A BIJECTION BETWEEN BOUNDED DOMINANT SHI REGIONS AND CORE PARTITIONS

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ABSTRACT. It is well-known that Catalan numbers $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$ count the number of bounded dominant regions in the Shi arrangement of type A_{n-1}, and that they also count partitions which are both n-cores as well as $(n-1)$-cores. These concepts all have natural extensions which share the same relationship. In this paper, we construct a bijection between bounded dominant regions of the m-Shi arrangement and partitions which are both n-cores as well as $(mn-1)$-cores. This result builds on the work in [7], and the reader is referred there for several details. The bijection is natural in the sense that it commutes with the action of the affine symmetric group.

1. INTRODUCTION

Let Δ be the root system of type A_{n-1}, with Weyl group W, and let m be a positive integer. Then let $\mathcal{S}_n^m$ be the arrangement of hyperplanes $H_{\alpha,k} = \{x \mid \langle \alpha \mid x \rangle = k \text{ for } -m+1 \leq k \leq m \text{ and } \alpha \in \Delta^+\}$. $\mathcal{S}_n^m$ is the mth extended Shi arrangement of type A_{n-1}, called here the m-Shi arrangement.

The main construction of the authors' previous paper [7] was a bijection between dominant regions of $\mathcal{S}_n^m$, and partitions that are n-cores as well as $(mn+1)$-cores. In this note, we modify the construction to give a direct bijection between the *bounded* dominant regions and partitions that are simultaneously n-cores and $(mn-1)$-cores.

We thank Nathan Reading who pointed out that the number of bounded dominant regions in the complement of the m-Shi arrangement is counted by $\frac{1}{s+t}\binom{s+t}{t}$ for appropriate s, t, and by [1] this number also counts the number of partitions that are s-cores as well as t-cores.

The number of such regions, called $N^+(\Delta, m)$ in [5], is also equal to the number of facets of the positive part of the generalized cluster complex which Fomin and Reading [8] associate to the pair (Δ, m).

Date: October 13, 2009.

Both authors wish to thank AIM and the SQuaREs program where this work was started.

Our bijection is W-equivariant in the following sense. In each bounded connected component of $\mathcal{S}_n^m$ there is exactly one "representative," or m-maximal, alcove farthest from the fundamental alcove $\mathcal{A}_0$. Since the affine Weyl group W acts freely and transitively on the set of alcoves, there is a natural way to associate an element $w \in W = \widehat{\mathfrak{S}}_n$ to any alcove $w^{-1}\mathcal{A}_0$, and to this one in particular. There is also a natural action of $\widehat{\mathfrak{S}}_n$ on partitions, whereby the orbit of the empty partition $\emptyset$ is precisely the n-cores. We will show that $w\emptyset$ is also an $(mn-1)$-core and that all such $(mn-1)$-cores that are also n-cores can be obtained this way.

Roughly speaking, to each n-core λ we can associate an integer vector $\vec{n}(\lambda)$ whose entries sum to zero, as in [9]. When λ is also an $(mn-1)$-core, these entries satisfy certain inequalities. On the other hand, these are precisely the inequalities that describe when a dominant alcove is m-maximal.

As a consequence, we show an n-core λ is automatically an $(mn-1)$-core if $\varphi_i(\lambda) \leq m$ for all $0 \leq i < n$, where $\varphi_i(\lambda)$ counts how many addable boxes of residue i the partition λ has. The paper [7] showed the related result, that an n-core λ is automatically an $(mn+1)$-core if $\varepsilon_i(\lambda) \leq m$ for all $0 \leq i < n$, where $\varepsilon_i(\lambda)$ counts the number of removable boxes of residue i.

