Painlevé Representations for Distribution Functions for Next-Largest, Next-Next-Largest, etc., Eigenvalues of GOE, GUE and GSE

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Figure 1: Paul Painlevé, 1863–1933.

Outline of Talk

I. Basic definitions and universality theorems

II. Painlevé representations for largest eigenvalue distributions: Orthogonal, unitary and symplectic ensembles

III. Next-largest, next-next-largest eigenvalue distributions for unitary ensemble

IV. Statement of results for orthogonal and symplectic ensembles

V. Applications to Wishart distribution

VI. Remarks on the proof for the orthogonal ensemble

VII. Open problems and future directions

BASIC DEFINITIONS

Given n-tuplets of random variables $\{\lambda_1, \ldots, \lambda_n\}$, define the joint density functions

$$\mathrm{P}_{n\beta}(\lambda_1, \ldots, \lambda_n) = C_{n\beta} \exp\left[-\frac{1}{2}\beta \sum_{i=1}^{n} \lambda_i^2\right] \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

$C_{n\beta}$ are normalization constants and $\beta_{GOE} = 1$, $\beta_{GUE} = 2$, $\beta_{GSE} = 4$. For $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, let

$$\hat{\lambda}_k^{(n)} = \frac{\lambda_k - \sqrt{2\,n}}{2^{-1/2}\,n^{-1/6}}$$

$\hat{\lambda}_k^{(n)}$ is the rescaled k^{th} eigenvalue measured from the EDGE OF THE SPECTRUM. We are interested in

$$F_\beta(s,k) = \lim_{n\to\infty} \mathrm{P}_{n\beta}\Big(\hat{\lambda}_k^{(n)} \leq s\Big)\,,\ \beta = 1, 2, 4.$$

UNIVERSALITY THEOREMS

Replace Gaussian ensembles by

$$C_{n\beta}\,\exp\left[-\sum_{i=1}^{n} V_\beta(\lambda_i)\right]\prod_{i<j}|\lambda_i-\lambda_j|^\beta$$

where V_β is a polynomial of even degree (with positive leading coefficient).

For $\beta = 2$ it is a result of Deift-Kriecherbauer-McLaughlin-Venakides-Zhou and for $\beta = 1, 4$ a result of Deift-Gioev that that the limiting distributions $F_\beta(s,k)$ are independent of V_β. (The centering and norming constants do depend on V_β.)

Soshnikov showed, for $\beta = 1, 2$, the same universality holds for WIGNER MATRICES. (Distribution on matrix elements has finite moments, odd moments zero.)

Painlevé Representations for $F_\beta(s,1)$

Tracy-Widom:

$$F_2(s,1) = \exp\left[-\int_s^\infty (x-s)\, q^2(x)\,d\,x\right]$$

$$F_1^2(s,1) = F_2(s,1)\,\exp\left[-\int_s^\infty q(x)\,d\,x\right]$$

$$F_4^2(s,1) = F_2(s,1)\,\cosh^2\left[-\frac{1}{2}\int_s^\infty q(x)\,d\,x\right]$$

