Integrable Differential Equations in Random Matrix Theory

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From Painlevé to Okamoto

June 9–13, 2008

The University of Tokyo

Determinants, Integrable DEs & RMM

1. Historical remarks
2. Random Matrix Models (RMM) with unitary symmetry
3. RMM with orthogonal symmetry
4. Wishart distributions
5. RMM and Extended Kernels
6. Extension of Largest Eigenvalue Distributions to general β—Dyson's Coulomb gas ensemble.

Figure 1: E. Ivar Fredholm, 1866–1927.

Figure 2: Paul Painlevé, 1863–1933.

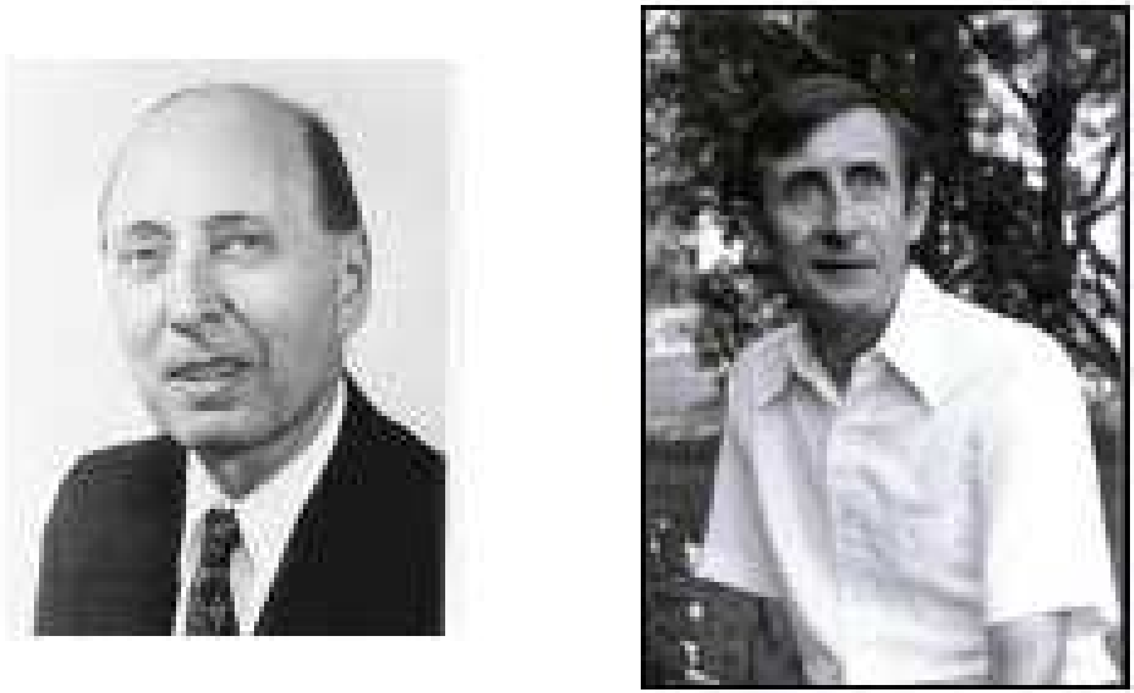

Figure 3: Eugene Wigner & Freeman Dyson

Figure 4: Tai Tsun Wu

Figure 5: Mikio Sato, Tetsuji Miwa & Michio Jimbo

Figure 6: Kazuo Okamoto

§1. Historical Remarks

2D Ising Model: First connection between

Toeplitz and Fredholm Dets $\rightsquigarrow$ Painlevé

Wu, McCoy, T, & Barouch (1973–77) [70]:

$$\lim_{\substack{T \to T_c^{\pm}, R^2 = M^2 + N^2 \to \infty \\ r = R/\xi(T) \text{fixed}}} \mathbb{E}\left(\sigma_{00}\sigma_{MN}\right) = \left\{ \begin{array}{c} \sinh \frac{1}{2}\psi(r) \\ \cosh \frac{1}{2}\psi(r) \end{array} \right\} \times$$

$$\exp\left(-\frac{1}{4}\int_r^\infty \left(\frac{d\psi}{dy}\right)^2 - \sinh^2\psi(y)\, dy\right)$$

where

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = \frac{1}{2}\sinh(2\psi), \;\; \psi(r) \sim \frac{2}{\pi} K_0(r), \;\; x \to \infty.$$

Note: $y(x) = \mathrm{e}^{-\psi(x)}$ is a particular **Painlevé III** transcendent. (See also Widom [69].)

