MIRKOVIC-VILONEN CYCLES AND POLYTOPES

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Abstract. We give an explicit description of the Mirkovic-Vilonen cycles on the affine Grassmannian for arbitrary reductive groups. We also give a combinatorial characterization of the MV polytopes. We prove that a polytope is an MV polytope if and only if every 2-face of it is a rank 2 MV polytope.

1. Introduction

Let G be a complex connected semisimple group. Let $G^\vee$ be its Langlands dual group. Let $\mathcal{K} = \mathbb{C}((t))$ denote the field of Laurent series and let $\mathcal{O} = \mathbb{C}[[t]]$ denote the ring of power series. The quotient $\mathcal{G}r = G(\mathcal{K})/G(\mathcal{O})$ is called the affine Grassmannian. The geometric Satake correspondence of Lusztig, Ginzburg, Beilinson-Drinfeld, and Mirkovič-Vilonen provides a connection between the geometry of $\mathcal{G}r$ and the representation theory of $G^\vee$.

In particular, Mirkovič-Vilonen [MV] identified a certain class of subvarieties of the affine Grassmannian, called Mirkovič-Vilonen cycles, which give bases for representations of $G^\vee$. These MV cycles are the components of intersections of opposite "semi-infinite cells" in $\mathcal{G}r$. Anderson [A2] studied the moment map images of the Mirkovič-Vilonen cycles, called Mirkovič-Vilonen polytopes, and showed that these polytopes could be used to understand the combinatorics of representations of $G^\vee$.

Because of this connection to the representation theory, some attempts have been made to give an explicit description of the MV cycles and polytopes. Gaussent-Littelmann [GL] associated an MV cycle to each Littelmann path. However, their method did not give equations cutting out the MV cycles and does not lead to a description of the MV polytopes. Anderson-Kogan [AK] studied MV cycles for GL_n by means of the lattice model for $\mathcal{G}r$. They gave a recipe for producing MV cycles and polytopes for GL_n, but not an explicit description of the cycles and polytopes.

Here we give an explicit combinatorial description of the MV cycles and polytopes uniform across all types. We show (Theorem 2.5) that a polytope is an MV polytope if and only if every 2-face is an MV polytope of the appropriate rank 2 group (of type $A_1 \times A_1, A_2, B_2, G_2$). A polygon is a rank 2 MV polytope if the distances of its sides from the origin satisfy a certain $(+, \max)$ equation called the "tropical Plücker relation" (15), (16), (17).

This result, combined with [BZ2], shows that Lusztig's canonical basis and the set of MV cycles are governed by the same combinatorics (Theorem 2.7). In the case of MV cycles, the tropical Plücker relations appear naturally, whereas their appearance in [BZ2] to describe the canonical basis was unexpected.

There is a close connection between our work and the Anderson-Kogan description of MV cycles and polytopes for GL_n and our work. In fact, their work served as an important source of motivation. The details of this connection will be explained in [K].

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The proof of our main theorem uses the results of Berenstein-Fomin-Zelevinsky [BFZ, BZ1, FZ] concerning generalized minors.

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2. Background and results

2.1. **Notation.** If G is complex connected semisimple group, then its affine Grassmannian is the disjoint union of $\pi_1(G)$ many copies of the affine Grassmannian of the simply-connected semisimple group with the same root system as G. So here we only consider the case G connected simply-connected semisimple. As another simplification, in this paper we consider only the case of G singly or doubly-laced. Extending our results to include G_2 factors is quite simple; it just requires including the extra cases of $a_{ij} = -3$ and $a_{ji} = -3$ in the statement of the tropical Plücker relation and in Propositions 4.1 and 4.2. The case $a_{ij} = -3$ appears in [BZ1] and the case $a_{ji} = -3$ can be easily derived from there.

Let G be a connected simply-connected semisimple complex group.

Let T be a maximal torus of G and let $X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times), X_*(T) = \mathrm{Hom}(\mathbb{C}^\times, T)$ denote the weight and coweight lattices of T. Let $\Delta \subset X^*(T)$ denote the set of roots of G. Let $W = N(T)/T$ denote the Weyl group.

Let B be a Borel subgroup of G containing T. Let $\alpha_1, \dots, \alpha_r$ and $\alpha_1^\vee, \dots, \alpha_r^\vee$ denote the simple roots and coroots of G with respect to B. Let N denote the unipotent radical of B. Let $\Lambda_1, \dots, \Lambda_r$ be the fundamental weights. Let $I = \{1, \dots, r\}$ denote the vertices of the Dynkin diagram of G. Let $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ denote the Cartan matrix. Let $\rho := \sum \Lambda_i,\ \rho^\vee := \sum \Lambda_i^\vee$ be the Weyl and dual Weyl vectors.

We use $\geq$ for the usual partial order on $X_*(T)$, so that $\mu \geq \nu$ if and only if $\mu - \nu$ is a sum of positive coroots. More generally, we have the twisted partial order $\geq_w$, where $\mu \geq_w \nu$ if and only if $w^{-1} \cdot \mu \geq w^{-1} \cdot \nu$.

Let $\mathfrak{t}_\mathbb{R} := X_*(T) \otimes \mathbb{R}$. For each w, we extend $\geq_w$ to a partial order on $\mathfrak{t}_\mathbb{R}$, so that $\beta \geq_w \alpha$ if and only if $\langle \beta - \alpha, w \cdot \Lambda_i \rangle \geq 0$ for all i.

For each $i \in I$, let $\psi_i : SL_2 \to G$ be denote the ith root subgroup of G.

For $w \in W$, let $\overline{w}$ denote the usual lift of w to G, using the lift of $\overline{s_i} := \psi_i\Big(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Big)$.

We will also need the Bruhat order on W, which we also denote by $\geq$. Let w_0 denote the longest element of W.

A **reduced word** for an element $w \in W$ is a sequence of indices $\mathbf{i} = (i_1, \dots, i_m)$ such that $w = s_{i_1} \cdots s_{i_m}$ is a reduced expression.

Let kpf denote the **Kostant partition function** on $X_*(T)$, so that $\mathrm{kpf}(\mu)$ is the number of ways to write μ as a sum of positive coroots.

2.2. **MV cycles.** For the purposes of this paper, it will be convenient to write the affine Grassmannian as the left quotient $\mathcal{G}r = G(\mathcal{O}) \setminus G(\mathcal{K})$. We view $\mathcal{G}r$ as an ind-scheme over $\mathbb{C}$ whose set of $\mathbb{C}$ points is $G(\mathcal{O}) \setminus G(\mathcal{K})$. Similarly, we view $G(\mathcal{K}), N(\mathcal{K}), \mathcal{K}^m$ as ind-schemes over $\mathbb{C}$, in fact they are the images of the formal loop space functor. For more details, see [FB, Sections 11.3.3, 20.3.3].

A coweight $\mu \in X_*(T)$ gives a homomorphism $\mathbb{C}^\times \to T$ and hence an element of $\mathcal{G}r$. We denote the corresponding element t^μ. It is easy to see that these t^μ are the fixed points for the action of $T(\mathbb{C})$ on $\mathcal{G}r$.

For $w \in W$, let $N_w = wNw^{-1}$. For $w \in W$ and $\mu \in X_*(T)$ define the **semi-infinite cells**

$$S_w^\mu := t^\mu N_w(\mathcal{K}). \tag{1}$$

To a certain extent, these semi-infinite cells behave like the Schubert cells on a finite dimensional flag variety. In particular, they are each attracting cells for a certain $\mathbb{C}^\times$ action on $\mathcal{G}r$,

$$S_w^\mu = \{L \in \mathcal{G}r : \lim_{s\to\infty} L \cdot (w \cdot \rho^\vee)(s) = t^\mu\}. \tag{2}$$

They have the simple containment relation (see [MV])

$$\overline{S_w^\mu} = \bigcup_{\nu \geq_w \mu} S_w^\nu. \tag{3}$$

Lemma 2.1. *If $S_w^\mu \cap S_v^\nu \neq \emptyset$ then $\nu \geq_w \mu$.*

Proof. Let $L \in \overline{S_w^\mu} \cap S_v^\nu$. Then by (2), $t^\nu = \lim_{s\to\infty} L \cdot (v \cdot \rho^\vee)(s)$. Since S_w^μ is T-invariant, this shows that $t^\nu \in \overline{S_w^\mu}$. So by (3), $\nu \geq_w \mu$. □

Following Anderson [A2], for $\mu \geq 0$ we define an **MV cycle of coweight** μ to be a component of $\overline{X(\mu)}$ where $X(\mu)$ is the intersection of two opposite semi-infinite cells,

$$X(\mu) = S_1^0 \cap S_{w_0}^\mu. \tag{4}$$

It is well-known that this intersection has $\mathrm{kpf}(\mu)$ components (for example this follows from [BFG, Section 13], or from [A2]). It is also known that this intersection has pure dimension $\langle \mu, \rho \rangle$, but we will not need this fact.

Following Anderson [A2], given a T-invariant closed subvariety A of the affine Grassmannian let $\Phi(A) \subset \mathfrak{t}_\mathbb{R}$ be the convex hull of $\{\mu \in X_*(T) : t^\mu \in A\}$. By [A2], this is the moment polytope for the T action on the A. If A is an MV cycle of coweight μ, then we say that $\Phi(A)$ is an **MV polytope of coweight** μ.

For example, by (3), we see that $\Phi(\overline{S_w^\mu}) = C_w^\mu := \{\alpha \in \mathfrak{t}_\mathbb{R} : \alpha \geq_w \mu\} = \{\alpha : \langle \alpha, w \cdot \Lambda_i \rangle \geq \langle \mu, w \cdot \Lambda_i \rangle$ for all $i\}$.

2.3. **pseudo-Weyl polytopes.** We will start our investigation by examining a larger family of polytopes, called pseudo-Weyl polytopes, which we will show contains the family of MV polytopes. We will show how to pick out the MV polytopes from the pseudo-Weyl polytopes.

For $\lambda \in X_*(T)$, $W_\lambda = \mathrm{conv}(W \cdot \lambda) \subset \mathfrak{t}_\mathbb{R}$ is called the λ-**Weyl polytope**. Recall that the Weyl polytope W_λ can be described in three different ways. It is the convex hull of the orbit of λ under the action of the Weyl group, it is the intersection of translated and reflected cones, and it is the intersection of half spaces. In particular, if λ is in the anti-dominant Weyl chamber, then

$$W_\lambda = \bigcap_w C_w^{w\cdot\lambda} = \{\alpha \in \mathfrak{t}_\mathbb{R} : \langle \alpha, w \cdot \Lambda_i \rangle \geq \langle \lambda, \Lambda_i \rangle \text{ for all } w \in W \text{ and } i \in I\}.$$

Following Berenstein-Zelevinsky, we call a weight $w \cdot \Lambda_i$ a **chamber weight** of level i. So the chamber weights $\Gamma := \bigcup_{w \in W, i \in I} w \cdot \Lambda_i$ are dual to the hyperplanes defining any Weyl polytope.

A **pseudo-Weyl polytope** is a polytope P along with a map $w \mapsto \mu_w$ from W onto the vertices of P such that

$$P = \bigcap_w C_w^{\mu_w}. \tag{5}$$

Pseudo-Weyl polytopes admit a dual description in terms of intersecting half spaces.

