Handout 4 ?

Notation

Let V, W be vector spaces / $\mathbb{F}$
$T: V \to W$ be a linear transformation,
Let $\underline{v} \in V$, $\underline{w} \in W$. Let $B = (\underline{v}_1, \ldots, \underline{v}_n)$ be a basis of V and $C = (\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_m)$ be a basis of W.

- If $\underline{v} = \sum a_i \underline{v}_i$ we write $[\underline{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ for the coordinate vector of v wrt B.

Notice $[\underline{v}_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $[\underline{v}_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$ $[\underline{v}_n]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

- Let $[T]_B^C$ be the matrix such that
$$[T]_B^C [\underline{v}]_B = [T(\underline{v})]_C \quad \forall \underline{v} \in V.$$

Claim
$$[T]_B^C = \begin{bmatrix} | & | & & | \\ [T(v_1)]_C & [T(v_2)]_C & \cdots & [T(v_n)]_C \\ | & | & & | \end{bmatrix}$$

Pf

We must have $[T]_B^C [\underline{v}_i]_B = [T(\underline{v}_i)]_C$.

On the other hand, $[\underline{v}_i]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i^{th}$ row, and so $[T]_B^C \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = i^{th}$ column of matrix.

This means $[T]_B^C$ is the matrix above.

- Change of basis. Let $I : V \to V$ be the identity map. Let D be another basis of V.

Then, by the above
$$[I]_B^D [\underline{v}]_B = [I(\underline{v})]_D = [\underline{v}]_D$$

So $P = [I]_B^D$ is the <u>change of basis</u> matrix from the basis B to the basis C.

Also by the above $P = \begin{bmatrix} | & | & & | \\ [\underline{v}_1]_D & [\underline{v}_2]_D & \cdots & [\underline{v}_n]_D \\ | & | & & | \end{bmatrix}$

- <u>ex'l</u> Let $V = \mathbb{R}^2$, $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, $D = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ the standard basis.

Then $[I]_B^S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $[I]_D^S = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$

$$[I]_B^D = [I]_S^D [I]_B^S = \left([I]_D^S\right)^{-1} [I]_B^S$$

A short cut is to row reduce $[D \mid B] \rightsquigarrow [I \mid [I]_B^D]$.

i.e. $\left[\begin{array}{cc|cc} 3 & -1 & 1 & 1 \\ 4 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -1/3 & 1/3 & 1/3 \\ 0 & 10/3 & -1/3 & -4/3 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 3/10 & 1/5 \\ 0 & 1 & -1/10 & -2/5 \end{array}\right]$

$\underbrace{\quad}_{= [I]_B^D}$

- <u>Exercise</u>: Verify for $T : V \to W$ $\quad [I]_C^{C'} [T]_B^C [I]_{B'}^B = [T]_{B'}^{C'}$.