Sato, Miwa & Jimbo, 1977–1980, (see, e.g., [33]) introduced

τ-functions and holonomic quantum fields,

a class of field theories that include the scaling limit of the Ising model and for which the expression of correlation functions in terms of solutions to holonomic differential equations is a general feature[a]

These developments led Jimbo-Miwa-Môri-Sato [36] to consider, in 1980, the *Fredholm determinant* and *Fredholm minors* of the operator whose kernel is the familiar **sine kernel**

$$\frac{1}{\pi}\,\frac{\sin \pi(x-y)}{x-y}$$

on the domain $\mathbb{J} = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n)$.

Their main interest was the *density matrix of the impenetrable Bose gas*, and only incidentally, random matrices.

[a] A **book length** account of these developments can be found in Palmer [51].

For $\mathbb{J} = (0, s)$, the **JMMS result** is

$$\det\left(I - \lambda K_{\text{sine}}\right) = \exp\left(-\int_0^{\pi s} \frac{\sigma(x;\lambda)}{x}\, dx\right)$$

where

$$(x\sigma'')^2 + 4(x\sigma' - \sigma)\left(x\sigma' - \sigma + (\sigma')^2\right) = 0$$

with boundary condition

$$\sigma(x, \lambda) = -\frac{\lambda}{\pi}\, x + \mathrm{O}(x^2), \;\; x \to 0.$$

- σ is expressible in terms of Painlevé V. An example of the *σ-form for Painlevé equations* [35, 45].
- Okamoto analyzed the τ-function associated to Painlevé equations [45] and produced his famous series of papers *Studies on Painlevé equations* [46, 47, 48, 49].

- For general $\mathbb{J}$, JMMS [36] obtains a compatible system of nonautonomous Hamiltonian equations generated by Poisson commuting Hamiltonians where the independent variables are the a_j, b_j—i.e. the endpoints of the intervals. (See also Harnad [28].)
- A simplified derivation of the JMMS equations can be found in TW [59].[a]
- Connections with quantum inverse scattering were developed by Its, Korepin and others. (See, e.g., [31, 41].)

[a] See Gangardt [27] for recent developments on the impenetrable Bose gas.

§2. RMM with Unitary Symmetry

Many RMM with unitary symmetry come down to the evaluation of Fredholm determinants $\det(I - \lambda K)$ where K has kernel of the form

$$\frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}\, \chi_{\mathbb{J}}(y)$$

where

$$\mathbb{J} = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n).$$

Examples:

- Sine kernel: $\varphi(x) = \sin \pi x$, $\psi(x) = \cos \pi x$.
- Airy kernel: $\varphi(x) = \mathrm{Ai}(x)$, $\psi(x) = \mathrm{Ai}'(x)$.
- Bessel kernel: $\varphi(x) = J_\alpha(\sqrt{x})$, $\psi(x) = x\varphi'(x)$.
- Hermite kernel: $\varphi(x) = (\frac{N}{2})^{1/4}\varphi_N(x), \psi(x) = (\frac{N}{2})^{1/4}\varphi_{N-1}(x)$ where $\varphi_k(x) = \frac{1}{\sqrt{2^k\, k!\pi^{1/2}}}\, \mathrm{e}^{-x^2/2}\, H_k(x)$.

