THE COHOMOLOGY OF THE THIRD FINITE SUBSET SPACE OF A CLOSED ORIENTABLE SURFACE

CHRISTOPHER TUFFLEY

Abstract. The kth finite subset space of a topological space X is the space $\exp_k X$ of nonempty finite subsets of X of size at most k, topologised as a quotient of X^k. The construction co-incides with the symmetric product $\mathrm{Sym}^k X = X^k/S_k$ for $k = 2$ but is a proper quotient of the symmetric product for $k \geq 3$. We use Mayer-Vietoris arguments and the ring structure of $H^*(\mathrm{Sym}^k \Sigma)$ to calculate the integer cohomology groups of the third finite subset space of a closed orientable surface Σ.

1. Introduction

The kth finite subset space of a topological space X is the space $\exp_k X$ of nonempty subsets of X of size at most k, topologised as a quotient of X^k via the map

$$(x_1, \ldots, x_k) \mapsto \{x_1\} \cup \cdots \cup \{x_k\}.$$

The first finite subset space is of course simply X, and the second finite subset space co-incides with the second symmetric product $\mathrm{Sym}^2 X = X^2/S_2$, but for $k \geq 3$ we have the proper quotient of $\mathrm{Sym}^k X = X^k/S_k$ obtained by forgetting multiplicities: both (a, a, b) and (a, b, b) in X^3 map to $\{a, b\}$ in $\exp_3 X$.

In our previous papers [9, 10] we studied the finite subset spaces of the circle, connected graphs, and punctured surfaces. The purpose of this note is to calculate the cohomology groups of the third finite subset space of a closed orientable surface, and our main result is the following:

Theorem 1. *Let Σ_g be a closed orientable surface of genus g. The cohomology group $H^i(\exp_3 \Sigma_g; \mathbf{Z})$ is given by*

		g		
		0	1	≥ 2
	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
	1	0	0	0
	2	0	$\mathbf{Z}$	$\mathbf{Z}^{\binom{2g}{2}}$
i	3	0	$\mathbf{Z}^5$	$\mathbf{Z}^{p(g)}$
	4	$\mathbf{Z}$	$\mathbf{Z}^4$	$\mathbf{Z}^{q(g)} \oplus [\mathbf{Z}/2\mathbf{Z}]^{2g}$
	5	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}^2 \oplus \mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}^{2g} \oplus \mathbf{Z}/2\mathbf{Z}$
	6	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$

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in which

$$p(g) = \binom{2g}{3} + \binom{2g}{2} + \binom{2g}{1}, \qquad q(g) = \binom{2g}{2} + 1$$

for $g \geq 2$*. The Euler characteristic is*

$$\chi(\exp_3 \Sigma_g) = \frac{-4g^3 + 12g^2 - 17g + 9}{3}.$$

In particular $\chi(\exp_3 S^2) = 3$*,* $\chi(\exp_3 T^2) = 0$*, and* $\chi(\exp_3 \Sigma_2) = -3$*.*

We will prove Theorem 1 using quite different techniques from those of [9, 10], in which we used explicit cell structures to study the finite subset spaces of a connected graph Γ. A key step in building the cell structures was ordering each subset of Γ in order to choose a preferred lift from $\exp_k \Gamma$ to Γ^k. Ordering each subset made use of the fact that a graph is one dimensional, but it appears that with some work a similar idea may apply to 2–complexes, using the lexicographic ordering on I^2. However, we do not pursue this approach here. Instead, we build a homotopy model for $\exp_3 \Sigma$ out of $\mathrm{Sym}^3 \Sigma$ and the mapping cylinder of $\Sigma^2 \to \mathrm{Sym}^2 \Sigma$, and use Mayer-Vietoris arguments and the calculation of $H^*(\mathrm{Sym}^k \Sigma)$ due to Macdonald [6] and Seroul [7, 8].

We adopt the convention that where a co-efficient group is not specified integer co-efficients should be assumed.