Let $M_\bullet = \left(M_\gamma\right)_{\gamma \in \Gamma}$ be a collection of integers, one for each chamber weight. Given such a collection, we can form $P(M_\bullet) := \{\alpha \in \mathfrak{t}_\mathbb{R} : \langle \alpha, \gamma \rangle \geq M_\gamma \text{ for all } \gamma \in \Gamma\}$. This is the polytope made by translating the hyperplanes defining the Weyl polytopes to distances M_γ from the origin.

Proposition 2.2. *Let $\mu_\bullet = \left(\mu_w\right)_{w \in W}$ be a collection of coweights. Then* $\operatorname{conv}(\mu_\bullet)$ *is a pseudo-Weyl polytope if and only if $\mu_v \geq_w \mu_w$ for all v, w.*

A pseudo-Weyl polytope has defining hyperplanes dual to the chamber weights. In particular, if P is a pseudo-Weyl polytope with vertices $\mu_\bullet$, then

$$P = P(M_\bullet) \tag{6}$$

where

$$M_{w \cdot \Lambda_i} = \langle \mu_w, w \cdot \Lambda_i \rangle. \tag{7}$$

*Moreover, the $M_\bullet$ satisfy the following condition which we call the **non-degeneracy inequalities**.*

For each $w \in W$ and $i \in I$, we have:

$$M_{ws_i \cdot \Lambda_i} + M_{w \cdot \Lambda_i} + \sum_{j \neq i} a_{ji} M_{w \cdot \Lambda_j} \leq 0 \tag{8}$$

Conversely, suppose that a collection of positive integers $\left(M_\gamma\right)_{\gamma \in \Gamma}$ satisfies the above condition. Then the polytope $P(M_\bullet)$ is pseudo-Weyl polytope with vertices given by

$$\mu_w = \sum_i M_{w \cdot \Lambda_i} w \cdot \alpha_i^\vee. \tag{9}$$

Another way to describe pseudo-Weyl polytopes is to say that they are polytopes whose dual fan is a coarsening of the Weyl fan. The proposition follows from a standard result concerning polytopes with fixed dual fans (see [F]).

Let P be a pseudo-Weyl polytope, $P = \operatorname{conv}(\mu_\bullet) = P(M_\bullet)$. For any $w \in W, i \in I$, there is an edge connecting μ_w and μ_{ws_i}. We have

$$\mu_{ws_i} - \mu_w = c w \cdot \alpha_i^\vee, \quad \text{where } c = -M_{w \cdot \Lambda_i} - M_{ws_i \cdot \Lambda_i} - \sum_{j \neq i} a_{ji} M_{w \cdot \Lambda_j}. \tag{10}$$

We call c the **length** of the edge from μ_w to μ_{ws_i}. Note that it is the negative of the left hand side of (8).

2.4. **GGMS strata.** The geometric version of the pseudo-Weyl polytopes are the Gelfand-Goresky-MacPherson-Serganova (GGMS) strata on the affine Grassmannian. These GGMS strata will be our candidates to be MV cycles.

Given any collection $\mu_\bullet = \big(\mu_w\big)_{w \in W}$ of coweights, we can form the GGMS stratum

$$A(\mu_\bullet) := \bigcap_{w \in W} S_w^{\mu_w}. \tag{11}$$

By Lemma 2.1, this intersection is empty unless $\mu_u \geq_w \mu_w$ for all u, w. Hence the intersection is empty unless $\operatorname{conv}(\mu_\bullet)$ is a pseudo-Weyl polytope.

We will prove that each MV cycle is the closure of $A(\mu_\bullet)$ for appropriate $\mu_\bullet$. Once we know which of these are MV cycles, we will also know the MV polytopes, since we have the following easy lemma, due to Anderson-Kogan in type A.

Lemma 2.3. *Let $\mu_\bullet$ be as above. Then $\Phi(\overline{A(\mu_\bullet)}) = \operatorname{conv}(\mu_\bullet)$ or $A(\mu_\bullet) = \emptyset$.*

Proof. Let $X = \overline{A(\mu_\bullet)}$. Assume that X is non-empty.

Let P denote the moment polytope of X. We know that P is the convex hull of the set $\{\mu \in X_*(T) : t^\mu \in X\}$. Let $L \in \cap_w S_w^{\mu_w}$. Since X is the closure of the intersection of T-invariant subsets, X is T-invariant. For each $w \in W$ consider the one parameter subgroup $w \cdot \rho^\vee : \mathbb{C}^\times \to T$. Since X is closed and T-invariant, $\lim_{s \to \infty} L \cdot (w \cdot \rho^\vee)(s) \in X$. But since $L \in S_w^{\mu_w}$, we see that $\lim_{s\to\infty} L \cdot (w \cdot \rho^\vee)(s) = t^{\mu_w}$.

Hence $t^{\mu_w} \in A$ for all $w \in W$. Hence $\operatorname{conv}(\mu_\bullet) \subset P$.

Conversely, if $t^\nu \in X$, then $t^\nu \in \overline{S_w^{\mu_w}}$ for each $w \in W$. So $\nu \in C_w^{\mu_w}$. Hence $\nu \in \cap_w C_w^{\mu_w}$. Since $\cap_w C_w^{\mu_w} = \operatorname{conv}(\mu_\bullet)$ is convex, this shows that $P \subset \operatorname{conv}(\mu_\bullet)$. □

For each $L \in \mathcal{G}r$, let $P(L)$ denote the pseudo-Weyl polytope corresponding to the GGMS stratum containing L.

We now introduce constructible functions on the affine Grassmannian which cut out the GGMS strata. These functions are new, but were motivated by the work of Speyer [S].

If U is a vector space over $\mathbb{C}$, the vector space $U \otimes \mathcal{K}$ has a filtration by $\cdots \subset U \otimes t\mathcal{O} \subset U \otimes \mathcal{O} \subset U \otimes t^{-1}\mathcal{O} \subset \cdots$. We use this filtration to define a valuation val on $U \otimes \mathcal{K}$, by $\operatorname{val}(u) = k$ if $u \in U \otimes t^k \mathcal{O}$ and $u \notin U \otimes t^{k+1}\mathcal{O}$.

Fix a high weight vector v_{Λ_i} in each fundamental representation V_{Λ_i} of G. For each chamber weight $\gamma = w \cdot \Lambda_i$, let $v_\gamma = \overline{w} \cdot v_{\Lambda_i}$. Since G acts on V_{Λ_i}, $G(\mathcal{K})$ acts on $V_{\Lambda_i} \otimes \mathcal{K}$.

For each $\gamma \in \Gamma$ define the function D_γ by:

$$\begin{aligned} \mathrm{D}_\gamma : \mathcal{G}r &\to \mathbb{Z} \\ [g] &\mapsto \operatorname{val}(g \cdot v_\gamma) \end{aligned} \tag{12}$$

This gives a well-defined function on $\mathcal{G}r = G(\mathcal{O}) \setminus G(\mathcal{K})$, since if $g \in G(\mathcal{O})$ and $u \in V_{\Lambda_i} \otimes \mathcal{K}$, then $\operatorname{val}(g \cdot u) = \operatorname{val}(u)$.

The functions D_γ have a simple structure with respect to the semi-infinite cells. To see this note that if $\gamma = w \cdot \Lambda_i$, then v_γ is invariant under N_w. This immediately implies the following lemma.

Lemma 2.4. *Let $w \in W$. The function $D_{w\cdot\Lambda_i}$ takes the constant value $\langle \mu, w \cdot \Lambda_i \rangle$ on S_w^μ. In fact,*

$$S_w^\mu = \{L \in \mathcal{G}r : D_{w \cdot \Lambda_i}(L) = \langle \mu, w \cdot \Lambda_i \rangle \text{ for all } i\}.$$

Let $M_\bullet$ be a collection of integers, one for each chamber weight. Then we consider the simultaneous level set of the functions $D_\bullet$,

$$A(M_\bullet) := \{L \in Gr : D_\gamma(L) = M_\gamma \text{ for all } \gamma\}. \tag{13}$$

Let $\mu_\bullet$ be a collection of coweights describing a pseudo-Weyl polytope. Let $M_\bullet$ be the corresponding collection of integers defined by (7). Then by Lemma 2.4, we have two descriptions of the GGMS stratum: $A(\mu_\bullet) = A(M_\bullet)$.

By Proposition 2.2, we also have two different descriptions of the pseudo-Weyl polytope: $\text{conv}(\mu_\bullet) = P(M_\bullet)$.

If the GGMS stratum is non-empty, then the GGMS stratum and the pseudo-Weyl polytope are related in two different ways:

$$A(\mu_\bullet) = A(M_\bullet) = \{L \in \mathcal{G}r : P(L) = \text{conv}(\mu_\bullet) = P(M_\bullet)\},$$
$$\Phi\big(\overline{A(\mu_\bullet)}\big) = \Phi\big(\overline{A(M_\bullet)}\big) = \text{conv}(\mu_\bullet) = P(M_\bullet),$$

where the first line of equations is by the definition of $P(L)$ and the second is by Lemma 2.3.

2.5. **BZ data.** Our goal is now to give neccesary and sufficient conditions on a collection $M_\bullet$ for $\overline{A(M_\bullet)}$ to be an MV cycle. For this purpose, it is necessary to understand better the functions $D_\bullet$. To that end, we consider the **generalized minors** of Berenstein-Zelevinsky [BZ1]. For each chamber weight γ of level i, they introduced the function

$$\begin{aligned} \Delta_\gamma : G &\to \mathbb{C} \\ g &\mapsto \langle g \cdot v_\gamma, v_{-\Lambda_i}\rangle \end{aligned} \tag{14}$$

(note that $v_{-\Lambda_i} \in V_{-w_0\cdot\Lambda_i} = V_{\Lambda_i}^\star$).

When $G = SL_n$, a chamber weight of level i is just an i element subset of $\{1, \dots, n\}$ and $\Delta_\gamma(g)$ is the minor of g using the first i rows and column set γ.

The function D_γ on the affine Grassmannian is closely related to the valuation of Δ_γ. In general, one can see that $\text{val}(\Delta_\gamma(g)) \ge \mathrm{D}_\gamma([g])$ (see the remarks at the beginning of Section 3.5). We will show (in the course of the proof of Theorem 3.5) that if $L \in \mathcal{G}r$ then there exists $g \in G(\mathcal{K})$ such that $[g] = L$ and $\mathrm{D}_\gamma(L) = \text{val}(\Delta_\gamma(g))$ for all γ.

Berenstein-Zelevinsky [BZ1] established certain three-term Plücker relations among these generalized minors. In general, the process of passing from relations among Laurent series to relations among integers using val is called tropicalization. The most naive way to tropicalize is to simply replace multiplication by addition and addition by min. This will not always work because cancellation of leading terms can occur in addition. However, we will show that this naive tropicalization is enough to understand the values of the D_γ on an open subset of each MV cycle. This motivates the following definition (which originally appeared - though with a different motivation - in [BZ2]).