A **general theory** of such Fredholm determinants was developed in TW [61] under the additional hypothesis that

$$m(x)\frac{d}{dx}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}A(x) & B(x)\\ -C(x) & -A(x)\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

where m, A, B and C are polynomials. For example, for the Airy kernel

$$m(x)=1,\ A(x)=0,\ B(x)=1,\ C(x)=-x.$$

The **basic objects** of the theory are

$$Q_j(x;\mathbb{J}) = (I-K)^{-1}\, x^j\varphi(x),\ \ P_j(x;\mathbb{J}) = (I-K)^{-1}\, x^j\psi(x),$$

and

$$u_j = (Q_j,\varphi)\,, v_j = (P_j,\varphi)\,, \tilde{v}_j = (Q_j,\psi)\,, w_j = (P_j,\psi)$$

where $(\cdot,\cdot)$ denotes the inner product. The **independent variables** are the endpoints a_j and b_j making up $\mathbb{J}$.

There are **two types of differential equations**:

- Universal equations, i.e. equations that hold *independently* of the differential equations for φ and ψ.
- Equations that depend upon m, A, B and C.

For $K = K_{\mathrm{Airy}}$ with $\mathbb{J} = (s, \infty)$, $q(s) := Q_0(s, \mathbb{J})$, $p(s) := P_0(s, \mathbb{J})$, $u = u_0$, $v = v_0$, the general theory reduces to the differential equations

$$\begin{aligned} \frac{dq}{ds} &= p - qu, \quad \frac{dp}{ds} = sq - 2qv + pu, \\ \frac{du}{ds} &= -q^2, \qquad \frac{dv}{ds} = -pq, \end{aligned}$$

together with

$$\frac{d}{ds} R(s,s) = -q^2, \quad \frac{d}{ds} \log\det(I - K) = -R(s,s),$$

where $R(x, y)$ is the resolvent kernel of K.

Using the first integral $u^2 - 2v = q^2$, one easily derives that q satisfies the **Painlevé II equation**

$$\frac{d^2 q}{ds^2} = sq + 2q^3.$$

This then leads to the distribution of the largest eigenvalue in GUE in the *edge scaling limit*[60]

$$F_2(x) = \exp\left(-\int_x^\infty (x-y)q(y)^2\,dy\right)$$

- **Key features** of the proof are simple expressions for $(m(x) \equiv 1)$

  $$[D, (I-K)^{-1}] \quad \text{and} \quad [M^k, (I-K)^{-1}]$$

  where D is differentiation with respect to the independent variable and M is multiplication by the independent variable.

For example, when $K = K_{\text{Airy}}$ with $\mathbb{J} = (s, \infty)$ we have

$$[D, (I - K)^{-1}] \doteq -Q(x)Q(y) + R(x, s)\rho(s, y)$$

- PALMER [50] and HARNAD & ITS [29] have an **isomondromic deformation** approach to these type of kernels.
- ADLER, SHIOTA, & VAN MOERBEKE'S [1] Virasoro algebra approach gives directly equations for the resolvent kernel $R(s, s)$.[a]
- Given the DE, e.g. P_{II}, one is faced with the **asymptotic analysis of the solutions** which involves finding **connection formulae**. One approach, the **RH method**, which has its origins in the **isomondromy deformation** methods of FLASCHKA & NEWELL [24] and JIMBO, MIWA & UENO [34] in 1980s, began with the work of DEIFT & ZHOU [15] when they

[a] The connection between these two approaches has been clarified by HARNAD [28] and RUMANOV [56].

proposed a nonlinear version of the classical steepest descent method for oscillatory RH problems. .

- A recent achievement [14] of the RH approach is a proof that as $s \to -\infty$,

$$\log\det(I - K_{\mathrm{Airy}}) = -\frac{s^3}{12} - \frac{1}{8}\log s + \kappa + \mathrm{O}(s^{-3/2})$$

where

$$\kappa = \frac{1}{24}\log 2 + \zeta'(-1)$$

and $\zeta(s)$ is the Riemann zeta function.

Remark: The first two terms follows from the Hastings-McLeod [30] solution of P_{II} [60]. The constant κ was conjectured in 1994 [60] and proved in 2006 [14].