2. A homotopy model for the third finite subset space of a surface

2.1. **The model.** To construct a homotopy model for $\exp_3 \Sigma$ we begin with the intermediate quotient $\mathrm{Sym}^3 \Sigma = \Sigma^3 / S_3$. Inside $\mathrm{Sym}^3 \Sigma$ we have the preimage of $\exp_2 \Sigma$, namely the branch locus $\mathcal{D}$ consisting of the quotient of the diagonals of $\Sigma \times \Sigma \times \Sigma$. Instead of quotienting $\mathcal{D}$ further to get $\exp_3 \Sigma$ we attach the mapping cylinder of $\mathcal{D} \to \exp_2 \Sigma$ to obtain a homotopy equivalent space $E_3 \Sigma$. Fortunately $\mathcal{D}$ is simply a copy of $\Sigma \times \Sigma$ and $\mathcal{D} \to \exp_2 \Sigma$ is the quotient map $\Sigma \times \Sigma \to \mathrm{Sym}^2 \Sigma$, so the construction is all in terms of known spaces and maps.

Concretely, let $q_k \colon \Sigma^k \to \mathrm{Sym}^k \Sigma$ be the quotient map and $\Delta \colon \Sigma \to \Sigma \times \Sigma$ the diagonal map. Then $\mathcal{D}$ is the image of $\Sigma \times \Sigma$ under the map $\iota = q_3 \circ \mathrm{id} \times \Delta$, and is homeomorphic to $\Sigma \times \Sigma$ since ι is injective, $\Sigma \times \Sigma$ is compact, and $\mathrm{Sym}^3 \Sigma$ is Hausdorff. Further, $\exp_3 \Sigma$ is obtained from $\mathrm{Sym}^3 \Sigma$ by identifying $\iota(a, b) = q_3(a, b, b)$ with $\iota(b, a) = q_3(b, a, a)$. Let

$$M_{q_2} = \frac{\Sigma \times \Sigma \times I \amalg \mathrm{Sym}^2 \Sigma}{(x, 1) \sim q_2(x)}$$

be the mapping cylinder of q_2, and note that we adopt the convention that mapping cylinders are attached to the target at $1 \in I$ rather than 0. We obtain our model for $\exp_3 \Sigma$ by attaching M_{q_2} to $\mathrm{Sym}^3 \Sigma$ along $\mathcal{D}$ and $\Sigma \times \Sigma \times \{0\}$, namely

$$\begin{aligned} E_3 \Sigma &= \mathrm{Sym}^3 \Sigma \cup_{\Sigma \times \Sigma} M_{q_2} \\ &= \frac{\mathrm{Sym}^3 \Sigma \amalg M_{q_2}}{\iota(x) \sim (x, 0)}. \end{aligned}$$

The space $E_3 \Sigma$ is shown schematically in figure 1. The subscript 3 reflects the hope that a similar construction may apply for $k \geq 4$, perhaps using several mapping cylinders to successively quotient $\mathrm{Sym}^k \Sigma$ to $\exp_k \Sigma$ in several stages. However,

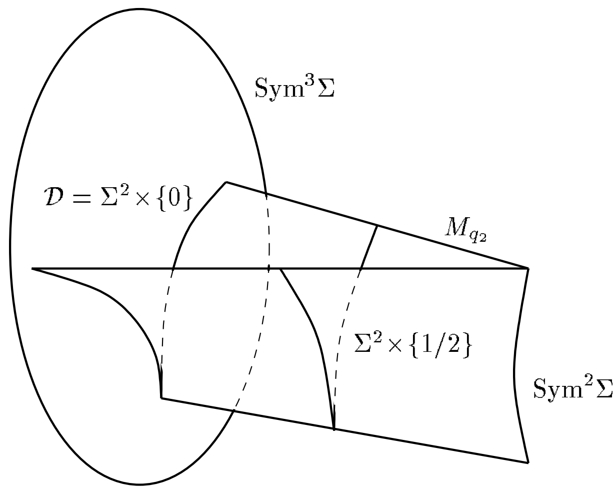

FIGURE 1. A schematic picture of the space $E_3\Sigma$. As we shall see in section 2.2, $\mathrm{Sym}^3\Sigma$ is a complex 3–manifold with the branch locus $\mathcal{D} \cong \Sigma^2$ embedded with a cusp along the diagonal. Σ^2 maps two-to-one to $\mathrm{Sym}^2\Sigma$ via q_2, and we form $E_3\Sigma$ by attaching the mapping cylinder M_{q_2} to $\mathcal{D}$ along $\Sigma^2 \times \{0\}$.

such a generalisation is complicated by the increasing complexity of the branch locus: for example, when $k = 4$ it is $\Sigma \times \mathrm{Sym}^2\Sigma$ with an embedded copy of $\Sigma \times \Sigma$ quotiented to $\mathrm{Sym}^2\Sigma$.