The article is organized as follows. In Section 2 we introduce notation and recall facts about Coxeter groups, root systems of type A, and inversion sets for elements of the affine symmetric group. Section 3 explains how the position of $w^{-1}\mathcal{A}_0$ relative to our system of affine hyperplanes is captured by the action of w on affine roots and that m-maximality can be expressed by certain inequalities on the entries of $w(0,0,\ldots,0)$. In Section 4 we review facts about core partitions and in particular remind the reader how to associate an element of the root lattice to each core. Our main theorem, the bijection between bounded dominant regions of the m-Shi arrangement and special cores, is in Section 5. Section 6 describes the effect of a related bijection on m-maximal alcoves and characterizes alcove walls in terms of addable and removable boxes. We note that much of the machinery developed in [7] is used here, and that the proofs of this paper's results are also very similar; consequently proofs here are often very terse or omitted.

2. Preliminaries

Please see [7] for all details. Let Δ be the root system for type A_{n-1}, with Weyl group the symmetric group $\mathfrak{S}_n$. Let $\widetilde{\Delta}$ be the affine root system of type $A_{n-1}^{(1)}$, with null root δ, and with Weyl group the affine symmetric group $\widehat{\mathfrak{S}}_n$. See [15] for more details. Δ spans a Euclidean space V with inner product $\langle \, | \, \rangle$. Let $Q \subseteq V$ denote the root lattice for Δ. Let m be a positive integer. The *m-Shi arrangement* is the collection of hyperplanes

$$\mathcal{S}_n^m = \{H_{\alpha,k} \mid \alpha \in \Delta^+, -m < k \leq m\},$$

where $H_{\alpha,k} = \{v \in V \mid \langle v \mid \alpha \rangle = k\}$. This arrangement can be defined for all types; here we are concerned with type A.

The arrangement dissects V into connected components we call regions. We refer to regions which are in the dominant chamber of V as *dominant* regions. Each connected component of $V \setminus \bigcup_{\substack{\alpha\in\Delta^+\\ k\in\mathbb{Z}}} H_{\alpha,k}$ is called an alcove and the *fundamental alcove* is denoted $\mathcal{A}_0$. We denote the (closed) half spaces $H^+_{\alpha,k} = \{v \in V \mid \langle v \mid \alpha \rangle \geq k\}$ and $H^-_{\alpha,k} = \{v \in V \mid \langle v \mid \alpha \rangle \leq k\}$. Note $\mathcal{A}_0$ is the interior of $H^-_{\theta,1} \cap \bigcap_{i=1}^{n-1} H^+_{\alpha_i,0}$ and the dominant chamber is $\bigcap_{i=1}^{n-1} H^+_{\alpha_i,0}$.

The affine symmetric group $\widehat{\mathfrak{S}}_n$ acts on V (preserving Q) via affine linear transformations, and acts freely and transitively on the set of alcoves. We thus identify each alcove $\mathcal{A}$ with the unique $w \in \widehat{\mathfrak{S}}_n$ such that $\mathcal{A} = w\mathcal{A}_0$. We also note that we may express any $w \in \widehat{\mathfrak{S}}_n$ as $w = ut_\gamma$ for unique $u \in \mathfrak{S}_n$, $\gamma \in Q$, or equivalently $w = t_{\gamma'}u$ where $\gamma' = u(\gamma)$. If we embed V into $\mathbb{R}^n$ by mapping α_i to $\varepsilon_i - \varepsilon_{i+1}$, note that $\gamma' = w(0, \ldots, 0)$.

For $w \in \widehat{\mathfrak{S}}_n$, we define the inversion set $\mathrm{Inv}(w) = \{\alpha \in \widetilde{\Delta^+} \mid w(\alpha) \in \widetilde{\Delta^-}\}$. Notice that the length $\ell(w) = |\mathrm{Inv}(w)|$ for $w \in \widehat{\mathfrak{S}}_n$ is just the minimal number of affine hyperplanes separating $w^{-1}\mathcal{A}_0$ from $\mathcal{A}_0$. We will need the following well-known proposition and corollary, both describing $\mathrm{Inv}(w)$ and both proved in [7].