Let $w \in W, i, j \in I$ be such that $ws_i > w, ws_j > w,\ i \ne j$. We say that a collection $\big(M_\gamma\big)_{\gamma\in\Gamma}$ satisfies the **tropical Plücker relation** at (w, i, j) if $a_{ij} = 0$ or if

(i) if $a_{ij} = a_{ji} = -1$, then

$$M_{ws_i\cdot\Lambda_i} + M_{ws_j\cdot\Lambda_j} = \min(M_{w\cdot\Lambda_i} + M_{ws_is_j\cdot\Lambda_j}, M_{ws_js_i\cdot\Lambda_i} + M_{w\cdot\Lambda_j}); \tag{15}$$

(ii) if $a_{ij} = -1, a_{ji} = -2$, then

$$
\begin{aligned}
M_{ws_j\cdot\Lambda_i} + M_{ws_is_j\cdot\Lambda_j} + M_{ws_i\cdot\Lambda_i} = \min\Big(&2M_{ws_is_j\cdot\Lambda_j} + M_{w\cdot\Lambda_i},\\
&2M_{w\cdot\Lambda_j} + M_{ws_is_js_i\cdot\Lambda_i},\\
&M_{w\cdot\Lambda_j} + M_{ws_js_is_j\cdot\Lambda_j} + M_{ws_i\cdot\Lambda_i}\Big),\\
M_{ws_js_i\cdot\Lambda_i} + 2M_{ws_is_j\cdot\Lambda_j} + M_{ws_i\cdot\Lambda_i} = \min\Big(&2M_{w\cdot\Lambda_j} + 2M_{ws_is_js_i\cdot\Lambda_i},\\
&2M_{ws_js_is_j\cdot\Lambda_j} + 2M_{ws_i\cdot\Lambda_i},\\
&M_{ws_is_js_i\cdot\Lambda_i} + 2M_{ws_is_j\cdot\Lambda_j} + M_{w\cdot\Lambda_i}\Big);
\end{aligned} \tag{16}
$$

(iii) if $a_{ij} = -2, a_{ji} = -1$, then

$$
\begin{aligned}
M_{ws_js_i\cdot\Lambda_i} + M_{ws_i\cdot\Lambda_i} + M_{ws_is_j\cdot\Lambda_j} = \min\Big(&2M_{ws_i\cdot\Lambda_i} + M_{ws_js_is_j\cdot\Lambda_j},\\
&2M_{ws_is_js_i\cdot\Lambda_i} + M_{w\cdot\Lambda_j},\\
&M_{ws_is_js_i\cdot\Lambda_i} + M_{w\cdot\Lambda_i} + M_{ws_is_j\cdot\Lambda_j}\Big),\\
M_{ws_j\cdot\Lambda_j} + 2M_{ws_i\cdot\Lambda_i} + M_{ws_is_j\cdot\Lambda_j} = \min\Big(&2M_{ws_is_js_i\cdot\Lambda_i} + 2M_{w\cdot\Lambda_j},\\
&2M_{w\cdot\Lambda_i} + 2M_{ws_is_j\cdot\Lambda_j},\\
&M_{w\cdot\Lambda_j} + 2M_{ws_i\cdot\Lambda_i} + M_{ws_js_is_j\cdot\Lambda_j}\Big).
\end{aligned} \tag{17}
$$

We say that a collection $M_\bullet = \big(M_\gamma\big)_{\gamma\in\Gamma}$ satisfies the **tropical Plücker relations** if it satisfies the tropical Plücker relation at each (w, i, j).

A collection $\big(M_\gamma\big)_{\gamma\in\Gamma}$ is called a **BZ datum** if:

(i) $M_\bullet$ satisfies the tropical Plücker relations.
(ii) $M_\bullet$ satisfies the non-degeneracy inequalities (8).
(iii) $M_{\Lambda_i} = 0$ for all i.

The **coweight** of a BZ datum is defined to be $\sum_i M_{w_0\cdot\Lambda_i} w_0 \cdot \alpha_i^\vee$. This is the w_0 vertex of the corresponding pseudo-Weyl polytope $P(M_\bullet)$.

Our main result, which will be proven is Sections 3 and 4, is the following characterization of MV cycles and polytopes.

Theorem 2.5. *Let $M_\bullet$ be a BZ datum of coweight μ. Then $\overline{A(M_\bullet)}$ is an MV cycle of coweight μ, and each MV cycle arises this way for a unique BZ datum $M_\bullet$.*

Hence a pseudo-Weyl polytope $P(M_\bullet)$ is an MV polytope if and only if $M_\bullet$ is a BZ datum.

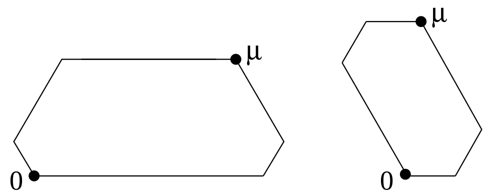
So BZ data $M_\bullet$ are the good sets of values for the $D_\bullet$, namely the sets of values such that the closure of the resulting level set $A(M_\bullet)$ is an MV cycle.

In the case of $G = SL_3$, it is possible to give a very explicit description of the BZ data and MV polytopes. In this case we have $\Gamma = \{1, 2, 3, 12, 13, 23\}$ where we identify 2 with $(0, 1, 0) \in X^*(T)$, 23 with $(0, 1, 1)$, etc.

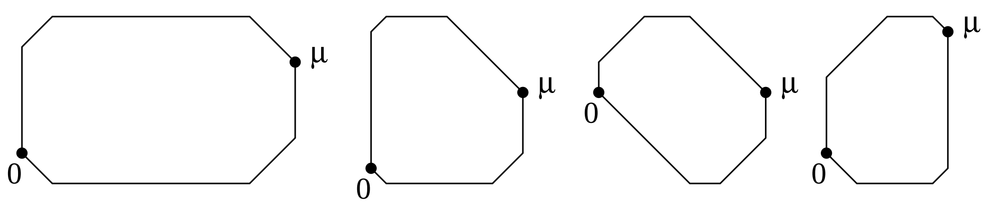
There is only one tropical Plücker relation (which occurs at $(w = 1, i = 1, j = 2)$),

$$M_2 + M_{13} = \min\{M_1 + M_{23}, M_3 + M_{12}\}. \tag{18}$$

Translated into the world of polytopes, we note that pseudo-Weyl polytopes for SL_3 are hexagons with every pair of opposite sides parallel and all sides meeting at 120°. The above relation (18) shows that a pseudo-Weyl polytope is an MV polytope if and only if the distance between the middle pair of opposite sides is the maximum of the distances between the other two pairs of opposite sides. Hence there are two possible forms for SL_3 MV polytopes, depending on which distance achieves this maximum. Here are example of each of the two kinds (where 0 marks the 1 vertex and μ marks the w_0 vertex).



In the case of $G = Sp_4$, there is again one tropical Plücker relation (which occurs at $(w = 1, i = 1, j = 2)$). Examining the possible cases in either (16) or (17) shows there there are the following four possible types of polytopes.



The MV polytopes for SL_3 and Sp_4 appeared without proof in Anderson's thesis [A1]. There they were expressed as the Minkowski sums of certain "prime" MV polytopes. In [K] we explain how to write MV polytopes (for all G) as Minkowski sums of finitely many "prime" MV polytopes.

Each tropical Plücker relation concerns the placement of the hyperplanes incident to a particular 2-face of the pseudo-Weyl polytope. Hence we see that if $\operatorname{rank}(G) > 2$, then a pseudo-Weyl polytope is an MV polytope if and only if all of its 2-faces are MV polytopes. So a pseudo-Weyl polytope is an MV polytopes if and only if all of its 2-faces are rectangles (the MV polytopes for $SL_2 \times SL_2$) or one of the above types. More generally, this shows that any face of an MV polytope is an MV polytope. In [K], we give an explanation of this phenomenom.

2.6. **Lusztig data.** Let $\mathbf{i} = (i_1, \dots, i_m)$ be a reduced word for w_0. Such an $\mathbf{i}$ gives a distinguished path $\mu_1, \mu_{s_{i_1}}, \mu_{s_{i_1} s_{i_2}}, \dots, \mu_{w_0}$ through the 1-skeleton of the pseudo-Weyl polytope $P = \operatorname{conv}(\mu_\bullet)$. Let $n_1, \dots, n_m$ be the sequence of lengths of the edges of this path. We call the vector $(n_1, \dots, n_m)$ the **i-Lusztig datum** of P. For MV polytopes, this is enough information to reconstruct the entire polytope. This follows from [BZ1], but in the course of this paper we will reprove this result.

Theorem 2.6. *The map*

$$\{MV polytopes\} \to \mathbb{N}^m$$
$$P \mapsto (n_1, \dots, n_m)$$

is a bijection.

Now, we can explain our strategy for the proof of the main theorem, Theorem 2.5. Fix a coweight $\mu \geq 0$. For any $\mathbf{i}$, we decompose $X(\mu)$ into a disjoint union according to $\mathbf{i}$-Lusztig datum (Section

3.2). We then prove (Theorem 3.2) that the closures of these pieces are the irreducible components of $X(\mu)$.

Next, we consider the problem of how the different decompositions for different $\mathbf{i}$ overlap. The key is to first consider reduced words $\mathbf{i}, \mathbf{i}'$ which differ by a d-move (Section 4.1). Using this knowledge, we are able to prove that the MV cycles are as in Theorem 2.5.

2.7. **Relation to the canonical basis.** Let $\mathcal{B}$ denote Lusztig's canonical basis for $U_+^\vee$, the upper triangular part of the quantized universal envelopping algebra of $G^\vee$. Lusztig showed that a choice of reduced word $\mathbf{i}$ for w_0 gives rise to a bijection $\mathcal{B} \to \mathbb{N}^m$ (see [BZ2] for more details). Following Berenstein-Zelevinsky, we call the image of this bijection the **i-Lusztig datum** of the canonical basis element.

By considering varying the choice of $\mathbf{i}$, Berenstein-Zelevinsky [BZ2] discovered BZ data. Their results can be expressed in our language as follows.

Theorem 2.7 ([BZ1, Theorem 4.3], [BZ2, Example 5.4])**.** *There is a coweight preserving bijection $b \mapsto P(b)$ between the canonical basis $\mathcal{B}$ and the set of MV polytopes. Under this bijection, the $\mathbf{i}$-Lusztig datum of b equals the $\mathbf{i}$-Lusztig datum of $P(b)$.*

In other words to find the $\mathbf{i}$-Lusztig datum of b, we can just look at the lengths of the edges in $P(b)$ along the path determined by $\mathbf{i}$.

Proof. We adopt the notation of [BZ1]. Example 5.4 in [BZ2] gives a bijection between $\mathcal{B}$ and $\{(t_k^{\mathbf{i}}) : t_k^{\mathbf{i}} \geq 0 \text{ for all } \mathbf{i}, k\} \subset \mathcal{L}(\mathbb{Z}_{\text{trop}})$, where $\mathcal{L}$ denotes the Lusztig variety. This map records the Lusztig datum of a canonical basis element with respect to each reduced word.