where q is the solution to Painlevé II

$$q'' = xq + 2q^3,\;\; q(x) \sim \mathrm{Ai}(x), x \to \infty$$

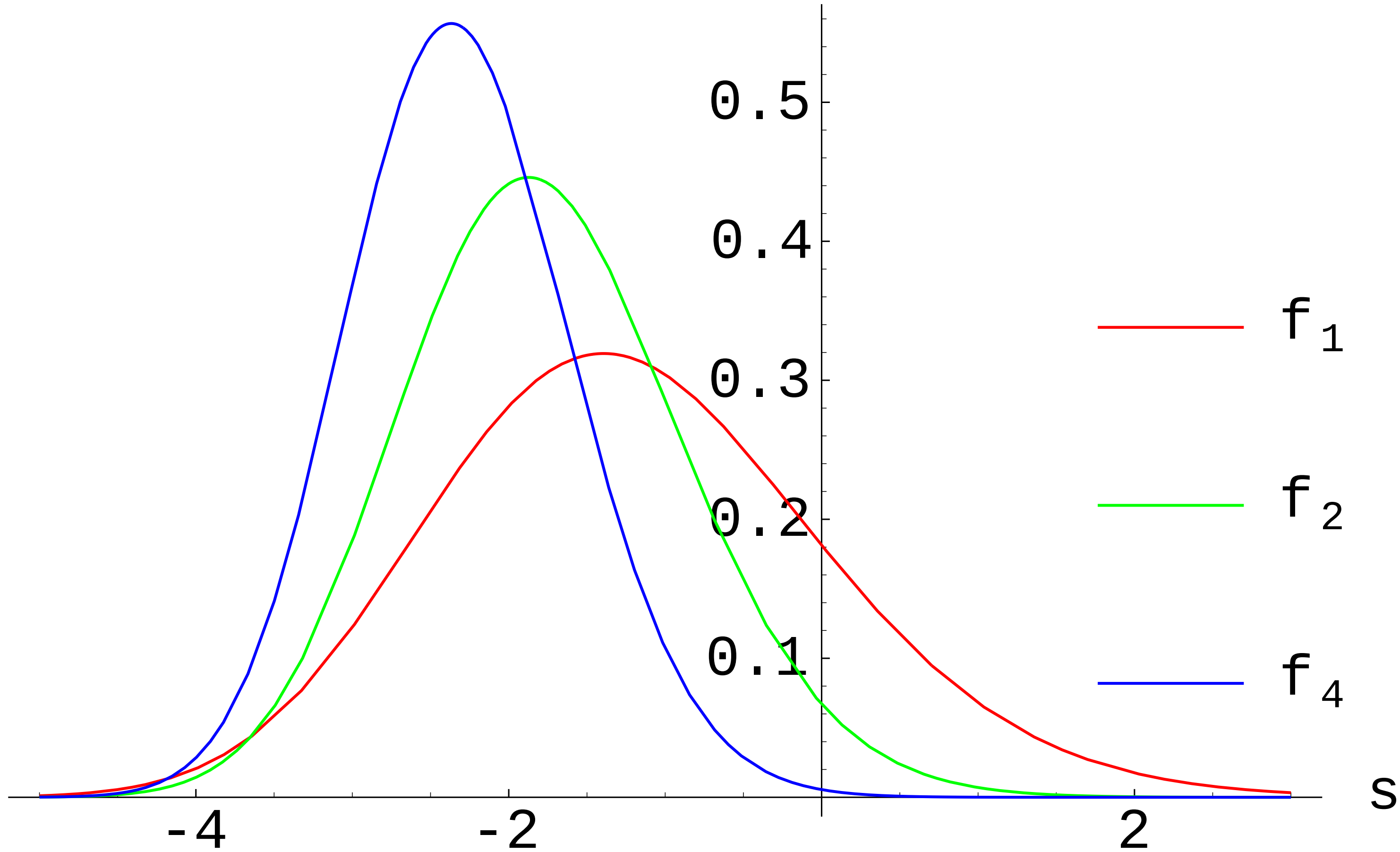

Figure 2: The TW density functions $f_\beta, \beta = 1, 2, 4$

Distributions $F_2(s,k)$ for Unitary Ensembles

Define

$$D_2(s,\lambda) = \det\left(I - \lambda K_{\text{Airy}}\right),\ 0 \le \lambda \le 1,$$

where K_{Airy} is the Airy kernel

$$K_{\text{Airy}}(x,y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \text{ on } L^2(s,\infty)$$

then

$$F_2(s,k+1) - F_2(s,k) = \frac{(-1)^k}{k\,!}\frac{\partial^k}{\partial\lambda^k} D_2(s,\lambda)\Big|_{\lambda=1^-} \quad k \ge 0, F_2(s,0) := 0$$

We have a Painlevé representation for $D(s,1)$.

What is the Painlevé representation for $D(s,\lambda)$?