Proposition 2.1. Let $w \in \widehat{\mathfrak{S}}_n$ and $\alpha + k\delta \in \widetilde{\Delta^+}$, i.e. $k > 0, \alpha \in \Delta$ or $k = 0, \alpha \in \Delta^+$. Then $\alpha + k\delta \in \mathrm{Inv}(w)$ iff $w^{-1}\mathcal{A}_0 \subseteq H^+_{-\alpha,k}$

Corollary 2.2. Suppose w is a minimal length left coset representative for $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$. Then $\mathrm{Inv}(w)$ consists only of roots of the form $-\alpha + k\delta$, $k \in \mathbb{Z}_{>0}, \alpha \in \Delta^+$. Further, if $-\alpha+k\delta \in \mathrm{Inv}(w)$ and $k > 1$ then $-\alpha+(k-1)\delta \in \mathrm{Inv}(w)$.

We also remind the reader that when w^{-1} is a minimal length right coset representative for $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$, then we may write $w^{-1} = t_{\gamma'}u$ where $u \in \mathfrak{S}_n$ and γ' is in the dominant chamber.

3. m-MAXIMAL ALCOVES

We can identify each bounded connected component of the complement of the m-Shi arrangement with the unique alcove $w\mathcal{A}_0$ contained in it such that $\ell(w)$ is largest. In this situation we will say the alcove $w\mathcal{A}_0$ is *m-maximal.* Note that for unbounded regions, no such alcove exists. See

Figure 2 below for a picture of the m-maximal alcoves of type A_2 for $m = 1, 2$.

The following proposition is useful. For a given alcove, it characterizes the affine hyperplanes containing its walls and which simple reflections flip it over those walls (by the right action). It can be found in [20] in slightly different notation.

Proposition 3.1. Suppose $w\mathcal{A}_0 \subseteq H^+_{\alpha,k}$ but $ws_i\mathcal{A}_0 \subseteq H^-_{\alpha,k}$.

(1) Then $w(\alpha_i) = \alpha - k\delta$.
(2) Let $\beta = w^{-1}(0, \dots, 0) \in V$. Then $\langle \beta \mid \alpha_i \rangle = -k$.

Using the coordinates of $V \subseteq \mathbb{R}^n$, we note $k = \gamma_{u(i)} - \gamma_{u(i+1)}$, where $w = t_\gamma u$.

Remark 3.2. Note, if $w\mathcal{A}_0$ is m-maximal, then whenever $k \in \mathbb{Z}_{\geq 0}$ and $w\mathcal{A}_0 \subseteq H^-_{\alpha,k}$ but $ws_i\mathcal{A}_0 \subseteq H^+_{\alpha,k}$ then we must have $k \leq m$ in the case $\alpha \in \Delta^+$ and $k \leq m-1$ in the case $\alpha \in \Delta^-$.

It is easy to see that the condition in Remark 3.2 is not only necessary but sufficient to describe when $w\mathcal{A}_0$ is m-maximal. Together with Proposition 2.1, Proposition 3.1 says that when $\alpha_i \in \mathrm{Inv}(w)$, $w(\alpha_i) = \alpha - k\delta$ then $k \geq -m$, and for $\beta = w^{-1}(0, \dots, 0)$ that $\langle \beta \mid \alpha_i \rangle \leq m$.

Applying Remark 3.2 to just positive α and alcoves in the dominant chamber, we get the following corollary.

Corollary 3.3. Suppose $w\mathcal{A}_0$ is in the dominant chamber and m-maximal.

(1) If $w\mathcal{A}_0 \subseteq H^-_{\alpha,k}$ but $ws_i\mathcal{A}_0 \subseteq H^+_{\alpha,k}$ for some $\alpha \in \Delta^+, k \in \mathbb{Z}_{\geq 0}$, then $k \leq m$.
(2) Let $\beta = w^{-1}(0, \dots, 0)$. Then $\langle \beta \mid \alpha_i \rangle \leq m$, for all i, and in particular $\langle \beta \mid \theta \rangle \geq -m + 1$.

4. Core partitions and their abacus diagrams

In this section we review some well-known facts about n-cores and review the useful tool of the abacus construction. Details can be found in [13].