Theorem 4.3 in [BZ1] gives a bijection between $\mathcal{L}(\mathbb{Z}_{\text{trop}})$ and $M(\mathbb{Z}_{\text{trop}})$, which is the set of all collections $M_\bullet$ satisfying the tropical Plücker relations and the normalization condition $M_{\Lambda_i} = 0$ for all i. The image of $\mathcal{B}$ under the composition of these two bijections is the subset of $M(\mathbb{Z}_{\text{trop}})$ satisfying the non-degeneracy inequalities (8). Hence we get a bijection from $\mathcal{B}$ to the set of BZ data, which we have proven to be the same as the set of MV polytopes. The equality of the Lusztig data is clear from the definitions.

□

2.8. **Tensor product multiplicities.** Recall that $G^\vee$ is the group with root datum dual to that of G. In particular, the weight lattice of $G^\vee$ is $X_*(T)$.

Building on the work of Mirkovič-Vilonen, Anderson showed that MV polytopes could be used to obtain a tensor product multiplicity formula for $G^\vee$ in terms of counting polytopes.

Theorem 2.8 ([A2, Theorem 1])**.** *Let $\lambda, \mu, \nu \in X_*(T)_+$. The tensor product multiplicity $c_{\lambda\mu}^\nu$ of V_ν inside $V_\lambda \otimes V_\mu$ is equal to the number of MV polytopes P such that*

(i) *P has coweight $\lambda + \mu - \nu$,*
(ii) *$P + \nu - \mu$ is contained in W_λ,*
(iii) *$P + \nu - \mu$ is contained in $-W_\mu + \nu$.*

Combining Theorem 2.8 with our Theorem 2.5, we immediately obtain the following result.

Theorem 2.9. *The multiplicity $c_{\lambda\mu}^\nu$ equals the number of BZ data satisfying:*

(i) *$M_{w_0 \cdot \Lambda_i} = \langle \lambda + \mu - \nu, w_0 \cdot \Lambda_i \rangle$ for all i,*
(ii) *$M_\gamma \geq \langle \mu - \nu, \gamma \rangle + \langle w_0 \cdot \lambda, \Lambda_i \rangle$ for all chamber weight γ of level i,*

(iii) $M_\gamma \geq \langle \mu, \gamma - \Lambda_i \rangle$ *for all chamber weights* γ *of level* i.

Theorem 2.9 is very similar to Theorem 5.16 in [BZ2]. Both results compute tensor product multiplicity by counting a certain subset of the BZ data. Once the notation is adjusted slightly, it is easy to see that our result imposes stronger condition on the BZ data, so *a priori* counts fewer BZ data. Since both theorems compute the same number, they must in fact be imposing the same conditions on the BZ data. It would be interesting to find a direct combinatorial proof that conditions (ii), (iii) above are equivalent to the conditions (3), (4) in [BZ2, Theorem 5.16]. We have been able to do this in type A_n, D_n but not in general (see [K] for more details).

3. Lusztig data decomposition

3.1. **Reduced words and paths.** Fix a reduced word $\mathbf{i} = (i_1, \dots, i_m)$ for an element $w \in W$. The word $\mathbf{i}$ determines a sequence of distinct Weyl group elements $w_k^{\mathbf{i}} := s_{i_1} \cdots s_{i_k}$ and distinct positive coroots $\beta_k^{\mathbf{i}} := w_{k-1}^{\mathbf{i}} \cdot \alpha_{i_k}^\vee$. In particular, when $w = w_0$, we get all the positive coroots this way.

We say that a chamber weight γ is an **i-chamber weight** if it is of the form $w_k^{\mathbf{i}} \cdot \Lambda_j$ for some k, j. We write $\Gamma^{\mathbf{i}}$ for the set of all $\mathbf{i}$-chamber weights. Let $\gamma_k^{\mathbf{i}} = w_k^{\mathbf{i}} \cdot \Lambda_{i_k}$.

Because of the relationship $s_j \cdot \Lambda_i = \Lambda_i$ for $j \neq i$, it is fairly easy to see that $\Gamma^{\mathbf{i}}$ consists of $m + r$ elements: the $\gamma_k^{\mathbf{i}}$ and the fundamental weights (see [BZ1, Prop 2.9]).

It is worth keeping in mind the polytope combinatorics associated to this choice of reduced word. Let $\Sigma := W_{-\rho^\vee}$ be the $-\rho^\vee$-Weyl polytope. We will refer to this polytope as the **permutahedron**. For each $w \in W$, it has a vertex $w \cdot -\rho^\vee$ which we call the w vertex of Σ. For each $w \in W$ and $i, j \in I$, there is an edge connecting the w vertex and the ws_i vertex. Understanding the faces of the permutahedron is enough to understand the faces of any pseudo-Weyl polytope since there is a map from the set of faces of the permutahedron onto the set of faces of any pseudo-Weyl polytope.

A reduced word $\mathbf{i}$ determines a distinguished path $w_0^{\mathbf{i}} = 1, w_1^{\mathbf{i}} = s_{i_1}, w_2^{\mathbf{i}}, \dots, w_m^{\mathbf{i}} = w$ through the 1-skeleton of Σ. The kth leg of this path is the vector $w_k^{\mathbf{i}} \cdot \rho - w_{k-1}^{\mathbf{i}} \cdot \rho = \beta_k^{\mathbf{i}}$. The $\mathbf{i}$-chamber weights are exactly those dual to hyperplanes incident to the vertices along this path.

Example 1. Consider $G = SL_3$. Let $\mathbf{i} = (1, 2, 1)$, then

$$w_1^{\mathbf{i}} = 213,\ w_2^{\mathbf{i}} = 231,\ w_3^{\mathbf{i}} = 321,$$

and

$$\beta_1^{\mathbf{i}} = (1, -1, 0),\ \beta_2^{\mathbf{i}} = (1, 0, -1),\ \beta_3^{\mathbf{i}} = (0, 1, -1).$$

Also,

$$\gamma_1^{\mathbf{i}} = 2,\ \gamma_2^{\mathbf{i}} = 23,\ \gamma_3^{\mathbf{i}} = 3,$$

where we write $(0, 1, 0)$ as 2, $(0, 1, 1)$ as 23, etc.

The fundamental weights $1, 12$ are also $\mathbf{i}$-chamber weights, so in fact every chamber weight is a $\mathbf{i}$-chamber weight except for 13.

In Figure 1, we show the permutahedron for SL_3 along with the distinguished path corresponding to $\mathbf{i}$ and the hyperplanes defined by each chamber weight.

3.2. **The decomposition.** With these considerations in mind, we proceed to discuss the decomposition according to Lusztig data. Fix a reduced word $\mathbf{i}$ for w_0 and a coweight $\mu \geq 0$.

Let $n_\bullet \in \mathbb{N}^m$. We say that $n_\bullet$ is an $\mathbf{i}$-Lusztig datum of coweight μ if $\mu = \sum_k n_k \beta_k^{\mathbf{i}}$. For such $n_\bullet$, let $\mathcal{P}^{\mathbf{i}}(n_\bullet)$ be the collection of pseudo-Weyl polytopes $P = \mathrm{conv}(\mu_\bullet)$, such that for all k, the length

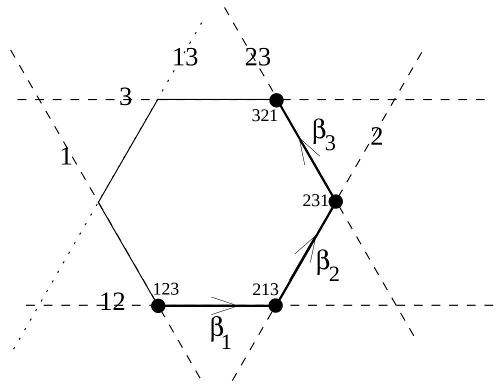
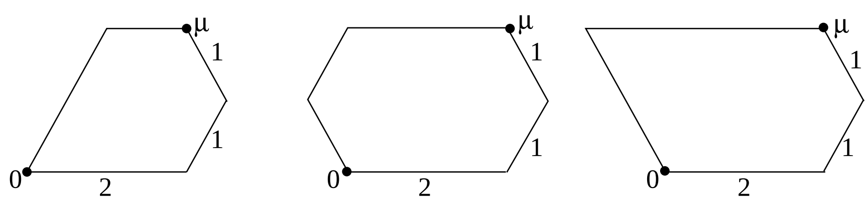


FIGURE 1. The permutahedron for SL_3.

of the edge from $\mu_{w^{\mathbf{i}}_{k-1}}$ to $\mu_{w^{\mathbf{i}}_k}$ is n_k (i.e. P has $\mathbf{i}$-Lusztig datum $n_\bullet$). Note that if $P \in \mathcal{P}(n_\bullet)$, then $\mu_{w_0} = \sum_k n_k \beta^{\mathbf{i}}_k = \mu$ is the coweight of the $\mathbf{i}$-Lusztig datum of P.

Example 2. Continuing as in Example 1, we see that there are three pseudo-Weyl polytopes with $\mathbf{i}$-Lusztig datum (2,1,1):



Let $A^{\mathbf{i}}(n_\bullet) := \{L \in X(\mu) : P(L) \in \mathcal{P}^{\mathbf{i}}(n_\bullet)\}$. Since each pseudo-Weyl polytope has some $\mathbf{i}$-Lusztig datum, we immediately have the following decomposition of $X(\mu)$ into locally closed subsets.

Proposition 3.1.

$$X(\mu) = \bigsqcup A^{\mathbf{i}}(n_\bullet)$$

where the union is over all $\mathbf{i}$*-Lusztig data* $n_\bullet$ *of coweight* μ*.*

Fix an $\mathbf{i}$-Lusztig datum $n_\bullet$ of coweight μ. Let $\mu_k = \sum_{l=1}^k n_l \beta^{\mathbf{i}}_l$. Suppose that P is a pseudo-Weyl polytope with $\mathbf{i}$-Lusztig datum $n_\bullet$. Then the $w^{\mathbf{i}}_k$ vertices of P are at position μ_k. So if $L \in A^{\mathbf{i}}(n_\bullet)$, then L lies in a GGMS stratum $A(\nu_\bullet)$ with $\nu_{w^{\mathbf{i}}_k} = \mu_k$. This shows that

$$A^{\mathbf{i}}(n_\bullet) = \bigcap_k S^{\mu_k}_{w^{\mathbf{i}}_k}.$$

Let $M_{\gamma^{\mathbf{i}}_k} = \langle \mu_k, \gamma^{\mathbf{i}}_k \rangle$. Then by the length formula (10), we see that $\big(M_\gamma\big)_{\gamma \in \Gamma^{\mathbf{i}}}$ is the unique solution to the system of equations

$$\begin{aligned} n_k &= -M_{w^{\mathbf{i}}_{k-1} \cdot \Lambda_{i_k}} - M_{w^{\mathbf{i}}_k \cdot \Lambda_{i_k}} - \sum_{j \neq i} a_{ji} M_{w^{\mathbf{i}}_k \cdot \Lambda_j} \text{ for all } k, \\ M_{\Lambda_i} &= 0 \text{ for all } i. \end{aligned} \tag{19}$$

This system is upper triangular (note that each $M_{\gamma_k^{\mathbf{i}}}$ shows up for the first time in the equation with n_k on the left hand side) and so such a solution is unique.