The answer (TW) is remarkably simple:

$$D_2(s,\lambda) = \exp\left[-\int_s^\infty (x-s)\, q^2(x,\lambda)d\,x\right]$$

where $q(x,\lambda)$ satisfies the same Painlevé II equation but with boundary condition

$$q(x,\lambda) \sim \sqrt{\lambda}\,\mathrm{Ai}(x),\ x \to \infty.$$

Thus $F_2(s,k)$ are expressible in terms of

$$q(s,1),\ \frac{\partial q}{\partial \lambda}(s,1),\ \ldots,\ \frac{\partial^k q}{\partial \lambda^k}(s,1)$$

Will same hold for orthogonal and symplectic ensembles?

i.e. Take $\lambda = 1$ results and simply make replacement

$$q(x) = q(x,1) \to q(x,\lambda)\,?$$

A hint that things are not so simple

Forrester-Rains: Eigenvalues of GSE_n are distributed like alternate even eigenvalues of GOE_{2n+1}.

This was conjectured earlier, in edge scaling, by Baik-Rains.

In particular, this says the distribution of next-largest eigenvalue of GOE (in edge scaling) equals the distribution of the largest eigenvalue of GSE (in edge scaling).

But this would imply a relationship between

$$q(s,1) \text{ and } \frac{\partial q}{\partial \lambda}(s,1)$$

Very Unlikely!

Let

$$D_1(s,\lambda) := \lim_{\text{Edge Scaling}} \det\left(I - \lambda K_{n,\text{GOE}}\right) = \det\nolimits_2\left(I - \lambda K_{1,\text{Airy}}\right)$$

$$D_4(s,\lambda) := \lim_{\text{Edge Scaling}} \det\left(I - \lambda K_{n,\text{GSE}}\right) = \det\left(I - \lambda K_{4,\text{Airy}}\right)$$

Remarks:

1. Convergence for $\beta = 4$ is in trace-class norm. For $\beta = 1$ convergence is to the regularized determinant, $\det_2$, in the Hilbert-Schmidt norm (TW).

2.

$$F_\beta(s, k+1) = F_\beta(s,k) + \frac{(-1)^k}{k!}\frac{\partial^k}{\partial \lambda^k} D_\beta^{1/2}(s,\lambda)\bigg|_{\lambda=1}, \ \beta = 1, 4,$$

with $F_\beta(s,0) := 0$.

Painlevé Representations for D_1 and D_4

Momar Dieng proved the following:

$$D_4(s,\lambda) = D_2(s,\lambda)\,\cosh^2\left(\frac{\mu(s,\lambda)}{2}\right)$$

$$D_1(s,\lambda) = D_2(s,\tilde{\lambda})\,\frac{\lambda - 1 - \cosh\mu(s,\tilde{\lambda}) + \sqrt{\tilde{\lambda}}\,\sinh\mu(s,\tilde{\lambda})}{\lambda - 2}$$

with

$$\mu(s,\lambda) := \int_s^\infty q(x,\lambda)\,dx \qquad \text{and} \qquad \tilde{\lambda} := 2\,\lambda - \lambda^2$$

In the symplectic case the prescription $q(x,1) \to q(x,\lambda)$ is valid; whereas for the orthogonal case, a new formula appears.

Note, in the orthogonal case, that D_2 and q are evaluated at $\tilde{\lambda}$.

Two Corollaries

I.

$$D_1(s,\lambda) = D_4(s,\tilde{\lambda})\,\left(1-\sqrt{\frac{\lambda}{2-\lambda}}\,\tanh\frac{\mu(s,\tilde{\lambda})}{2}\right)^2$$

II.

$$\frac{(-1)^n}{n!}\,\frac{\partial^n}{\partial\,\lambda^n}\,D_4^{1/2}(s,\lambda)\,\Big|_{\lambda=1} =$$

$$\left[-\frac{1}{(2\,n+1)!}\,\frac{\partial^{2\,n+1}}{\partial\,\lambda^{2\,n+1}}+\frac{1}{(2\,n)!}\,\frac{\partial^{2\,n}}{\partial\,\lambda^{2\,n}}\right]\,D_1^{1/2}(s,\lambda)\,\Big|_{\lambda=1}$$

which implies (and gives a new proof of)