- Choup [9] has given explicit Painlevé representations for **corrections to edge scaling** for both finite n GUE and LUE.

§3. RMM with Orthogonal Symmetry

The added difficulty with RMM with orthogonal symmetry is that the kernels are **matrix kernels** [21, 43, 62, 64, 68]. For example, for finite N GOE the operator is

$$K_1 = \chi \begin{pmatrix} K_2 + \psi \otimes \varepsilon\varphi & K_2 D - \psi \otimes \varphi \\ \varepsilon K_2 - \varepsilon + \varepsilon\psi \otimes \varepsilon\varphi & K_2 + \varepsilon\varphi \otimes \psi \end{pmatrix} \chi$$

where

$$K_2 \doteq \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y),$$

ε is the operator with kernel $\frac{1}{2}\mathrm{sgn}(x-y)$, D is the differentiation operator, and χ is the indicator function for the domain $\mathbb{J}$.
Notation: $A \otimes B \doteq A(x)B(y)$.

The idea of the proof in TW [62] is to factor out the GUE part

$$(I - K_2\chi)$$

and through various determinant manipulations show that the **remaining part is a finite rank perturbation**. Thus one ends up with formulas like

$$\det(I - K_1) = \det(I - K_2\chi)\,\det\left(I - \sum_{j=1}^{k} \alpha_j \otimes \beta_j\right)$$

For the case $\mathbb{J} = (s, \infty)$, an asymptotic analysis shows that as $N \to \infty$ the distribution of the scaled largest eigenvalue in GOE is expressible in terms of the **same** P_{II} function appearing in GUE. The resulting GOE and GSE largest eigenvalue distribution functions are

$$
\begin{aligned}
F_1(x) &= \exp\left(-\frac{1}{2}\int_x^\infty q(y)\,dy\right)(F_2(x))^{1/2} \\
F_4(x/\sqrt{2}) &= \cosh\left(\frac{1}{2}\int_x^\infty q(y)\,dy\right)(F_2(x))^{1/2}
\end{aligned}
$$

where

$$F_2(x) = \exp\left(-\int_x^\infty (x-y)q(y)^2\,dy\right)$$

and q is the Hastings-McLeod solution of P_{II}.

- The **edge scaling limit** is more **subtle for GOE** than for GUE or GSE. For GUE and GSE we have **convergence in trace norm** to limiting operators $K_{2,\text{Airy}}$ and $K_{4,\text{Airy}}$, but for GOE the **convergence is to a regularized determinant**, i.e. $\det_2$. The subtleness is due to the presence of the ε. The lack of trace norm convergence basically explains why the limit $N \to \infty$ was taken at the end in [62]. The pointwise limit of finite N K_1 was worked out by Ferrari [22] and by Forrester, Nagao & Honner [25]. The convergence at the operator level is in TW [66].

- Recently Ferrari & Spohn [23] gave a different determinantal expression for edge scaling in GOE. It would be interesting to explore further their approach.

UNIVERSALITY THEOREMS

Though not a part of this survey proper, it's important to mention that these *same* distribution functions (and hence integrable DEs) arise for a much wider class of models than the Gaussian cases discussed here.

Invariant Measures: $\mathrm{e}^{-\mathrm{Tr}(A^2)} \longrightarrow \mathrm{e}^{-N\mathrm{Tr}(V(A))}$

Unitary:

- BLEHER & ITS [8], $V(x) = \frac{1}{2}tx^2 + \frac{1}{4}gx^4$, $g > 0, t < 0$.
- DEIFT, KRIECHERBAUER, MCLAUGHLIN, VENAKIDES, ZHOU [12, 13], V real analytic and $V/\log|x| \to +\infty$ as $|x| \to \infty$.

Orthogonal & Symplectic: DEIFT & GIOEV [11], poly. V

Noninvariant Measures:

- SOSHNIKOV [57], Real symmetric and complex Hermitian Wigner matrices.