The cornerstone of this paper is the following lemma:

Lemma 1. *The spaces $E_3\Sigma$ and $\exp_3\Sigma$ are homotopy equivalent.*

We prove the lemma after a brief digression on known facts about symmetric products of surfaces. Not all of what follows is central to our argument, but it nonetheless serves to give a fuller picture of the construction.

2.2. **Symmetric products of surfaces.** We recall that the symmetric product of an orientable surface is a manifold, and moreover that a complex structure on Σ leads to a complex structure on $\mathrm{Sym}^k\Sigma$. Local co-ordinates about $\{p_1, \ldots, p_k\}$ are given by the elementary symmetric functions in the local complex co-ordinates about the p_i, and are obtained by regarding the points as the zeroes of a polynomial (see Griffiths and Harris [3, p. 326]). In particular $\mathrm{Sym}^n \mathbf{C}P^1$ may be identified with the nonvanishing homogeneous polynomials of degree n in two variables, modulo scaling, and as such is equal to $\mathbf{C}P^n$.

A polynomial p has repeated roots if and only if its discriminant is zero. The discriminant is a polynomial in the co-efficients of p (see Lang [5, pp. 192–194]) and it follows that the branch locus, the image of the diagonals of Σ^k, is locally given by the vanishing of a polynomial and is therefore an algebraic variety. Specialising to $k = 3$, for a suitable choice of complex co-ordinates (a, b, c) the branch locus $\mathcal{D}$ is locally the set $b^2 = c^3$. We see that $\mathcal{D}$ has a cusp along $b = c = 0$, which is precisely the image of the main diagonal in $\Sigma \times \Sigma \times \Sigma$.

2.3. **The proof of Lemma 1.** Consider the mapping cylinder of $q\colon \mathrm{Sym}^3\Sigma \to \exp_3\Sigma$. This deformation retracts to $\exp_3\Sigma$ and contains $E_3\Sigma$ as a subspace, and our aim is to show that it also deformation retracts to $E_3\Sigma$. To do this it suffices to show that $\mathrm{Sym}^3\Sigma \times I$ deformation retracts to $\mathrm{Sym}^3\Sigma \times \{0\} \cup \mathcal{D} \times I$, as such a homotopy will descend setwise to the quotient M_q and be continuous there.

The existence of a deformation retraction from $\mathrm{Sym}^3\Sigma\times I$ to $\mathrm{Sym}^3\Sigma\times\{0\}\cup\mathcal{D}\times I$ follows from results in Bredon [1, pp. 431–432] and Dugundji [2, pp. 327–328] and the existence of a neighbourhood $U \supseteq \mathcal{D}$ that strongly deforms to $\mathcal{D}$ in $\mathrm{Sym}^3\Sigma$. We will prove it in this way, using the fact that $\mathrm{Sym}^3\Sigma$ and $\mathcal{D}$ are both compact manifolds to construct U. However, an approach of perhaps greater generality might be to realise $\mathcal{D}$ as a subcomplex of $\mathrm{Sym}^3\Sigma$ and appeal to Hatcher [4, Prop. 0.16]. Hatcher [4, pp. 482–483] gives a construction of an S_k–equivariant simplicial structure on X^k for simplicial X, and Lemma 1 would follow from checking whether this contains the diagonals as a subcomplex.

Although $\mathrm{Sym}^3\Sigma$ and $\mathcal{D}$ are both manifolds the existence of the desired neighbourhood U does not simply follow from the tubular neighbourhood theorem, since $\mathcal{D}$ is not smoothly embedded. We therefore resort to more hands-on means, and use the fact that manifolds are Euclidean neighbourhood retracts. Embed $M = \mathrm{Sym}^3\Sigma$ in some $\mathbf{R}^n$. Then there are neighbourhoods V of M, W of $\mathcal{D}$, and retractions $r_M\colon V \to M$, $r_W\colon W \to \mathcal{D}$. W may be taken sufficiently small that the linear homotopy from $W \hookrightarrow \mathbf{R}^n$ to r_W remains in V, and we post-compose this with r_M and intersect W with M to get the desired neighbourhood and deformation.