There is a well-known bijection $\mathcal{C} : \{n\text{-cores}\} \to Q$ that commutes with the action of $\widehat{\mathfrak{S}}_n$. One can use the $\widehat{\mathfrak{S}}_n$-action to define the bijection, or describe it directly from the combinatorics of partitions via the work of Garvan-Kim-Stanton's $\vec{n}$-vectors [9], Lascoux in [14], or as described in terms of balanced abaci as in [6]. Here, we will recall the description from [6] as well as remind the reader of the $\widehat{\mathfrak{S}}_n$-action on n-cores.

We identify a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ with its Young diagram, the array of boxes with coordinates $\{(i, j) \mid 1 \leq j \leq \lambda_i\}$. We say the box

$(i, j) \in \lambda$ has *residue* $j - i \bmod n$, and in that case, we often refer to it as a $(j - i \bmod n)$-box. Its *hook length* $h^{\lambda}_{(i,j)}$ is 1+ the number of boxes to the right of and below (i, j).

An n-*core* is a partition λ such that $n \nmid h^{\lambda}_{(i,j)}$ for all $(i, j) \in \lambda$.

We say a box is *removable* from λ if its removal results in a partition. Equivalently its hook length is 1. A box not in λ is *addable* if its union with λ results in a partition.

Claim 4.1. Let λ be an n-core. Suppose λ has a removable i-box. Then it has no addable i-boxes. Likewise, if λ has an addable i-box it has no removable i-boxes.

$\widehat{\mathfrak{S}}_n$ acts transitively on the set of n-cores as follows. Let λ be an n-core. Then

$$s_i\lambda = \begin{cases} \lambda \cup \text{all addable } i\text{-boxes} & \exists \text{ any addable } i\text{-box,} \\ \lambda \setminus \text{all removable } i\text{-boxes} & \exists \text{ any removable } i\text{-box,} \\ \lambda & \text{else.} \end{cases}$$

It is easy to check $s_i\lambda$ is an n-core.

4.1. **Abacus diagrams.** We can associate to each partition λ its abacus diagram. When λ is an n-core, its abacus has a particularly nice form, and then can be used to construct an element of Q.

Each partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is determined by its hook lengths in the first column, the $\beta_k = h^{\lambda}_{(k,1)}$.

An *abacus diagram* is a diagram, with entries from $\mathbb{Z}$ arranged in n columns labeled $0, 1, \ldots, n - 1$, called *runners*. The horizontal cross-sections or rows will be called *levels* and runner k contains the entry labeled by $rn + k$ on level r where $-\infty < r < \infty$. We draw the abacus so that each runner is vertical, oriented with $-\infty$ at the top and ∞ at the bottom, and we always put runner 0 in the leftmost position, increasing to runner $n - 1$ in the rightmost position. Entries in the abacus diagram may be circled; such circled elements are called *beads*. Entries which are not circled will be called *gaps*. We shall say two abaci are equivalent if they differ by adding a constant to all entries. (Note, in this case we must cyclically permute the runners so that runner 0 is leftmost.) Given a partition λ its abacus is any abacus diagram equivalent to the one obtained by placing beads at entries $\beta_k = h^{\lambda}_{(k,1)}$ and all $j \in \mathbb{Z}_{<0}$.

Remark 4.2. It is well-known that λ is an n-core if and only if its abacus is *flush*, that is to say whenever there is a bead at entry j there is also a bead at $j - n$.

We define the *balance number* of an abacus to be the sum over all runners of the largest level in that runner which contains a bead. We say that an abacus is *balanced* if its balance number is zero. Note that there is a unique abacus which represents a given n-core λ for each balance number. Given a flush abacus, that is, the abacus of an n-core λ, we can associate to it the vector whose i^{th} entry is the largest level in runner $i-1$ which contains a bead. Note that the sum of the entries in this vector is the balance number of the abacus. When the abacus is balanced, we will call this vector $\vec{n}(\lambda)$, in keeping with the notation of [9]. We note that $\vec{n}(\lambda) \in Q$, when we identify Q with $\{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid \sum_i a_i = 0\}$. We recall the following claim, which can be found in [6].