Next Largest, Next-Next Largest, ... Eigenvalue Distributions

$$\mathbf{D}_\beta(\mathbf{s},\lambda) := \mathbf{det}\,(\mathbf{I} - \lambda \mathbf{K}_{\beta,\mathrm{Airy}}),\ \ \beta = 1,2,4,\ 0 \le \lambda \le 1.$$

($\det_2$ for $\beta = 1$.) One needs

$$\left.\frac{\partial^j D_\beta(s,\lambda)}{\partial \lambda^j}\right|_{\lambda=1}$$

for next largest, next-next largest eigenvalue, etc. distributions.

For $\beta = 2,4$ there is a **simple answer**: Let

$$q(x) \longrightarrow q(x,\lambda)$$

in $\lambda = 1$ distributions where now q satisfies **same Painlevé II** equation but with boundary condition

$$q(x,\lambda) \sim \sqrt{\lambda}\,\mathrm{Ai}(x), \quad x \to \infty.$$

Not So for Orthogonal Symmetry!

DIENG [18] proved (see also [19])

$$D_1(s,\lambda) = D_2(s,\tilde{\lambda})\,\frac{\lambda - 1 - \cosh\mu(s,\tilde{\lambda}) + \sqrt{\tilde{\lambda}}\sinh\mu(s,\lambda)}{\lambda - 2}$$

with

$$\mu(s,\lambda) = \int_s^\infty q(x,\lambda)\,dx \quad \text{and} \quad \tilde{\lambda} := 2\lambda - \lambda^2.$$

Note evaluation at $\tilde{\lambda}$ in above. For $\lambda = 1$ this reduces to TW [62].

From this follows distribution functions for next-largest, next-next largest, etc. for GOE universality class in terms of Painlevé II function q and derivatives