3. The calculation of $H^*(\exp_3\Sigma;\mathbf{Z})$

3.1. **Introduction.** To calculate the cohomology of $E_3\Sigma$ we use the Mayer-Vietoris sequence and the obvious decomposition

$$E_3\Sigma = (E_3\Sigma \setminus \mathrm{Sym}^2\Sigma) \cup (E_3\Sigma \setminus \mathrm{Sym}^3\Sigma).$$

The pieces are homotopy equivalent to $\mathrm{Sym}^3\Sigma$ and $\mathrm{Sym}^2\Sigma$ respectively, and intersect in $\Sigma\times\Sigma\times(0,1) \simeq \Sigma\times\Sigma$, leading to a long exact sequence

$$\cdots \to H^i(E_3\Sigma) \to H^i(\mathrm{Sym}^3\Sigma)\oplus H^i(\mathrm{Sym}^2\Sigma) \to H^i(\Sigma^2) \to H^{i+1}(E_3\Sigma) \to \cdots.$$

Before proceeding we describe the rings $H^*(\Sigma^k)$ and $H^*(\mathrm{Sym}^k\Sigma)$. Recall that integer co-efficients are to be assumed except where specified otherwise.

3.2. **The rings $H^*(\Sigma^k)$ and $H^*(\mathrm{Sym}^k\Sigma)$.** Let $\alpha_1,\ldots,\alpha_{2g}$ be generators for $H^1(\Sigma)$ such that

$$\alpha_i\alpha_j = \begin{cases} 0 & |i-j| \neq g, \\ \beta & j = i+g, \end{cases}$$

where β is the generator of $H^2(\Sigma)$ coming from the orientation of Σ. Since $H^*(\Sigma)$ is finitely generated and free the Künneth formula applies and

$$H^*(\Sigma^k) \cong H^*(\Sigma)^{\otimes k}.$$

The cohomology ring of $\mathrm{Sym}^k\Sigma$ has been calculated by Macdonald [6] and Seroul [7]. In addition Seroul's paper [8] gives a sketch of his argument. Macdonald uses methods from algebraic geometry to give generators and relations for $H^*(\mathrm{Sym}^k\Sigma;K)$ over a field K of characteristic zero, and to show that $H^*(\mathrm{Sym}^k\Sigma)$

is torsion free. He then states incorrectly that this implies the same elements generate over the integers. Seroul confirms Macdonald's answer, using purely algebraic-topological techniques to find $H^*(\mathrm{Sym}^k\Sigma;\mathbf{Z})$ directly. In part the result may be stated as follows; we omit the statement of the relations as we will do all ring multiplication in $H^*(\mathrm{Sym}^k\Sigma)$.

Theorem 2 (Macdonald [6] and Seroul [7, 8])**.** *The map*

$$q_k^*\colon\ H^*(\mathrm{Sym}^k\Sigma;R)\to H^*(\Sigma^k;R)$$

is an isomorphism of $H^*(\mathrm{Sym}^k\Sigma;R)$ onto $H^*(\Sigma^k;R)^{S_k}$, the subring of cohomology fixed by S_k, for R a field of characteristic zero, and is injective for $R=\mathbf{Z}$. $H^*(\mathrm{Sym}^k\Sigma;\mathbf{Z})$ is generated by elements $\xi_1,\dots,\xi_{2g}$ in degree 1 and η in degree 2 such that

$$q_k^*\xi_i=\sum_{j=1}^{k}\pi_j^*\alpha_i, \qquad\qquad q_k^*\eta=\sum_{j=1}^{k}\pi_j^*\beta,$$

where $\pi_j\colon\ \Sigma^k\to\Sigma$ is projection on the jth factor. A basis for $H^r(\mathrm{Sym}^k\Sigma;\mathbf{Z})$ is given by the monomials $\xi_{i_1}\cdots\xi_{i_m}\eta^n$ for which $m+2n=r$, $i_1<\cdots<i_m$, and $m\le\min\{r,2k-r\}$.

We remark that q_k is a degree $k!$ map, so q_k^* is certainly not onto $H^*(\Sigma^k)^{S_k}$ with integer co-efficients.