By Lemma 2.4 it follows that

$$A^{\mathbf{i}}(n_\bullet) = \{L \in \mathcal{G}r : \mathrm{D}_\gamma(L) = M_\gamma \text{ for all } \mathbf{i}\text{-chamber weights } \gamma\}. \tag{20}$$

Example 3. Continuing as in Example 1, we see that in this case

$$\mu_1 = (n_1, -n_1, 0),\ \mu_2 = (n_1 + n_2, -n_1, -n_2),\ \mu_3 = (n_1 + n_2, n_3 - n_1, -n_2 - n_3),$$
$$n_1 = -M_2,\ n_2 = -M_{23} + M_2,\ n_3 = -M_2 - M_3 + M_{23}.$$

The goal of this section is to prove the following result.

Theorem 3.2. *For each* $\mathbf{i}$*-Lusztig data of coweight* μ*,* $\overline{A^{\mathbf{i}}(n_\bullet)}$ *is an irreducible component of* $\overline{X(\mu)}$*. Moreover each component of* $\overline{X(\mu)}$ *appears exactly once this way.*

The following elementary algebraic geometry lemma will prove quite useful.

Lemma 3.3. *Let* X *be a reducible algebraic set with* n *components. Suppose that* $X = \sqcup C_k$ *is a decomposition of* X *into* n *irreducible locally closed subvarieties. Then* $\overline{C_1}, \dots, \overline{C_n}$ *are the distinct irreducible components of* $\overline{X}$*.*

Proof. Let $A_1, \dots, A_n$ denote the irreducible components of $\overline{X}$. Let $B_i = \overline{C_i}$. Then $\overline{X} = \cup B_i$, so

$$A_j = \bigcup A_j \cap B_i.$$

Since A_j is irreducible and each $A_j \cap B_i$ is closed, $A_j = A_j \cap B_i$ for some i. So $A_j \subset B_i$. By similar reasoning, there exists k such that $B_i \subset A_k$. Hence $A_j \subset B_i \subset A_k$. Since the A_j are the components, each listed once, $j = k$ and so $A_j = B_i$. Continuing this argument shows that there exists a map σ of $\{1, \dots, n\}$ to $\{1, \dots, n\}$ such that $A_j = B_{\sigma(j)}$. This map is injective since the A_j are distinct. Hence it is bijective as desired. ☐

The number of $\mathbf{i}$-Lusztig data of coweight μ is $\mathrm{kpf}(\mu)$ which equals the number of components of $\overline{X(\mu)}$. So prove Theorem 3.2, it suffices to show that $A^{\mathbf{i}}(n_\bullet)$ is irreducible for each Lusztig datum $n_\bullet$. To prove this, we will use another basic algebraic geometry fact, that the image of an irreducible variety is irreducible. Hence our goal is to construct a surjective map from an irreducible variety onto $A^{\mathbf{i}}(n_\bullet)$. To that end, it will be necessary to examine certain parametrizations of N introduced by Lusztig and Berenstein-Fomin-Zelevinsky.

3.3. Parametrizations of N.

Fix $w \in W$. Following Berenstein-Zelevinsky [BZ1], we will define the **twist** automorphism $\eta_w : N \cap B_- w B_- \to N \cap B_- w B_-$. First, let $x \mapsto x^T$ be the involutive Lie algebra anti-automorphism of $\mathfrak{g}$ given by

$$e_i^T = f_i, \quad f_i^T = e_i, \quad h_i^T = h_i,$$

where e_i, f_i, h_i denote the standard Chevalley generators of $\mathfrak{g}$. We use the same notation $g \mapsto g^T$ for the corresponding involutive anti-automorphism of G.

For $y \in N \cap B_- w B_-$, we define $\eta_w(y)$ to be the unique element in the intersection $N \cap B_- \overline{w} y^T$. See [BZ1] for proof that this function is well-defined.

For each $i \in I$, we have the ith map ψ_i of SL_2 into G. Then define $\mathbf{x}_i : \mathbb{C} \to N$ by

$$\mathbf{x}_i(a) = \psi_i\Big(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\Big).$$

Let $\mathbf{i}$ be a reduced word for w. Following [BZ1, FZ], we define regular maps $\mathbf{x_i}$ and $\mathbf{y_i}$ from $(\mathbb{C}^\times)^m$ to N,

$$\mathbf{x_i}(b_1, \ldots, b_m) = \mathbf{x}_{i_m}(b_m) \cdots \mathbf{x}_{i_1}(b_1),$$
$$\mathbf{y_i}(b_1, \ldots, b_m) = \eta_{w^{-1}}^{-1}(\mathbf{x_i}(b_1, \ldots, b_m)).$$

Berenstein-Fomin-Zelevinsky established the following result, which they call the **Chamber Ansatz**, which provides an inverse for y.

Theorem 3.4. *Let $y = \mathbf{y_i}(b_1, \ldots, b_m)$. Then*

$$b_k = \frac{1}{\Delta_{w^{\mathbf{i}}_{k-1} \cdot \Lambda_{i_k}}(y) \Delta_{w^{\mathbf{i}}_k \cdot \Lambda_{i_k}}(y)} \prod_{j \neq i_k} \Delta_{w^{\mathbf{i}}_k \cdot \Lambda_j}(y)^{-a_{j,i_k}} \text{ for all } k. \tag{21}$$

Conversely, $\Delta_\gamma(y)$ is a monomial in the b_k whenever γ is a $\mathbf{i}$-chamber weight.

Moreover, if $w = w_0$, then $\mathbf{y_i}$ is a biregular isomorphism onto $\{g \in N : \Delta_\gamma(g) \neq 0$ for all $\mathbf{i}$-chamber weights $\gamma\}$.

Proof. The first part of this theorem is exactly Theorem 1.4 in [BZ1] and Theorem 2.19 in [FZ], except we have switched the order of the reduced word.

The system (21) is the same as the system (19), except it is written multiplicatively instead of additively. We have already observed that (19) is invertible, hence so is (21) and so $\Delta_\gamma(y)$ is a monomial in the b_k. The explicit form of this monomial is given in Thoerem 4.3 in [BZ1].

To prove the last statement, let $U = \{y \in N : \Delta_\gamma(y) \neq 0$ for all $\mathbf{i}$-chamber weights $\gamma\}$. The first half of the theorem provides a map $U \to (\mathbb{C}^\times)^m$ which is a left inverse to $\mathbf{y_i}$. Hence $\mathbf{y_i}$ is injective.

So it suffices to show that $\mathbf{y_i}$ is surjective. Let $y \in U$ and determine b_k from y by (21). Let $y' = \mathbf{y_i}(b_\bullet)$. By the above observations, the generalized minors Δ_γ take the same values on y, y' for each $\mathbf{i}$-chamber weight γ. But by the results of [BZ1], every function on N is a rational function of the Δ_γ for γ an $\mathbf{i}$-chamber weight. Hence every function on N takes the same values on y' and y. Since N is affine, this shows that $y' = y$ and so $\mathbf{y_i}$ is surjective. □

Example 4. We continue from Example 3. In this case:

$$\mathbf{x_i}(b_1, b_2, b_3) = \begin{bmatrix} 1 & b_1 + b_3 & b_2 b_3 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{y_i}(b_1, b_2, b_3) = \begin{bmatrix} 1 & \frac{1}{b_1} & \frac{1}{b_2 b_3} \\ 0 & 1 & \frac{b_1 + b_3}{b_2 b_3} \\ 0 & 0 & 1 \end{bmatrix}.$$

So

$$b_1 = \frac{1}{\Delta_2(y)}, \; b_2 = \frac{\Delta_2(y)}{\Delta_{23}(y)}, \; b_3 = \frac{\Delta_{23}(y)}{\Delta_2(y)\Delta_3(y)}$$

as in Theorem 3.4.

Note that the map $\mathbf{y_i}$ is a map of varieties over $\mathbb{C}$. By the formal loop space functor, there is a corresponding map of ind-schemes over $\mathbb{C}$, $\mathcal{K}^m \to N(\mathcal{K})$. Moreover, the obvious analogue of Theorem 3.4 holds in this setting.

3.4. **Mapping onto the MV cycles.** Fix a reduced word $\mathbf{i}$ for w_0. Let $n_\bullet$ be a Lusztig datum of coweight μ. Let M_γ be determined from the $n_\bullet$ by (19).

Let

$$B(n_\bullet) := \{(b_1, \dots, b_m) \in \mathcal{K}^m : \mathrm{val}(b_k) = n_k \text{ for all } k\}.$$

The goal of the rest of this section is to prove the following theorem.

Theorem 3.5. *If $b_\bullet \in B(n_\bullet)$, then $[\mathbf{y_i}(b_\bullet)] \in A^{\mathbf{i}}(n_\bullet)$. Moreover, each $L \in A^{\mathbf{i}}(n_\bullet)$ has a representative of the form $\mathbf{y_i}(b_\bullet)$ for some $b_\bullet \in B(n_\bullet)$. Hence the restriction of $\mathbf{y_i}$ to $B(n_\bullet)$ combined with the surjection $G(\mathcal{K})$ to $\mathcal{G}r$ provides a surjective morphism $B(n_\bullet) \to A^{\mathbf{i}}(n_\bullet)$.*

Note that $B(n_\bullet)$ is irreducible, since it is isomorphic to a product of m copies of $\mathbb{C}^\times$ times m copies of $\mathcal{O}$. Hence by the remarks following Lemma 3.3, proving Theorem 3.5 will complete proof of Theorem 3.2.

As a first step towards Theorem 3.5, we establish the following lemma.

Lemma 3.6. *Let $b_\bullet \in \mathcal{K}^m$. Let $y = \mathbf{y_i}(b_\bullet)$.*

Then $b_\bullet \in B(n_\bullet)$ if and only if $\mathrm{val}(\Delta_\gamma(y)) = M_\gamma$ for all $\mathbf{i}$-chamber weights γ.

Proof. By Theorem 3.4, we see that

$$\mathrm{val}(b_k) = -\mathrm{val}(\Delta_{w^{\mathbf{i}}_{k-1}\cdot\Lambda_{i_k}}(y)) - \mathrm{val}(\Delta_{w^{\mathbf{i}}_k\cdot\Lambda_{i_k}}(y)) - \sum_{j \neq i_k} \mathrm{val}(\Delta_{w^{\mathbf{i}}_k\cdot\Lambda_j}(y)) \tag{22}$$

for all k. Also since $y \in N(\mathcal{K})$, $\Delta_{\Lambda_i}(y) = 1$ and so $\mathrm{val}(\Delta_{\Lambda_i}(y)) = 0$ for all i.

This is the same system of equations as (19), with $\mathrm{val}(b_k)$ instead of n_k and $\mathrm{val}(\Delta_\gamma(y))$ instead of M_γ. Since (19) is an invertible linear system, this shows that $\mathrm{val}(b_k) = n_k$ for all k if and only if $\mathrm{val}(\Delta_\gamma(y)) = M_\gamma$ for all $\mathbf{i}$-chamber weights γ. ☐

3.5. **Off-minors.** To complete the proof of Theorem 3.5, we will need a further investigation of relation between the function D_γ and the valuation of Δ_γ.

