}$$

- The number of BE curves, n, appears only as a parameter in the Painlevé V DE.

- Similar Painlevé V representation for $\mathbb{P}(X_n(\tau) < x)$.

[a] $r = -\sigma$, $\nu_0 = n$, $\nu_1 = n + 1/2$, $\nu_2 = \nu_3 = 0$.

Limit Theorems as $\mathbf{n} \to \infty$

• Look at scaling of bottom curve as $n \to \infty$:

$$x, y \to \sqrt{\frac{2n(1-\tau)}{\tau}}\, x, y, \quad J_k \to \sqrt{\frac{2n(1-\tau)}{\tau}}\, J_k$$

$$\tau_k \to \tau + \frac{\tau(1-\tau)}{2n}\, \tau_k$$

Then extended BE kernel approaches (with trace norm convergence) the *extended Bessel kernel* with $\alpha = 1/2$:

$$\frac{2}{\pi} \int_0^1 e^{(\tau_k - \tau_\ell)t^2/2} \sin xt \, \sin yt \, dt, \quad k \geq \ell$$

$$-\frac{2}{\pi} \int_1^\infty e^{(\tau_k - \tau_\ell)t^2/2} \sin xt \, \sin yt \, dt, \quad k < \ell.$$

• Expect top curve to scale to *Airy process.* Top curve does not "feel presence of the wall" as $n \to \infty$. Checked that diagonal matrix elements approach (with trace norm convergence) the Airy kernel.

Area under Bottom and Top Curves

Define

$$\mathcal{A}_{n,L} = \int_0^1 X_1(\tau)\, d\tau, \quad \mathcal{A}_{n,H} = \int_0^1 X_n(\tau)\, d\tau.$$

- Expected area

$$\begin{aligned}
\mathbb{E}(\mathcal{A}_{n,L}) &= \int_0^1 \mathbb{E}(X_1(\tau))\, d\tau \\
&= \int_0^1 \int_0^\infty \mathbb{P}(X_1(\tau) \geq x)\, dx\, d\tau \\
&= \int_0^1 \sqrt{2\tau(1-\tau)}\, d\tau \cdot \int_0^\infty \det\left(I - K\chi_{(0,s)}\right)\, ds \\
&= \frac{\pi}{4\sqrt{2}} \int_0^\infty \det\left(\delta_{j,k} - (\Psi_j, \Psi_k)\right)_{j,k=0}^{n-1}\, ds
\end{aligned}$$

- Last expression good for numerical evaluation for small n.

- Higher moments are expressible in terms of integrals over Fredholm dets with extended kernel.

Expected Area Asymptotics

Using convergence of BE kernel to Bessel kernel we derive (with some additional estimates to get convergence of the moments)

$$\mathbb{E}(\mathcal{A}_{n,L}) \sim \frac{c_L}{\sqrt{n}}, \quad n \to \infty \;\; \text{where}$$

$$c_L = \frac{\pi}{8\sqrt{2}} \int_0^\infty \det\left(I - K_{\text{Bessel}} \chi_{(0,s^2)}\right) ds \simeq 0.682808.$$

Compare with $\sqrt{n}\, \mathbb{E}(\mathcal{A}_{n,L})$ for $n = 5, \ldots, 11$

0.667334, 0.669708, 0.671449, 0.672784, 0.673838, 0.674691, 0.675396

- Top curve

$$\mathbb{E}(\mathcal{A}_{n,H}) = \frac{\pi}{2^{3/2}} \sqrt{n} + \frac{c_H}{n^{1/6}} + \mathrm{o}(n^{-1/6})$$

$$c_H = \frac{\pi}{8 \cdot 2^{1/6}} \mu_2 \simeq -0.619\,623\,767\,170$$

Here μ_2 is first moment of F_{GUE} largest eigenvalue distribution.