To avoid confusion we give different names to the generators of $H^*(\mathrm{Sym}^2\Sigma)$ and $H^*(\mathrm{Sym}^3\Sigma)$. Let

$$\begin{aligned}\zeta_i&=\alpha_i\otimes1+1\otimes\alpha_i,\\ \theta&=\beta\otimes1+1\otimes\beta,\\ \xi_i&=\alpha_i\otimes1\otimes1+1\otimes\alpha_i\otimes1+1\otimes1\otimes\alpha_i,\\ \eta&=\beta\otimes1\otimes1+1\otimes\beta\otimes1+1\otimes1\otimes\beta.\end{aligned}$$

Since q_k^* is injective we shall abuse notation and not take care to distinguish between elements of $H^*(\mathrm{Sym}^k\Sigma)$ and their images in $H^*(\Sigma^k)$, and will regard the ζ_i and θ as generators of $H^*(\mathrm{Sym}^2\Sigma)$, and the ξ_i and η as generators of $H^*(\mathrm{Sym}^3\Sigma)$.

3.3. **The cohomology calculation.** Returning to the Mayer-Vietoris sequence, letting

$$\Phi_i\colon\ H^i(\mathrm{Sym}^3\Sigma)\oplus H^i(\mathrm{Sym}^2\Sigma)\to H^i(\Sigma\times\Sigma)$$

be the map $\iota^*\oplus q_2^*=(q_3\circ\mathrm{id}\times\Delta)^*\oplus q_2^*$ we have the short exact sequence

$$0\to\operatorname{coker}\Phi_{i-1}\to H^i(E_3\Sigma)\to\ker\Phi_i\to0.$$

Since $H^i(\mathrm{Sym}^3\Sigma)\oplus H^i(\mathrm{Sym}^2\Sigma)$ is free the kernel of Φ_i is too, so the sequence splits and we get

$$H^i(E_3\Sigma)\cong\operatorname{coker}\Phi_{i-1}\oplus\ker\Phi_i. \tag{1}$$

In what follows we calculate the kernel and cokernel of each Φ_i.

Dimension one. Both $H^1(\mathrm{Sym}^3\Sigma)\oplus H^1(\mathrm{Sym}^2\Sigma)$ and $H^1(\Sigma\times\Sigma)$ have rank $4g$, with bases $\{\xi_i\}\cup\{\zeta_i\}$ and $\{\alpha_i\otimes 1\}\cup\{1\otimes\alpha_i\}$ respectively. For $\alpha\in H^j(\Sigma)$ we have

$$\begin{aligned}(\mathrm{id}\times\Delta)^*(1\otimes 1\otimes\alpha) &= (\mathrm{id}\times\Delta)^*\pi_3^*\alpha\\ &= (\pi_3\circ\mathrm{id}\times\Delta)^*\alpha\\ &= \pi_2^*\alpha = 1\otimes\alpha,\end{aligned}$$

and similarly $(\mathrm{id}\times\Delta)^*(1\otimes\alpha\otimes 1)=1\otimes\alpha$, $(\mathrm{id}\times\Delta)^*(\alpha\otimes 1\otimes 1)=\alpha\otimes 1$. Consequently

$$\begin{aligned}\Phi_1(\xi_i) &= (\mathrm{id}\times\Delta)^*(\alpha_i\otimes 1\otimes 1+1\otimes\alpha_i\otimes 1+1\otimes 1\otimes\alpha_i)\\ &= \alpha_i\otimes 1+2\otimes\alpha_i.\end{aligned}$$

Since $\Phi_1(\zeta_i)=q_2^*\zeta_i=\alpha_i\otimes 1+1\otimes\alpha_i$ and $\det\begin{bmatrix}1&1\\2&1\end{bmatrix}=-1$, Φ_1 maps $\mathrm{span}\{\xi_i,\zeta_i\}$ isomorphically onto $\mathrm{span}\{\alpha_i\otimes 1, 1\otimes\alpha_i\}$. Thus

$$\ker\Phi_1\cong\mathrm{coker}\,\Phi_1\cong\{0\}.$$

Dimension two. $H^2(\mathrm{Sym}^3\Sigma)\oplus H^2(\mathrm{Sym}^2\Sigma)$ has basis