Claim 4.3. The map $\lambda \mapsto \vec{n}(\lambda)$ is an $\widehat{\mathfrak{S}}_n$-equivariant bijection $\{n\text{-cores}\} \to Q$.

We recall here results of Anderson [1], which describe the abacus of an n-core that is also a t-core, for t relatively prime to n. When $t = mn - 1$, this takes a particularly nice form.

Proposition 4.4 (Anderson)**.** Let λ be an n-core. Suppose t is relatively prime to n. Let $M = nt - n - t$. Consider the grid of points $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with $0 \leq x \leq n-1$, $0 \leq y$ labelled by $M - xt - yn$. Circle a point in this grid if and only if its label is obtained from the first column hooklengths of λ or its label is in $\mathbb{Z}_{<0}$. Then λ is a t-core if and only if the following three conditions hold.

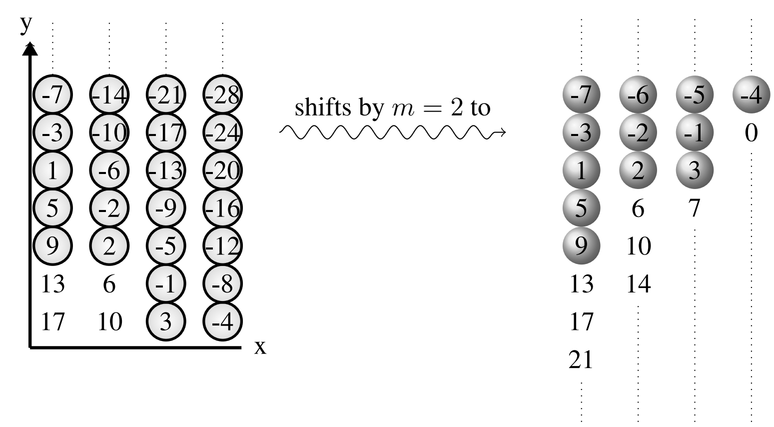
(1) All beads in the abacus of λ are at entries $\leq M$, in other words at (x, y) with $0 \leq x \leq n-1$, $0 \leq y$;
(2) The circled points in the grid are upwards flush, in other words if (x, y) is circled, so is $(x, y-1)$;
(3) The circled points in the grid are flush to the right, in other words if (x, y) is circled and $x \leq n-2$, so is $(x+1, y)$.

Note that the columns of this grid are exactly the runners of λ's abacus, written out of order, with each runner shifted up or down relative to its new left neighbor. The runners have also been truncated, which is irrelevant given condition (1) above. This shifting is performed exactly so labels in the same row are congruent $\bmod t$. This explains why the circles must be flush to the right as well as upwards flush.

In the special case $t = mn - 1$, the columns of the grid are the runners of λ's abacus, cyclically shifted so the 0-runner is now rightmost versus leftmost. Furthermore, each runner has been shifted m units down with respect to its left neighbor. So the condition of being flush to the right on Anderson's grid is given by requiring on the abacus that if the largest circled entry on runner i is at level r then runner $i+1$ must have a circled entry at level $r - m$. In other words, if $(a_1, \ldots, a_n) = \vec{n}(\lambda)$, then we require $a_{i+1} + m - a_i \geq 0$, i.e. $m \geq \langle \vec{n}(\lambda) \mid \alpha_i \rangle$ for $0 < i < n$. Recall the 0^{th} and $(n-1)^{st}$ and runners must also have this relationship (adding a constant to all entries in the abacus cyclically permutes the runners). This condition becomes $a_1 + m - 1 - a_n \geq 0$, i.e. $\langle \vec{n}(\lambda) \mid \theta \rangle \geq -m + 1$.

Corollary 4.5. Let λ be an n-core. Then λ is an $(mn-1)$-core if and only if $\langle \vec{n}(\lambda) \mid \alpha_i \rangle \leq m$ for $0 < i < n$ and $\langle \vec{n}(\lambda) \mid \theta \rangle \geq -m + 1$.