$$\left.\frac{\partial^k q(x,\lambda)}{\partial\lambda^k}\right|_{\lambda=1}$$

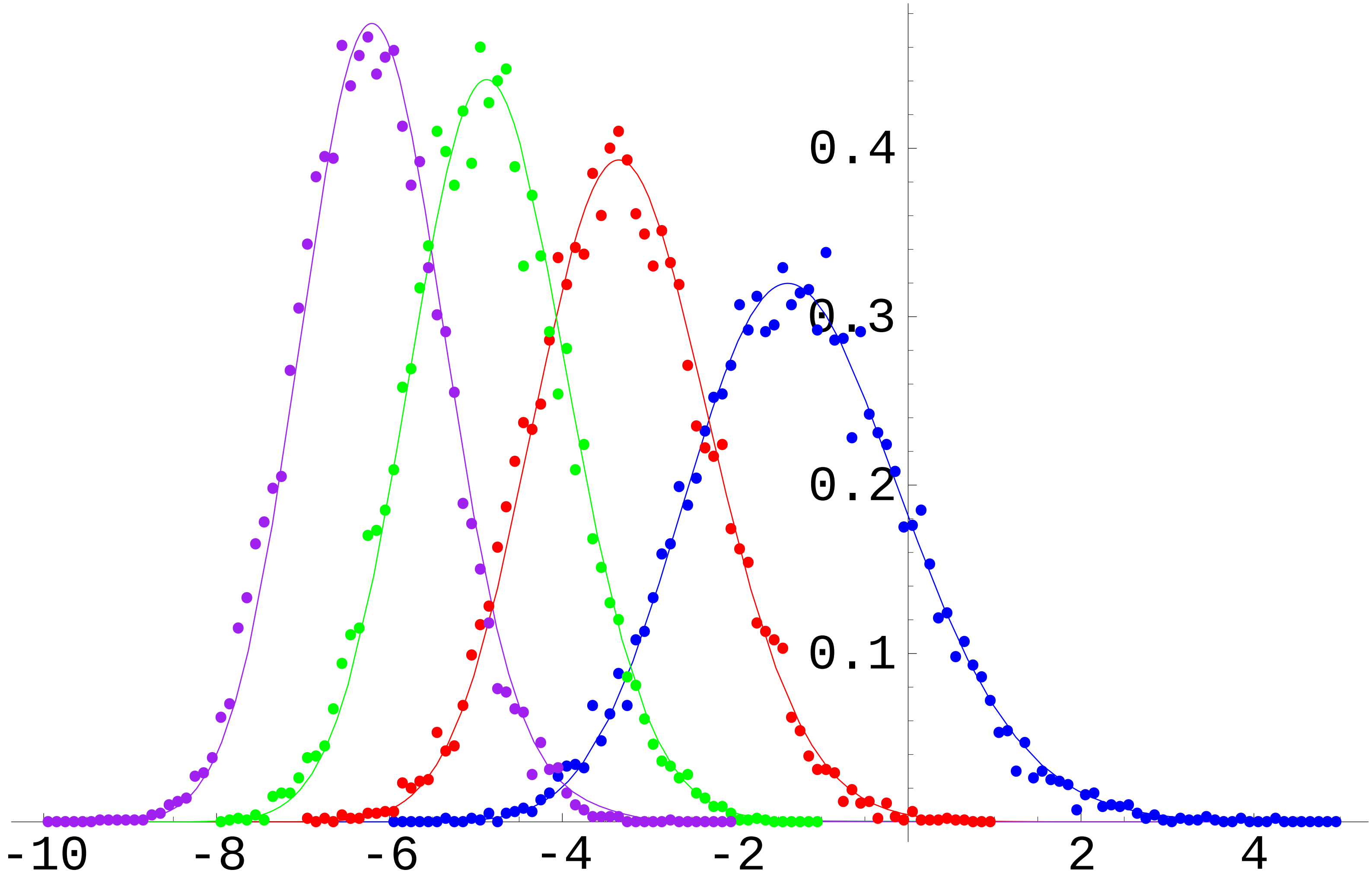

Figure 7: 10^4 realizations of $10^3 \times 10^3$ GOE matrices

§4. Wishart Distributions

If

$$A = X^T X$$

where the $n \times p$ matrix X is $N_p(0, I_n \otimes \Sigma)$, $\Sigma > 0$, then A is said to have **Wishart distribution** with n degrees of freedom and covariance Σ. The Wishart distribution [4, 44] is the multivariate generalization of the χ^2-distribution. We will say A is $W_p(n, \Sigma)$.

The quantity $\frac{1}{n}A$ is also called the **sample covariance matrix**.

Eigenvalues of a Wishart Matrix

Theorem: If A is $W_p(n, \Sigma)$, $n \geq p$, the **joint density function** for the eigenvalues $\ell_1, \ldots, \ell_p$ of A is

$$c_{p,n,\Sigma} \prod_{j=1}^{p} \ell_j^{(n-p-1)/2} \prod_{j<k} |\ell_j - \ell_k| \times$$

$$\int_{\mathcal{O}(p)} \mathrm{e}^{-\frac{1}{2}\,\mathrm{tr}(\Sigma^{-1} Q L Q^T)}\, dQ,$$

where $L = \mathrm{diag}(\ell_1, \ldots, \ell_p)$ and dQ is normalized Haar measure.

Corollary: If A is $W_p(n, I_p)$, then the integral over the orthogonal group in the previous theorem is

$$\mathrm{e}^{-\frac{1}{2}\sum_j \ell_j}\,.$$

- One is interested in **limit laws** as $n, p \to \infty$. For $\Sigma = I_p$, Johnstone [38] proved, using RMT methods, for centering and scaling constants

$$\mu_{np} = \left(\sqrt{n-1} + \sqrt{p}\right)^2,$$

$$\sigma_{np} = \left(\sqrt{n-1} + \sqrt{p}\right)\left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{p}}\right)^{1/3}$$

that

$$\frac{\ell_1 - \mu_{np}}{\sigma_{np}}$$

converges in distribution as $n, p \to \infty$, $n/p \to \gamma < \infty$, to the **GOE largest eigenvalue distribution** F_1.

- El Karoui [39] has extended the result to $\gamma \leq \infty$. The case $p \gg n$ appears, for example, in microarray data.