$$\{\xi_i\xi_j|i<j\}\cup\{\zeta_i\zeta_j|i<j\}\cup\{\eta,\theta\}$$

and rank $2\binom{2g}{2}+2$, while $H^2(\Sigma\times\Sigma)$ has basis

$$\{\alpha_i\otimes\alpha_j\}\cup\{\beta\otimes 1, 1\otimes\beta\}$$

and rank $4g^2+2$. Under Φ_2 we have

$$\begin{aligned}\xi_i\xi_j &\mapsto (\alpha_i\otimes 1+2\otimes\alpha_i)(\alpha_j\otimes 1+2\otimes\alpha_j)\\ &= \begin{cases}2(\alpha_i\otimes\alpha_j-\alpha_j\otimes\alpha_i) & |i-j|\neq g,\\ 2(\alpha_i\otimes\alpha_j-\alpha_j\otimes\alpha_i)+\beta\otimes 1+4\otimes\beta & j=i+g,\end{cases}\\ \zeta_i\zeta_j &\mapsto (\alpha_i\otimes 1+1\otimes\alpha_i)(\alpha_j\otimes 1+1\otimes\alpha_j)\\ &= \begin{cases}\alpha_i\otimes\alpha_j-\alpha_j\otimes\alpha_i & |i-j|\neq g,\\ \alpha_i\otimes\alpha_j-\alpha_j\otimes\alpha_i+\beta\otimes 1+1\otimes\beta & j=i+g,\end{cases}\\ \eta &\mapsto \beta\otimes 1+2\otimes\beta,\\ \theta &\mapsto \beta\otimes 1+1\otimes\beta.\end{aligned}$$

Clearly the image of Φ_2 is the span of

$$\{\beta\otimes 1, 1\otimes\beta\}\cup\{\alpha_i\otimes\alpha_j-\alpha_j\otimes\alpha_i|i<j\},\tag{2}$$

a subspace of rank $\binom{2g}{2}+2$. Thus the kernel of Φ_2 has rank $\binom{2g}{2}$. The set in (2) may be augmented to a basis for $H^2(\Sigma\times\Sigma)$, so the cokernel of Φ_2 is free of rank $4g^2+2-\binom{2g}{2}-2=\binom{2g}{2}+2g$. Hence

$$\ker\Phi_2\cong\mathbf{Z}^{\binom{2g}{2}},\qquad\qquad \mathrm{coker}\,\Phi_2\cong\mathbf{Z}^{\binom{2g}{2}+2g}.$$

Dimension three. A basis for $H^3(\mathrm{Sym}^3\Sigma)\oplus H^3(\mathrm{Sym}^2\Sigma)$ is given by

$$\{\xi_i\xi_j\xi_k | i<j<k\}\cup\{\xi_i\eta\}\cup\{\zeta_i\theta\}.$$

If the genus of Σ is greater than one the rank is $\binom{2g}{3}+4g$, but in genus equal to one there are only two distinct ξ_i, so the leftmost set in this union is empty and the rank of $H^3(\mathrm{Sym}^3\Sigma)\oplus H^3(\mathrm{Sym}^2\Sigma)$ is $4g=4$. In either case $H^3(\Sigma\times\Sigma)$ has basis

$$\{\alpha_i\otimes\beta\}\cup\{\beta\otimes\alpha_i\}$$

and rank $4g$. We have

$$\begin{aligned}\zeta_i\theta &\mapsto (\alpha_i\otimes 1+1\otimes\alpha_i)(\beta\otimes 1+1\otimes\beta)\\ &=\alpha_i\otimes\beta+\beta\otimes\alpha_i,\\ \xi_i\eta &\mapsto (\alpha_i\otimes 1+2\otimes\alpha_i)(\beta\otimes 1+2\otimes\beta)\\ &=2(\alpha_i\otimes\beta+\beta\otimes\alpha_i),\end{aligned}$$

and in genus one it follows that the kernel and cokernel of Φ_3 both have rank two. When $g\geq 2$ the triple product $\xi_i\xi_j\xi_k$ maps to 0 if i,j,k are distinct mod g, while

$$\begin{aligned}\xi_i\xi_{i+g}\xi_j &\mapsto (2(\alpha_i\otimes\alpha_{i+g}-\alpha_{i+g}\otimes\alpha_i)+\beta\otimes 1+4\otimes\beta)(\alpha_j\otimes 1+2\otimes\alpha_j)\\ &=2\beta\otimes\alpha_j+4\alpha_j\otimes\beta\end{aligned}$$

for $i\neq j\neq i+g$. Considering the images of $\zeta_i\theta$ and $\xi_i\xi_{i+g}\xi_j$ we see that the image of Φ_3 has rank $4g$ and that