Let U be a finite-dimensional vector space over $\mathbb{C}$. Earlier, we defined a function $\mathrm{val} : U \otimes \mathcal{K} \to \mathbb{Z}$. Note that if $u \in U \otimes \mathcal{K}$, then

$$\mathrm{val}(u) = \min_{\xi \in U^\star} \mathrm{val}(\langle u, \xi \rangle),$$

where on the right, val denotes the usual valuation map on $\mathcal{K}$. In fact, it is enough to take the min over a basis for $U^\star$.

Let us apply the above result to our situation. We see that if γ is a chamber weight of level i, then

$$\mathrm{D}_\gamma([y]) = \mathrm{val}(y \cdot v_\gamma) = \min_{\xi \in V^\star_{\Lambda_i}} \mathrm{val}(\langle y \cdot v_\gamma, \xi \rangle). \tag{23}$$

In particular, $\xi = v_{-\Lambda_i}$ shows up on the right hand side and so $\mathrm{val}(\Delta_\gamma(y))$ appears in the min (see (14)). We would like to show that the minimum is attained there when $y = \mathbf{y_i}(b_\bullet)$ and $b_\bullet \in B(n_\bullet)$.

Using a Bruhat decomposition of $G(\mathcal{K})$ it is possible to show that we need to take only extremal weight vectors ξ in the min above. However, we will not need this.

We call $\langle y \cdot v_\gamma, \xi \rangle$ an **off-minor** of y. In the case $G = SL_n$ it is the minor of y using γ as the set of columns and ξ as the set of rows (where we identify the usual basis for $V^\star_{\Lambda_i}$ with i element subsets of $\{1, \dots, n\}$).

The following lemma is a generalization of Lemma 3.1.3 from [BFZ], which dealt with the case $G = SL_n$.

Lemma 3.7. *Let $w \in W$. Let $\xi \in V_{\Lambda_i}^\star$. Let $x \in N \cap B_- w^{-1} B_-$ and $y = \eta_{w^{-1}}(x)$. Then*

$$\frac{\langle y \cdot v_{w \cdot \Lambda_i}, \xi \rangle}{\Delta_{w\cdot\Lambda_i}(y)} = \frac{\langle x^T \cdot v_{\Lambda_i}, \xi\rangle}{\langle v_{\Lambda_i}, v_{-\Lambda_i}\rangle}$$

Proof. Since $x = \eta_{w^{-1}}(y)$, there exists $p \in N_-$ and $d \in T$ such that $pdx = \overline{w^{-1}} y^T$. Note that $\overline{w}^{-1} = \overline{w}^T$ (since $\overline{s_i}^{-1} = \overline{s_i}^T$ by an SL_2 calculation). Hence, $y = x^T d^T p^T \overline{w^{-1}}$, and so

$$\langle y \cdot v_{w\cdot\Lambda_i}, \xi\rangle = \langle x^T d^T p^T \overline{w^{-1}} \cdot v_{w\cdot\Lambda_i}, \xi\rangle = \Lambda_i(rd)\langle x^T \cdot v_{\Lambda_i}, \xi\rangle, \tag{24}$$

where $r = \overline{w^{-1}}\,\overline{w} \in T$.

Similarly,

$$\langle y \cdot v_{w\cdot\Lambda_i}, v_{-\Lambda_i}\rangle = \Lambda_i(rd)\langle x^T \cdot v_{\Lambda_i}, v_{-\Lambda_i}\rangle = \Lambda_i(rd)\langle v_{\Lambda_i}, (x^T)^{-1} \cdot v_{-\Lambda_i}\rangle = \Lambda_i(rd)\langle v_{\Lambda_i}, v_{-\Lambda_i}\rangle \tag{25}$$

since $x^T \in N_-$, so $(x^T)^{-1} \in N_-$ and hence $(x^T)^{-1} \cdot v_{-\Lambda_i} = v_{-\Lambda_i}$.

Taking the ratio of (24) and (25) gives the desired result. □

This result allows us to express certain off-minors of y in terms of x. To express them all, we will also need the following lemma from Berenstein-Zelevinsky.

Lemma 3.8 ([BZ1, Proposition 5.4])**.** *Let $\mathbf{i}$ be a reduced word for w_0, let $1 \le k \le m$, let $w = w_k^{\mathbf{i}}$, and let $y = \mathbf{y_i}(b_1, \dots, b_m)$. Then y admits a factorization $y = y'y''$ where $y' = \mathbf{y}_{(i_1,\dots,i_k)}(b_1,\dots,b_k)$, and $y'' \in wNw^{-1}$.*

These last two lemmas combine in the following result describing the off minors.

Proposition 3.9. *Let $\mathbf{i}$ be a reduced word for w_0, let $\xi \in V_{\Lambda_i}^\star$, and let γ be an $\mathbf{i}$-chamber weight of level i. Let $y = \mathbf{y_i}(b_1, \dots, b_m)$. Then*

$$\frac{\langle y \cdot v_\gamma, \xi\rangle}{\Delta_\gamma(y)}$$

is a polynomial in the b_k.

Proof. Since γ is an $\mathbf{i}$-chamber weight, $\gamma = w_k^{\mathbf{i}} \cdot \Lambda_i$ for some k. Let $w = w_k^{\mathbf{i}}$. By the previous lemma, $y = y'y''$, where $y' = \mathbf{y}_{(i_1,\dots,i_k)}(b_1,\dots,b_k)$ and $y'' \in wNw^{-1}$.

Then $y \cdot v_\gamma = y'y'' \cdot v_\gamma = y' \cdot v_\gamma$ since $\gamma = w \cdot \Lambda_i$ and $y'' \in wNw^{-1}$.

So

$$\frac{\langle y \cdot v_\gamma, \xi\rangle}{\Delta_\gamma(y)} = \frac{\langle y' \cdot v_\gamma, \xi\rangle}{\Delta_\gamma(y')} = \frac{\langle {x'}^T \cdot v_{\Lambda_i}, \xi\rangle}{\langle v_{\Lambda_i}, v_{-\Lambda_i}\rangle},$$

where $x' = \mathbf{x}_{(i_1,\dots,i_k)}(b_1, \dots b_k)$. The first equality is by the above analysis and the second is by Lemma 3.7.

Any regular function of ${x'}^T$ is a polynomial in the b_k (since the extension of $\mathbf{x}_{(i_1,\dots,i_k)}$ to $\mathbb{C}^k$ is regular) and so the result follows. □

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. First, we will show that if $b_\bullet \in B(n_\bullet)$, then $[\mathbf{y_i}(b_\bullet)] \in A^{\mathbf{i}}(n_\bullet)$.

Fix $b_\bullet \in B(n_\bullet)$ and let $y = \mathbf{y_i}(b_\bullet)$.

By (20), $[y] \in A^{\mathbf{i}}(n_\bullet)$ if $\mathrm{D}_\gamma([y]) = M_\gamma$ for all $\mathbf{i}$-chamber weights γ.

By Lemma 3.6, $\mathrm{val}(\Delta_\gamma(y)) = M_\gamma$. So to prove that $[y] \in A^{\mathbf{i}}(n_\bullet)$, it suffices to show that $\mathrm{val}(\Delta_\gamma(y)) = \mathrm{D}_\gamma([y])$.

By (23), it suffices to show that $\mathrm{val}(\langle y \cdot v_\gamma, \xi\rangle) \geq \mathrm{val}(\Delta_\gamma(y))$ for any $\xi \in V_{\Lambda_i}^\star$. By Proposition 3.9,

$$\frac{\langle y \cdot v_\gamma, \xi\rangle}{\Delta_\gamma(y)} = P(b_1, \dots, b_m)$$

for some polynomial P. But $\mathrm{val}(b_k) = n_k \geq 0$ for all k, so $\mathrm{val}(P(b_1, \dots, b_m)) \geq 0$. Hence $\mathrm{val}(\langle y \cdot v_\gamma, \xi\rangle) - \mathrm{val}(\Delta_\gamma(y)) \geq 0$ as desired.

So we conclude that $[y] \in A^{\mathbf{i}}(n_\bullet)$, as desired.

Next, we need to check that if $L \in A^{\mathbf{i}}(n_\bullet)$, then $L = [\mathbf{y_i}(b_\bullet)]$ for some $b_\bullet \in B(n_\bullet)$. Suppose we know that there exists $y \in N(\mathcal{K})$ such that $L = [y]$ and $\mathrm{val}(\Delta_\gamma(y)) = \mathrm{D}_\gamma(L)$ for all γ. Then if γ is a $\mathbf{i}$-chamber weight, by (20) $\mathrm{D}_\gamma(L) = M_\gamma$, so $\mathrm{val}(\Delta_\gamma(y)) = M_\gamma$. In particular, $\Delta_\gamma(y)$ is non-zero for all $\mathbf{i}$-chamber weights γ. Hence by Theorem 3.4, there exist $(b_1, \dots, b_m) \in (\mathcal{K}^\times)^m$ such that $y = \mathbf{y_i}(b_1, \dots, b_m)$. By Lemma 3.6, we see that $b_k \in B(n_\bullet)$ as desired. Hence this completes the proof of the theorem.

So now we will prove the existence of y as above. Since $A(n_\bullet) \subset S_1^0$, L has a representative $g \in N(\mathcal{K})$. Let $h \in N(\mathbb{C})$. So $[h^{-1}g] = [g] = L$. We would like to find h such that $\mathrm{val}(\Delta_\gamma(h^{-1}g)) = \mathrm{D}_\gamma(L)$ for all chamber weights γ.

Let γ be a chamber weight of level i and let $d = \mathrm{D}_\gamma([g])$. Let $u_1, \dots, u_N$ be a basis for V_{Λ_i} with dual basis $u_1^\star, \dots, u_N^\star$ for $V_{\Lambda_i}^\star$.

Then

$$\Delta_\gamma(h^{-1}g) = \langle h^{-1}g \cdot v_\gamma, v_{-\Lambda_i}\rangle = \langle g \cdot v_\gamma, h \cdot v_{-\Lambda_i}\rangle.$$

Now $h \cdot v_{-\Lambda_i} = \sum_s c_s u_s^\star$ for some $c_s \in \mathbb{C}$. Hence

$$\Delta_\gamma(h^{-1}g) = \sum_s c_s \langle g \cdot v_\gamma, u_s^\star\rangle.$$

Let p_s be the coefficient of t^d in $\langle g \cdot v_\gamma, u_s^\star\rangle$. Since $d = \mathrm{D}_\gamma([g]) = \min_s \mathrm{val}(\langle g \cdot v_\gamma, u_s^\star\rangle)$, we see that p_s is nonzero for some s. Extracting the coefficient of t^d from the above equation shows that $\mathrm{val}(\Delta_\gamma(h^{-1}g)) = d$ if and only if $\sum_s p_s c_s \neq 0$.

Now, $c_s(h) = \langle u_s, h \cdot v_{-\Lambda_i}\rangle$. So finding h, such that $\mathrm{val}(\Delta_\gamma(h^{-1}g)) = \mathrm{D}_\gamma(L)$ for all chamber weights γ, is equivalent to finding h such that none of the corresponding linear forms $\sum_s p_s c_s$ vanish. It is well known that the ring of functions on N is an integral domain, is generated by the functions c_s, and that all relations are quadratic in these matrix coefficients. Hence the product of the linear forms $\sum_s p_s c_s$ does not vanish identically on N. So we can choose h as desired.