$$F_4(s,k) = F_1(s,2k),\;\; k\geq 1.$$

Simulations

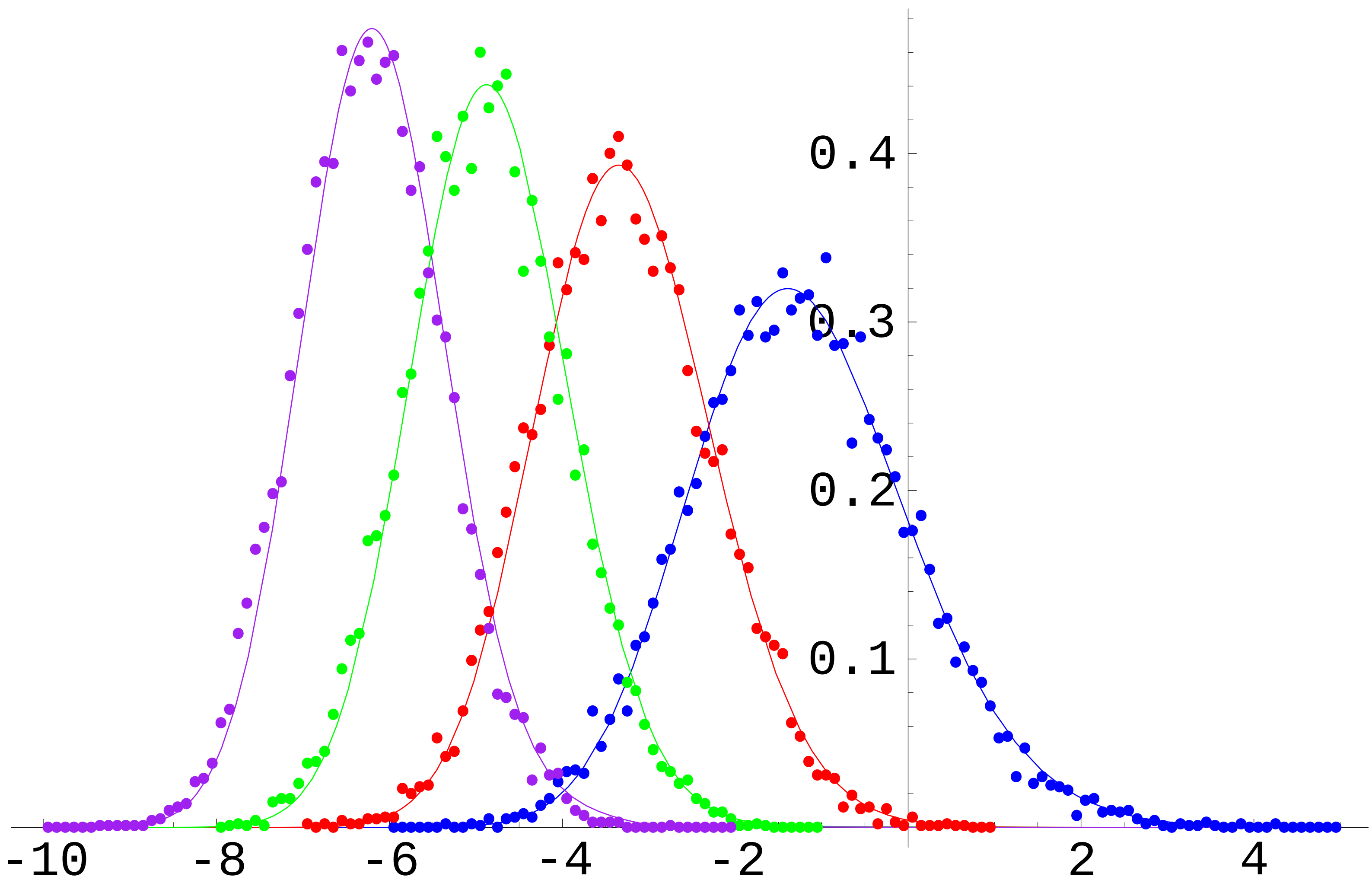

Figure 3: 10^4 realizations of $10^3 \times 10^3$ GOE matrices

Applications to Wishart Distribution

Let X denote an $n \times p$ data matrix whose rows are independent $N_p(0, \Sigma)$ random variables. The matrix

$$\frac{1}{n} X^t X,$$

called the SAMPLE COVARIANCE MATRIX, is said to have WISHART DISTRIBUTION $W_p(n, \Sigma)$. The *null case* corresponds to the choice $\Sigma = \mathrm{id}$. Let $\lambda_1 > \cdots > \lambda_n$ denote the eigenvalues of $X^t X$.