Example 4.6. We consider the 4-core $\lambda = (5, 2, 1, 1, 1)$, whose first column hooklengths are $9, 5, 3, 2, 1$. On the left, we draw Anderson's grid for the values $n = 4$, $t = 7 = 2 \times 4 - 1$ so that $M = 17$, and in this case $m = 2$. On the right, we shift the ith column up $m(i-1)$ units, and so change our setting from columns of a grid to runners of an abacus.



Note that the circles are flush to the right in the grid on the left, which shows us λ is a 7-core. The circles are also upwards flush since λ is a 4-core. The abacus on the right has each runner shifted up $m = 2$ units relative to its left neighbor in the grid.

5. The bijection between cores and alcoves

In this section, we will describe a bijection between the set of partitions that are both n-cores and $(mn-1)$-cores and the bounded connected components of the m-Shi hyperplane arrangement complement that lie in the dominant chamber, or more specifically, the dominant m-maximal alcoves. Furthermore, this bijection commutes with the action of $\widehat{\mathfrak{S}}_n$. (We note the minor technicality that the action on cores is a left action, but we take the right action on alcoves when discussing the Shi arrangement.)

In particular, this map is just the restriction of the $\widehat{\mathfrak{S}}_n$-equivariant map

$$\{n\text{-cores}\} \to \{\text{alcoves in the dominant chamber}\}$$
$$w\emptyset \mapsto w^{-1}\mathcal{A}_0.$$

Theorem 5.1. The map $\Phi : w\emptyset \mapsto w^{-1}\mathcal{A}_0$ for w a minimal length left coset representative of $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ induces a bijection from the set of n-cores that are also $(mn-1)$-cores to the set of m-maximal alcoves in the dominant chamber.

Proof. Let λ be an n-core and write $\lambda = w\emptyset$ for $w \in \widehat{\mathfrak{S}}_n$ a minimal length left coset representative for $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$. Recall that $\vec{n}(\lambda) = w(0,0,\ldots,0) \in Q$. Note, that since w is a minimal length left coset representative, w^{-1} is a minimal length right coset representative and in particular, $w^{-1}\mathcal{A}_0$ is in the dominant chamber. Recall by Corollary 3.3 that in this case $w^{-1}\mathcal{A}_0$ is m-maximal if and only if $\langle \beta \mid \alpha_i \rangle \leq m$ for $0 < i < n$ and $\langle \beta \mid \theta \rangle \geq -m+1$, where $\beta = w(0,\ldots,0) = \vec{n}(\lambda)$.

In Corollary 4.5 above, we have λ is an $(mn-1)$-core iff the conditions above hold for $\beta = \vec{n}(\lambda)$. $\square$

The bijection is pictured below in Figure 1.

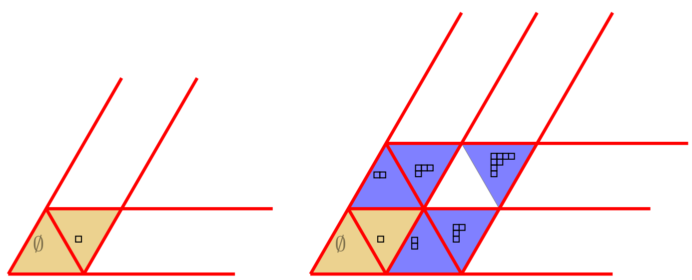


FIGURE 1. m-maximal alcoves $w^{-1}\mathcal{A}_0$ in the dominant chamber of the m-Shi arrangement of type A_2, filled with the 3-core partition $w\emptyset$. On the left ($m = 1$), they are also 2-cores, and on the right ($m = 2$), they are also 5-cores.