Thus, we can choose a representative $y \in N(\mathcal{K})$ for L such that $\mathrm{val}(\Delta_\gamma(y)) = \mathrm{D}_\gamma(L)$ for all γ. □

4. Piecing together

Thanks to Theorem 3.2, we now have a decomposition of $X(\mu)$ for each $\mathbf{i}$-chamber weight. To complete the proof of Theorem 2.5, it will be necessary to understand how these different decompositions fit together. In this section, a reduced word will always mean a reduced word for w_0.

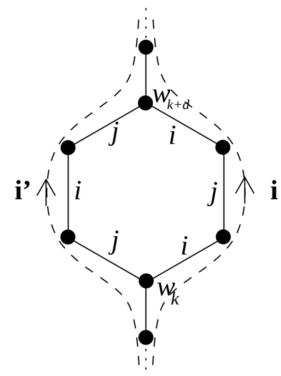


FIGURE 2. Two reduced words related by a 3-move.

4.1. **Local picture.** Two reduced words $\mathbf{i}, \mathbf{i}'$ are said to be related by a d-**move** involving i, j, starting at position k, if

$$\mathbf{i} = (\ldots, i_k, i, j, i, \ldots, i_{k+d+1}, \ldots),$$
$$\mathbf{i}' = (\ldots, i_k, j, i, j, \ldots, i_{k+d+1}, \ldots),$$

where d is the order of $s_i s_j$.

Recall that reduced words correspond to paths from the 1 vertex to the w_0 vertex of the permutahedron. If $\mathbf{i}, \mathbf{i}'$ are related as above, then $w_l^{\mathbf{i}} = w_l^{\mathbf{i}'}$, for $l \notin \{k+1, \ldots, k+d-1\}$. So the two paths agree for the first k vertices and then agree again at vertex $k+d$ and later. Moreover, the $w_l^{\mathbf{i}}$ and $w_l^{\mathbf{i}'}$ vertices for $l \in \{k, \ldots, k+d\}$ all lie on the same 2-face of the permutahedron. Namely, they lie on the 2-face which contains w vertex and is dual to the chamber weights $w \cdot \Lambda_p$ for $p \neq i, j$, where $w = w_k^{\mathbf{i}}$. This 2-face will be a 2d-gon (see Figure 2).

Following Lusztig, Berenstein-Zelevinsky studied the relationship between $\mathbf{y_i}$ and $\mathbf{y_{i'}}$.

Proposition 4.1 ([BZ1, Theorem 3.1]). *Let $\mathbf{i}, \mathbf{i}'$ be as above. Suppose that $\mathbf{y_i}(b_\bullet) = \mathbf{y_{i'}}(b'_\bullet)$.*

For $l \notin \{k+1, \ldots, k+d\}$, $b_l = b'_l$. For other l we have the following case by case formulas.

(i) *If $a_{ij} = 0$, so $d = 2$. Then*

$$b'_{k+1} = b_{k+2}, \; b'_{k+2} = b_{k+1}.$$

(ii) *If $a_{ij} = a_{ji} = -1$, so $d = 3$. Then*

$$b'_{k+1} = \frac{b_{k+2} b_{k+3}}{\pi}, \; b'_{k+2} = b_{k+1} + b_{k+3}, \; b'_{k+3} = \frac{b_{k+1} b_{k+2}}{\pi}, \quad \text{where } \; \pi = b_{k+1} + b_{k+3}. \tag{26}$$

(iii) *If $a_{ij} = -1, a_{ji} = -2$, so $d = 4$. Then*

$$b'_{k+1} = \frac{b_{k+2} b_{k+3} b_{k+4}}{\pi_1}, \; b'_{k+2} = \frac{\pi_1^2}{\pi_2}, \; b'_{k+3} = \frac{\pi_2}{\pi_1}, \; b'_{k+4} = \frac{b_{k+1} b_{k+2}^2 b_{k+3}}{\pi_2}, \quad \text{where } \; \pi_1 = b_{k+1} b_{k+2} + (b_{k+1} + b_{k+3}) b_{k+4}, \; \pi_2 = b_{k+1}(b_{k+2} + b_{k+4})^2 + b_{k+3} b_{k+4}^2. \tag{27}$$

(iv) *If* $a_{ij} = -2, a_{ji} = -1$, *so* $d = 4$. *Then*

$$b'_{k+1} = \frac{b_{k+2}b^2_{k+3}b_{k+4}}{\pi_2},\ b'_{k+2} = \frac{\pi_2}{\pi_1},\ b'_{k+3} = \frac{\pi_1^2}{\pi_2},\ b'_{k+4} = \frac{b_{k+1}b_{k+2}b_{k+3}}{\pi_1}, \tag{28}$$
$$\text{where}\ \ \pi_1 = b_{k+1}b_{k+2} + (b_{k+1} + b_{k+3})b_{k+4},\ \ \pi_2 = b^2_{k+1}b_{k+2} + (b_{k+1} + b_{k+3})^2 b_{k+4}.$$

Conversely, suppose that $b_\bullet \in (\mathbb{C}^\times)^m$ *is such that the denominators in the above expressions do not vanish. Define* $b'_\bullet$ *by the above expressions. Then* $\mathbf{y}_\mathbf{i}(b_\bullet) = \mathbf{y}_{\mathbf{i}'}(b'_\bullet)$.

The first part of this proposition is directly from [BZ1]. The last statement follows from the same reasoning as in our proof of the second statement of Theorem 3.4.

Note that this proposition holds over $\mathcal{K}$ as well.

Let $n_\bullet$ be an $\mathbf{i}$-Lusztig datum of coweight μ.

Proposition 4.2. *There exists a non-empty open subset* U *of* $B(n_\bullet)$ *such that for each* $b_\bullet \in U$, *there exists* $b'_\bullet \in \mathcal{K}^m$ *such that* $\mathbf{y}_\mathbf{i}(b_\bullet) = \mathbf{y}_{\mathbf{i}'}(b'_\bullet)$ *and the following formulas holds for* $n'_l := \mathrm{val}(b'_l)$.

(i) *If* $a_{ij} = 0$, *so* $d = 2$. *Then*

$$n'_{k+1} = n_{k+2},\ n'_{k+2} = n_{k+1}.$$

(ii) *If* $a_{ij} = a_{ji} = -1$, *so* $d = 3$. *Then*

$$n'_{k+1} = n_{k+2} + n_{k+3} - p,\ n'_{k+2} = p,\ n'_{k+3} = n_{k+1} + n_{k+2} - p, \tag{29}$$
$$\text{where}\ \ p = \min(n_{k+1}, n_{k+3}).$$

(iii) *If* $a_{ij} = -1, a_{ji} = -2$, *so* $d = 4$. *Then*

$$\begin{aligned} n'_{k+1} &= n_{k+2} + n_{k+3} + n_{k+4} - p_1,\ n'_{k+2} = 2p_1 - p_2,\\ n'_{k+3} &= p_2 - p_1,\ n'_{k+4} = n_{k+1} + 2n_{k+2} + n_{k+3} - p_2\\ \text{where}\ \ p_1 &= \min(n_{k+1} + n_{k+2}, n_{k+1} + n_{k+4}, n_{k+3} + n_{k+4}),\\ p_2 &= \min(n_{k+1} + 2n_{k+2}, n_{k+1} + 2n_{k+4}, n_{k+3} + 2n_{k+4}). \end{aligned} \tag{30}$$

(iv) *If* $a_{ij} = -2, a_{ji} = -1$, *so* $d = 4$. *Then*

$$\begin{aligned} n'_{k+1} &= n_{k+2} + 2n_{k+3} + n_{k+4} - p_2,\ n'_{k+2} = p_2 - p_1,\\ n'_{k+3} &= 2p_1 - p_2,\ n'_{k+4} = n_{k+1} + n_{k+2} + n_{k+3} - p_1\\ \text{where}\ \ p_1 &= \min(n_{k+1} + n_{k+2}, n_{k+1} + n_{k+4}, n_{k+3} + n_{k+4}),\\ p_2 &= \min(2n_{k+1} + n_{k+2}, 2n_{k+1} + n_{k+4}, 2n_{k+3} + n_{k+4}). \end{aligned} \tag{31}$$

Proof. If $a_{ij} = 0$ then the result holds with $U = B(n_\bullet)$.

Suppose that $a_{ij} = a_{ji} = -1$. Let

$$U := \{b_\bullet \in B(n_\bullet) : b^0_{k+1} + b^0_{k+3} \neq 0\},$$

where b^0_l is the coefficient t^{n_l} in b_l.

If $b_\bullet \in U$, then let $b'_\bullet, \pi$ be determined from $b_\bullet$ by (26). Since $\pi = b_{k+1} + b_{k+3}$, $\mathrm{val}(\pi) = p$ as the leading terms of b_{k+1} and b_{k+3} don't cancel. In particular, the denominator π doesn't vanish. Hence if $b'_\bullet$ is given by (26), then by Proposition 4.1, $\mathbf{y}_{\mathbf{i}'}(b'_\bullet) = \mathbf{y}_\mathbf{i}(b_\bullet)$. Moreover, the valuation of the b'_l are given by (29), since $\mathrm{val}(\pi) = p$.

The other cases follow similarly. □

Now, we transport our results from $G(\mathcal{K})$ to $\mathcal{G}r$. Let $n'_\bullet$ be the sequence of integers obtained from the $n_\bullet$ by the formulas in Proposition 4.2. It is easy to see that $n'_\bullet$ is an $\mathbf{i}'$-Lusztig datum of coweight μ.

Theorem 4.3. *The intersection $A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet)$ is open and dense in $A^{\mathbf{i}}(n_\bullet)$.*

Proof. Let U be the non-empty open subset of $B(n_\bullet)$ from Proposition 4.2. Since the map from $B(n_\bullet)$ to $A^{\mathbf{i}}(n_\bullet)$ is surjective (Theorem 3.5), the set $Y = \{[\mathbf{y_i}(b_\bullet)] : b_\bullet \in U\}$ is dense in $A^{\mathbf{i}}(n_\bullet)$. By Proposition 4.1, if $L \in Y$, then L has a representative $\mathbf{y}_{\mathbf{i}'}(b'_\bullet)$ for $b'_\bullet \in B(n'_\bullet)$. Hence by Theorem 3.5, $Y \subset A^{\mathbf{i}'}(n'_\bullet)$. Hence the interesection is dense. Also, it is locally closed, hence it is open. □

The reader familiar with parametrizations of the canonical basis will notice that we have proven that the transformation between $\mathbf{i}$, $\mathbf{i}'$-Lusztig data for MV cycles matches the transformation between $\mathbf{i}$, $\mathbf{i}'$-Lusztig data for the canonical basis (see Theorem 5.2 in [BZ2]). Hence from this stage, we could apply the machinery of [BZ2] to prove our main theorem. However in our case, there is a simpler, more geometric approach available which we will now explain.