Results of Johnstone for $k = 1$ and Soshnikov for $k > 1$ show that in the null case, as $n, p \to \infty$, $n/p \to \gamma$, $0 \leq \gamma < \infty$

$$\frac{\lambda_k - \mu_{np}}{\sigma_{np}} \xrightarrow{\mathcal{D}} F_1(s, k)$$

with explicit expressions for the centering and norming constants.

Further Developments for Wishart Distribution

1. El Karoui in null case for the largest eigenvalue, proves the limit law for $0 \le \gamma \le \infty$. This requires additional estimates to allow $\gamma = \infty$. Soshnikov's theorem for $k > 1$ has not been extended to the $\gamma = \infty$ case.

2. Soshnikov removes Gaussian assumption on the distribution of the matrix elements of X and only requires odd moments are zero and even moments satisfy a Gaussian type bound. Then for the null case and under the restriction that as $n, p \to \infty$ that

$$n - p = \mathrm{O}(p^{1/3})$$

we get the same limit law described by $F_1(s, k)$.

Remarks on the Proof for the Orthogonal Ensemble

One of the main ideas of TW was to rewrite the the 2×2 matrix $K_{1,n}$ with operator entries so that the $\det(I - K_{1,n})$ was equal to the determinant of an operator of the form

$$(I - K_{2,n})\ (I - B)$$

where

$$B = \text{rank two operator}$$

Once in this form the determinant of the first factor gives, in the edge scaling limit, the distribution F_2 while the determinant of the second factor gives

$$\mu(s,1) = \exp\left[-\int_s^\infty q(x,1)\, dx\right]$$

The same method worked in the case of GSE.

Try same idea for the λ-dependent determinants

For GSE everything remains pretty much the same and the result, in the end, is simply replacing

$$q(x,1) \to q(x,\lambda)$$

However, if one follows directly the proof for the orthogonal case, one finds the operator B is *not* of finite rank. That is,

$$\det\left(I - \lambda K_{1,n}\right) = \det\left(I - \lambda K_{2,n}\right) \det\left(I - B\right)$$

but B is not of finite rank (and hence unable to relate to q). What Dieng showed was that a different factorization works provided one factors out $I - \tilde{\lambda} K_{2,n}$, $\tilde{\lambda} = 2\lambda - \lambda^2$, i.e.

$$\det(I - \lambda K_{1,n}) = \det\left(I - \tilde{\lambda} K_{2,n}\right) \det(I - B)$$

where now

$$B = \text{rank three operator}$$

Open Problems and Future Directions

1. TW used WKB to find $x \to -\infty$ asymptotics of

$$\left(\frac{\partial^k q}{\partial \lambda^k}\right)(x,1),\ k \geq 1.$$

Develop a RH approach to this general problem for Painlevé functions.

2. TW showed

$$F_2(s) \sim \frac{\tau_0}{(-s)^{1/8}} \exp(s^3/12),\ s \to -\infty.$$

The constant τ_0 is conjectured to equal

$$e^{\zeta'(-1)}\, 2^{1/24}.$$

3. Lift restriction

$$n - p = \mathrm{O}(p^{1/3})$$

in Soshnikov's Wishart universality theorem.

4. For Wishart distribution, the problem is to go beyond the null case $\Sigma = \text{id}$.

   (a) Baik, Ben Arous, Peche have solved this problem in the *complex* case when Σ is a finite rank perturbation of the identity. A key feature of their analysis is the use of the Harish-Chandra/Iyzykson-Zuber integral. It is an important remark that their results are expressible in terms of the basic Painlevé II function q.

   (b) The difficulty in the *real* case is lack of an analog to the HCIZ integral. This is a fundamental problem.

Thank You for your Attention