6. FURTHER RESULTS

6.1. **A bijection on alcoves.** Here we describe what the bijection $w\mathcal{A}_0 \mapsto w^{-1}\mathcal{A}_0$ does to the m-maximal alcoves. In particular, we do not limit ourselves to dominant alcoves. Let

$$\mathfrak{a}_m = \{v \in V \mid \langle v \mid \alpha_i \rangle \leq m \text{ for } 1 \leq i < n, \langle v \mid \theta \rangle \geq -m+1\}.$$

Theorem 6.1. (1) The map $w\mathcal{A}_0 \mapsto w^{-1}\mathcal{A}_0$ restricts to a bijection between alcoves in the region $\mathfrak{a}_m$ and m-maximal alcoves.

(2) The map $w(0,\dots,0) \mapsto w^{-1}\mathcal{A}_0$ restricts to a bijection between $Q \cap \mathfrak{a}_m$ and m-maximal alcoves in the dominant chamber.

We omit the proof, as it is very similar to the one given in [7].

The bijection is illustrated below, the first part comparing Figure 2 to Figure 3, and the second part from restricting our attention to the lattice points.

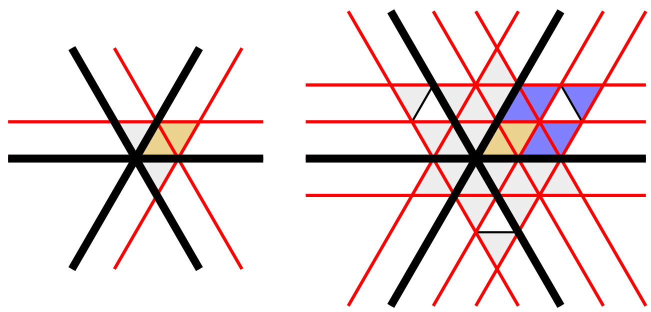


FIGURE 2. m-maximal alcoves in the m-Shi arrangement for $m = 1$ ($m = 2$). Dominant alcoves are shaded yellow (and/or blue, respectively), whereas other m-maximal alcoves are shaded gray.

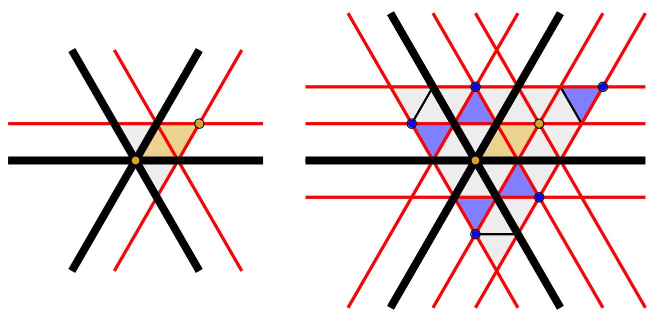


FIGURE 3. $w\mathcal{A}_0$ for the m-maximal alcoves $w^{-1}\mathcal{A}_0$ in Figure 2 above, $m = 1, 2$. Note $\bigcup w\bar{\mathcal{A}}_0 = \mathfrak{a}_m$. Each $\gamma = w(0,\dots,0) \in Q \cap \mathfrak{a}_m$ is in precisely one yellow/blue alcove, so this illustrates the second statement of Theorem 6.1.

6.2. **Alcove walls correspond to addable and removable boxes for cores.** For those readers familiar with the realization of the basic crystal $B(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_n$ as having nodes parameterized by n-regular partitions,

$$s_i\lambda = \begin{cases} \tilde{f}_i^{\langle h_i, \mathrm{wt}(\lambda)\rangle}(\lambda) & \langle h_i, \mathrm{wt}(\lambda)\rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, \mathrm{wt}(\lambda)\rangle}(\lambda) & \langle h_i, \mathrm{wt}(\lambda)\rangle \leq 0, \end{cases}$$

where

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{(x,y)\in\lambda} \alpha_{y-x \bmod n}, \tag{6.1}$$

and h_i is the co-root corresponding to α_i. Then the n-cores are exactly the $\widehat{\mathfrak{S}}_n$-orbit on the highest weight node, which is the empty partition $\emptyset$.