Note that the tropical Plücker relation (15), (16), (17) at $(w = w_k^{\mathbf{i}}, i, j)$ only involves M_γ for γ an $\mathbf{i}$ or $\mathbf{i}'$-chamber weight. This observation leads to the following result.

Proposition 4.4. *Let $L \in A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet)$. Then the collection $\big(M_\gamma := \mathrm{D}_\gamma(L)\big)_{\gamma \in \Gamma^{\mathbf{i}} \cup \Gamma^{\mathbf{i}'}}$ satisfies the tropical Plücker relation at (w, i, j).*

Proof. If $L \in A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet)$, then we know $\mathrm{D}_\gamma(L)$ for γ an $\mathbf{i}$ or $\mathbf{i}'$-chamber weight. Since these are the only chamber weights which show up in the tropical Plücker relation, we just need to make a simple computation to check that the relation between $n_\bullet$ and $n'_\bullet$ in Proposition 4.2 matches the tropical Plücker relation at (w, i, j).

The case $d = 2$ is trivial because there is no tropical Plücker relation (in fact, in this case $\Gamma^{\mathbf{i}} = \Gamma^{\mathbf{i}'}$).

Consider the case $a_{ij} = a_{ji} = -1$. Then by the length formula (19),

$$n'_{k+2} = -M_{w\cdot\Lambda_i} - M_{ws_js_i\cdot\Lambda_i} + M_{ws_j\cdot\Lambda_j} - \sum_{l\neq i,j} a_{li} M_{w\cdot\Lambda_l},$$

$$n_{k+1} = -M_{w\cdot\Lambda_i} - M_{ws_i\cdot\Lambda_i} + M_{w\cdot\Lambda_j} - \sum_{l\neq i,j} a_{li} M_{w\cdot\Lambda_l},$$

$$n_{k+3} = -M_{ws_i\cdot\Lambda_i} - M_{ws_js_i\cdot\Lambda_i} + M_{ws_is_j\cdot\Lambda_j} - \sum_{l\neq i,j} a_{li} M_{w\cdot\Lambda_l}.$$

By (29), $n'_{k+2} = \min(n_{k+1}, n_{k+3})$. Substituting the above expressing into this equation gives

$$-M_{w\cdot\Lambda_i} - M_{ws_js_i\cdot\Lambda_i} + M_{ws_j\cdot\Lambda_j} = \min(-M_{w\cdot\Lambda_i} - M_{ws_i\cdot\Lambda_i} + M_{w\cdot\Lambda_j}, -M_{ws_i\cdot\Lambda_i} - M_{ws_js_i\cdot\Lambda_i} + M_{ws_is_j\cdot\Lambda_j})$$

which is equivalent to the tropical Plücker relation (15).

The other cases are similar. □

It is easy to see that the converse of this Proposition holds, but we will not need this.

4.2. **Global picture.** We are now ready to prove the main theorem. Let $\mu \geq 0$ be a coweight.

Let $\mathbf{i}, \mathbf{i}'$ be two reduced words related by a d-move involving i, j, starting at position k. Let $L \in X(\mu)$ and let $n_\bullet$, $n'_\bullet$ be the $\mathbf{i}, \mathbf{i}'$-Lusztig data of $P(L)$. So $L \in A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet)$. We say that L is $\mathbf{i}, \mathbf{i}'$**-generic** if $n_\bullet$ and $n'_\bullet$ are related as in Proposition 4.2. By Proposition 4.4, if L is $\mathbf{i}, \mathbf{i}'$-generic, then $\mathrm{D}_\bullet(L)$ satisfies the tropical Plücker relation at $(w_k^{\mathbf{i}}, i, j)$.

We say that $L \in X(\mu)$ is **generic** if L is $\mathbf{i}, \mathbf{i}'$-generic for every pair of reduced words $\mathbf{i}, \mathbf{i}'$ related by a d-move.

If $w \in W, i, j \in I$ are such that $ws_i > w$ and $ws_j > w$, then there exist a pair of reduced words $\mathbf{i}, \mathbf{i}'$ related by a d-move starting at position k, involving i, j such that $w_k^{\mathbf{i}} = w$. Visually, for any 2-face in the permutahedron, there exist reduced words $\mathbf{i}, \mathbf{i}'$ such that this 2-face is the transition between them. Hence if L is generic, then $\mathrm{D}_\bullet(L)$ satisfies all the tropical Plücker relations.

Proof of Theorem 2.5. Let $\mu \geq 0$ be a coweight and let $M_\bullet$ be a BZ datum of coweight μ.

Let $\mathbf{i}$ be a reduced word for w_0. Let $n_\bullet$ be the $\mathbf{i}$-Lusztig datum corresponding to $M_\bullet$ under (19).

We claim that $\{L \in A^{\mathbf{i}}(n_\bullet) : L \text{ is generic } \}$ is dense in $A^{\mathbf{i}}(n_\bullet)$.

To prove this, for any reduced word $\mathbf{j}$ and any $\mathbf{j}$-Lusztig datum $m_\bullet$, define $A_k^{\mathbf{j}}(m_\bullet)$ recursively by $A_0^{\mathbf{j}}(m_\bullet) := A^{\mathbf{j}}(m_\bullet)$ and

$$A_k^{\mathbf{j}}(m_\bullet) := A_{k-1}^{\mathbf{j}}(m_\bullet) \cap \bigcap_{\mathbf{j}'} A_{k-1}^{\mathbf{j}'}(m'_\bullet),$$

where the intersection is over all reduced words $\mathbf{j}'$ which are related to $\mathbf{j}$ by a d-move and where $m'_\bullet$ is the $\mathbf{j}'$-Lusztig datum corresponding to $m_\bullet$ under Proposition 4.2.

We claim that for each k, $A_k^{\mathbf{j}}(m_\bullet)$ is open dense in $A_{k-1}^{\mathbf{j}}(m_\bullet)$ and in each $A_{k-1}^{\mathbf{j}'}(m'_\bullet)$ whenever $\mathbf{j}$ and $\mathbf{j}'$ are related by a d-move and $m_\bullet$ and $m'_\bullet$ are related as in Proposition 4.2. We proceed by induction.

By Theorem 4.3, $A^{\mathbf{j}}(m_\bullet) \cap A^{\mathbf{j}'}(m'_\bullet)$ is open dense in $A^{\mathbf{j}}(m_\bullet)$ and $A^{\mathbf{j}'}(m'_\bullet)$. So $A_1^{\mathbf{j}}(m_\bullet)$ is the intersection of subsets of $A^{\mathbf{j}}(m_\bullet)$ which are open dense in $A^{\mathbf{j}}(m_\bullet)$ and hence it is open dense in $A^{\mathbf{j}}(m_\bullet)$. This also shows that $A_1^{\mathbf{j}}(m_\bullet)$ is open dense in $A^{\mathbf{j}}(m_\bullet) \cap A^{\mathbf{j}'}(m'_\bullet)$ and hence in $A^{\mathbf{j}'}(m'_\bullet)$. This establishes the base case.

By the inductive hypothesis, $A_k^{\mathbf{j}}(m_\bullet)$ and $A_k^{\mathbf{j}'}(m'_\bullet)$ are each open dense in each of $A_{k-1}^{\mathbf{j}}(m_\bullet)$ and $A_{k-1}^{\mathbf{j}'}(m'_\bullet)$. Hence $A_k^{\mathbf{j}}(m_\bullet) \cap A_k^{\mathbf{j}'}(m'_\bullet)$ is open dense in $A_{k-1}^{\mathbf{j}}(m_\bullet) \cap A_{k-1}^{\mathbf{j}'}(m_\bullet)$ and so $A_k^{\mathbf{j}}(m_\bullet) \cap A_k^{\mathbf{j}'}(m'_\bullet)$ is open dense in $A_k^{\mathbf{j}}(m_\bullet)$ and in $A_k^{\mathbf{j}}(m'_\bullet)$. From here, the inductive step follows the same reasoning as the base case.

In these arguments, we are repeatedly using the fact that if $U \subset V \subset X$ and if U is open dense in X, then U is open dense in V.

Now, specialize to $\mathbf{j} = \mathbf{i}, m_\bullet = n_\bullet$. Let $\mathbf{j}, \mathbf{j}'$ be two reduced words which are related by a d-move and such that $\mathbf{j}$ is connected to $\mathbf{i}$ by less than k d-moves. Suppose that $L \in A_k^{\mathbf{i}}(n_\bullet)$, then by induction on k, we see that L is $\mathbf{j}, \mathbf{j}'$-generic. Hence if k is larger than the largest number of d-moves needed to connect any two reduced words, then $A_k^{\mathbf{i}}(n_\bullet) \subset \{L \in A^{\mathbf{i}}(n_\bullet) : L \text{ is generic } \}$. By a chain of dense inclusions, we see that $A_k^{\mathbf{i}}(n_\bullet)$ is dense in $A^{\mathbf{i}}(n_\bullet)$ and hence $\{L \in A^{\mathbf{i}}(n_\bullet) : L \text{ is generic } \}$ is dense is $A^{\mathbf{i}}(n_\bullet)$.

If $L \in A^{\mathbf{i}}(n_\bullet)$ is generic, then $\mathrm{D}_\bullet(L)$ and $M_\bullet$ both obey the tropical Plücker relations. Moreover, they have the same values whenever γ is an $\mathbf{i}$-chamber weight. Suppose that $\mathbf{i}'$ is another reduced word, related to $\mathbf{i}$ by a d-move involving i, j starting at position k. Then since both obey the tropical Plücker relation for $(w_k^{\mathbf{i}}, i, j)$, we see that $\mathrm{D}_\gamma(L) = M_\gamma$ whenever γ is a $\mathbf{i}'$-chamber weight as well. Continuing this argument (and using the fact that any reduced word is connected to $\mathbf{i}$ by a sequence of d-moves), we see that $\mathrm{D}_\gamma(L) = M_\gamma$ for all chamber weights γ. So $L \in A(M_\bullet)$.

Hence we see that

$$\overline{\{L \in A^{\mathbf{i}}(n_\bullet) : L \text{ is generic } \}} = \overline{A(M_\bullet)} = \overline{A^{\mathbf{i}}(n_\bullet)}.$$

By Theorem 3.2, $\overline{A^{\mathbf{i}}(n_\bullet)}$ is a component of $X(\mu)$, so $\overline{A(M_\bullet)}$ is a component. So $\overline{A(M_\bullet)}$ is an MV cycle of coweight μ.

Conversely, if Z is a component of $\overline{X(\mu)}$, then $Z = \overline{A^{\mathbf{i}}(n_\bullet)}$ for some $n_\bullet$ by Theorem 3.2. Let $L \in A^{\mathbf{i}}(n_\bullet)$ be generic. By the above analysis $Z = \overline{A(M_\bullet)}$. Since L is generic, $(M_\gamma = \mathrm{D}_\gamma(L))$ satisfies the tropical Plücker relations. Also $P(L) = P(M_\bullet)$ is a pseudo-Weyl polytope, so $M_\bullet$ satisfies the non-degeneracy inequalities. Finally, $M_{\Lambda_i} = 0$ for all i, since $L \in X(\mu) \subset S_1^0$. Hence $M_\bullet$ is a BZ datum. So all MV cycles are of the desired form.

□

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