- Soshnikov [58] and Péché [53] have **removed the Gaussian assumption**.

- For $\Sigma \neq I_p$, the difficulty in establishing limit theorms comes from the integral

$$\int_{\mathcal{O}(p)} \mathrm{e}^{-\frac{1}{2}\,\mathrm{tr}(\Sigma^{-1} Q \Lambda Q^T)} \,(dQ).$$

Using **zonal polynomials**, infinite series expansions have been derived for this integral, but these expansions are difficult to analyze and converge slowly [44].

- For **complex Gaussian data matrices** X similar density formulas are known for the eigenvalues of X^*X. Limit theorems for $\Sigma \neq I_p$ are known since the analogous group integral, now over the unitary group, is known explicitly—the Harish Chandra–Itzykson–Zuber integral. See the work of Baik, Ben Arous & Péché [5, 6] and El Karoui [40].

The BBP phase transition.

- These RMT developments have had recent application to the analysis of **genetic data**; in particular, determining if the samples are from a homogeneous population [52].

§5. RMM & Extended Kernels

Airy Process: The Airy process $\mathcal{A}(\tau)$, introduced by Prähoffer & Spohn [54] and Johansson [37], is a continuous stochastic process whose distribution functions are given by

$$\mathbb{P}\left(\mathcal{A}(\tau_1) < a_1, \ldots, \mathcal{A}(\tau_m) < a_m\right) = \det\left(I - K\right)$$

for $\tau_1 < \cdots < \tau_m$. Here K is the operator with $m \times m$ matrix kernel (K_{ij}) where

$$K_{ij}(x, y) = L_{ij}(x, y)\chi_{(a_j,\infty)}(y)$$

$$L_{ij}(x,y) = \begin{cases} \int_0^\infty \mathrm{e}^{-z(\tau_i - \tau_j)} \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\, dz & i \ge j, \\ -\int_{-\infty}^0 \mathrm{e}^{-z(\tau_i - \tau_j)} \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\, dx & i < j \end{cases}$$

For $m = 1$ this reduces to the Airy kernel (independent of τ).

Extended kernels are more difficult than "integrable kernels" in unitary ensembles. Nevertheless, it is possible to find (complicated!) systems of integrable differential equations: See Adler & van Moerbeke [2, 3] and TW [65, 67].

Much analysis remains to be done on these equations

§6. Dyson's Hermite β Ensemble

Dyson's *Hermite β ensemble* H_n^β is defined by the probability density

$$\mathbb{P}_\beta(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} e^{-\beta \sum_k \lambda_k^2/4} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

When $\beta = 1, 2$ or 4 this is the joint density of eigevaules in GOE, GUE and GSE, respectively. Building on earlier work of Dumitriu & Edelman[20], Ramírez, Rider & Virág [55] proved the following theorem:

Define the *stochastic Airy operator*

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}}\, b'_x$$

where b'_x is white noise.

THEOREM[55]: With probability one, for each $k \geq 0$ the set of eigenvalues of $\mathcal{H}_\beta$ has a well-defined $(k+1)$st lowest element $\Lambda_k(\beta)$. Moreover, let $\lambda_{\beta,1} \geq \lambda_{\beta,2} \geq \cdots$ denote the eigenvalues of the Hermite β-ensemble H_n^β. Then the vector

$$\left(n^{1/6}(2\sqrt{n} - \lambda_{\beta,\ell})\right)_{\ell=1,\ldots,k}$$

converges in distribution as $n \to \infty$ to $(\Lambda_0(\beta), \Lambda_1(\beta), \ldots, \Lambda_{k-1}(\beta))$.

Thus

$$F_\beta(x) = \mathbb{P}\left(-\Lambda_0(\beta) < x\right), \beta > 0$$

generalizes the largest eigenvalue distributions to general β

OPEN PROBLEM: For $\beta = 1, 2$ and 4 we can express F_β in terms of a solution to P_{II}. Is there a corresponding relation to integrable differential equations for general β?

To Professor Okamoto

Happy 60th Birthday!

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