$$\mathrm{coker}\,\Phi_3\cong\frac{\mathrm{span}\{\beta\otimes\alpha_j+2\alpha_j\otimes\beta\}}{\mathrm{span}\{2(\beta\otimes\alpha_j+2\alpha_j\otimes\beta)\}}\cong[\mathbf{Z}/2\mathbf{Z}]^{2g},$$

so that

$$\ker\Phi_3\cong\begin{cases}\mathbf{Z}^2 & g=1,\\ \mathbf{Z}^{\binom{2g}{3}} & g\neq 1,\end{cases}\qquad \mathrm{coker}\,\Phi_3\cong\begin{cases}\mathbf{Z}^2 & g=1,\\ [\mathbf{Z}/2\mathbf{Z}]^{2g} & g\neq 1.\end{cases}$$

Dimension four. $H^4(\mathrm{Sym}^3\Sigma)\oplus H^4(\mathrm{Sym}^2\Sigma)$ has rank $\binom{2g}{2}+2$ and basis

$$\{\xi_i\xi_j\eta | i<j\}\cup\{\eta^2,\theta^2\},$$

while $H^4(\Sigma\times\Sigma)$ has rank one and basis $\{\beta\otimes\beta\}$. Under Φ_4 we have

$$\begin{aligned}\theta^2 &\mapsto (\beta\otimes 1+1\otimes\beta)^2\\ &=2\beta\otimes\beta,\\ \eta^2 &\mapsto (\beta\otimes 1+2\otimes\beta)^2\\ &=4\beta\otimes\beta,\\ \xi_i\xi_j\eta &\mapsto 2(\alpha_i\otimes\beta+\beta\otimes\alpha_i)(\alpha_j\otimes 1+2\otimes\alpha_j)\\ &=\begin{cases}0 & |j-i|\neq g,\\ 6\beta\otimes\beta & j=i+g.\end{cases}\end{aligned}$$

Clearly

$$\ker\Phi_4\cong\mathbf{Z}^{\binom{2g}{2}+1},\qquad \mathrm{coker}\,\Phi_4\cong\mathbf{Z}/2\mathbf{Z}.$$

Dimensions five and six. $\Sigma \times \Sigma$ and $\mathrm{Sym}^2\Sigma$ have no cohomology in dimensions five and six so the cokernel of Φ_i is trivial and the kernel is $H^i(\mathrm{Sym}^3\Sigma)$ for $i = 5, 6$. $H^5(\mathrm{Sym}^3\Sigma) = \mathrm{span}\{\xi_i\eta^2\}$ has rank $2g$ and $H^6(\mathrm{Sym}^3\Sigma) = \mathrm{span}\{\eta^3\}$ has rank one, so

$$\begin{aligned} \ker \Phi_5 &\cong \mathbf{Z}^{2g}, & \mathrm{coker}\, \Phi_5 &\cong \{0\}, \\ \ker \Phi_6 &\cong \mathbf{Z}, & \mathrm{coker}\, \Phi_6 &\cong \{0\}. \end{aligned}$$

Completing the proof of Theorem 1. Putting the kernels and cokernels calculated above together using equation (1) gives the table in Theorem 1. Taking alternating sums of Betti numbers gives

$$\begin{aligned} \chi(\exp_3\Sigma_g) &= 3 - 4g + \binom{2g}{2} - \binom{2g}{3} \\ &= \frac{-4g^3 + 12g^2 - 17g + 9}{3} \end{aligned} \tag{3}$$

for $g \geq 2$, and direct substitution shows it holds for $g \geq 0$ also. As a check we calculate the Euler characteristic using

$$\chi(E_3\Sigma) = \chi(\mathrm{Sym}^3\Sigma) + \chi(\mathrm{Sym}^2\Sigma) - \chi(\Sigma \times \Sigma).$$

Macdonald gives $\chi(\mathrm{Sym}^n\Sigma) = (-1)^n\binom{2g-2}{n}$, so

$$\begin{aligned} \chi(E_3\Sigma) &= -\binom{2g-2}{3} + \binom{2g-2}{2} - (2-2g)^2 \\ &= \frac{-4g^3 + 12g^2 - 17g + 9}{3}, \end{aligned}$$

in agreement with (3).

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Department of Mathematics, University of California, Berkeley CA 94720, U.S.A.
E-mail address: `tuffley@math.berkeley.edu`