It is well-known that $s_i\lambda = \mu$ iff $s_i\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$ where the action of $\widehat{\mathfrak{S}}_n$ on the weight lattice is given by

$$s_i(\gamma) = \gamma - \langle \gamma \mid \alpha_i \rangle \alpha_i.$$

We refer the reader to Chapters 5,6 of [15] for details on the affine weight lattice, definition of Λ_0 and so on. For computational purposes, all we need remind the reader of is that $\langle \Lambda_0 \mid \alpha_i \rangle = \delta_{i,0}$ and $\langle \alpha_0 \mid \alpha_i \rangle = 2\delta_{i,0} - \delta_{i,1} - \delta_{i,n-1}$.

It is useful to recall the following notation from the theory of crystal graphs. In the case s_i removes k boxes of residue i from the core λ, write $\varepsilon_i(\lambda) = k, \varphi_i(\lambda) = 0$. In the case s_i adds r boxes to λ to obtain μ, write $\varepsilon_i(\lambda) = 0, \varphi_i(\lambda) = r$.

Remark 6.2. In other words, Equation (6.1) says that if s_i removes k boxes (of residue i) from λ, or adds $-k$ boxes to λ to obtain μ, then $\mathrm{wt}(\mu) = s_i(\mathrm{wt}(\lambda)) = \mathrm{wt}(\lambda) - k\alpha_i$. In either case, $\mathrm{wt}(\mu) = \mathrm{wt}(\lambda) + (\varphi_i(\lambda) - \varepsilon_i(\lambda))\,\alpha_i$.

A straightforward rephrasing of Proposition 3.1 is then:

Proposition 6.3. Let λ be an n-core, $k \in \mathbb{Z}_{>0}$, and $w \in \widehat{\mathfrak{S}}_n$ of minimal length such that $w\emptyset = \lambda$. Fix $0 \le i < n$. The following are equivalent

(1) $\varphi_i(\lambda) = k$,
(2) $\langle \vec{n}(\lambda) \mid \alpha_i \rangle = k$ for $i \neq 0$, $\quad \langle \vec{n}(\lambda) \mid \theta \rangle = -k + 1$ for $i = 0$,
(3) $w^{-1}\mathcal{A}_0 \subseteq H^-_{\alpha,k}$, $\quad w^{-1}s_i\mathcal{A}_0 \subseteq H^+_{\alpha,k}$ where $w^{-1}(\alpha_i) = -\alpha + k\delta$.

When we rephrase Corollary 3.3 in this context, it says:

Suppose $\lambda = w\emptyset$ is the n-core associated to the dominant alcove $\mathcal{A} = w^{-1}\mathcal{A}_0$. Then $\mathcal{A}$ is m-maximal iff whenever λ has exactly k addable boxes of residue i then $k \le m$. (And in this case, λ is also an $(mn-1)$-core.)

As a consequence, note an n-core λ is automatically an $(mn-1)$-core if $\varphi_i(\lambda) \le m$ for all $0 \le i < n$.

In [5], Athanasiades and Tzanaki define $h^+_k(\Delta, m)$, $0 \le k < n$ as the number of bounded dominant regions of $\mathcal{S}^m_n$ for which exactly $n - 1 - k$ hyperplanes of the form $H_{\alpha,m}$, $\alpha \in \Delta^+$ are walls (i.e. support a facet) of that region and do *not* separate it from the fundamental alcove $\mathcal{A}_0$.

By the definition of m-maximal, we can replace a bounded region by its unique m-maximal alcove and consider its walls instead. In other words, to calculate $h^+_k(\Delta, m)$, we count how many m-maximal alcoves $\mathcal{A} = w^{-1}\mathcal{A}_0$

satisfy that for exactly $n-1-k$ positive roots α, there exists an i such that $w^{-1}\mathcal{A}_0 \subseteq H^-_{\alpha,m}$ but $w^{-1}s_i\mathcal{A}_0 \subseteq H^+_{\alpha,m}$.

By Proposition 6.3 above, $h^+_k(\Delta, m)$ equivalently counts how many n-cores λ that are also $(mn-1)$-cores have exactly $n-1-k$ distinct residues i such that λ has precisely m addable i-boxes.

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