THE IRREDUCIBLE COMPLEX REPRESENTATIONS OF THE BRAID GROUP ON N STRINGS OF DEGREE $\leq$ N

EDWARD FORMANEK[1], WOO LEE, INNA SYSOEVA[1], AND MONICA VAZIRANI[2]

ABSTRACT. Let B_n be the braid group on n strings. Earlier work of E. Formanek, W. Lee and I. Sysoeva classified the irreducible complex representations of B_n of degree $\leq n$, except for $n = 5, 6, 7, 8$. Here the classification is completed using the theory of Hecke algebra representations.

1. INTRODUCTION.

Let B_n be the braid group on n strings. In a series of papers [F,L,S], the irreducible complex representations of B_n of degree $\leq n$ were classified, except for the n-dimensional representations when $n = 5, 6, 7, 8$. In this paper we classify the irreducible n-dimensional complex representations in the remaining four cases.

We say that an irreducible complex representation of B_n is of Burau, standard, or Hecke type if it is equivalent to the tensor product of a one-dimensional representation and, respectively, a composition factor of degree ≥ 2 of a specialization of the Burau representation, a specialization of the standard representation, or a representation which factors through the Hecke algebra. (See Section 2 for definitions.) The main result of [F] is that for $n \geq 7$, the irreducible complex representations of B_n of degree $\leq n-1$ are either one-dimensional or of Burau type, and the main result of [S] is that for $n \geq 9$, the irreducible complex representations of B_n of degree n are of standard type. However, other irreducible representations of degree $\leq n$ appear for smaller n, and ad hoc arguments are needed to classify them. The worst case, in the sense that the most anomalies occur, is $n = 4$, which required a lengthy analysis in [L].

The classification for $n = 5, 6, 7, 8$ has two steps. First, we show that for these n an irreducible complex representation of B_n of degree n is either of standard type or Hecke type. (The irreducible representations of B_n of degree $< n$ were classified in [F, Theorem 23].) Then we use recent algorithms for Hecke algebra representations to find those of Hecke type. The full classification is given in Theorem 14.

2. NOTATION.

The braid group on n strings, B_n, is the abstract group with presentation

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1, \ldots, n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2\rangle.$$

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The generators $\sigma_1, ..., \sigma_{n-1}$ are called the standard generators of B_n. For $r \le n$, the natural map $B_r \to B_n$ is an inclusion, and we will identify B_r with the subgroup of B_n generated by $\sigma_1, \dots, \sigma_{r-1}$.

Let t be an indeterminate, and let $\mathbb{C}[t^{\pm 1}]$ be a Laurent polynomial ring over the complex numbers. We define the following representations of B_n by matrices over $\mathbb{C}[t^{\pm 1}]$.

Definition. The *character*, $\chi(t)$, is the degree one representation defined by $\chi(t)(\sigma_i) = t$, for all σ_i. We regard $\chi(t)$ as a representation of all B_n simultaneously.

The *reduced Burau representation* $\beta_n(t) : B_n \to GL_{n-1}(\mathbb{C}[t^{\pm 1}])$ is defined by

$$\beta_n(t)(\sigma_1) = \left(\begin{array}{cc|c} -t & 0 & \multirow{2}{*}{0} \\ -1 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-3} \end{array}\right), \quad \beta_n(t)(\sigma_{n-1}) = \left(\begin{array}{c|cc} I_{n-3} & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 1 & -t \\ & 0 & -t \end{array}\right),$$

$$\beta_n(t)(\sigma_i) = \left(\begin{array}{c|ccc|c} I_{i-2} & & 0 & & 0 \\ \hline & 1 & -t & 0 & \\ 0 & 0 & -t & 0 & 0 \\ & 0 & -1 & 1 & \\ \hline 0 & & 0 & & I_{n-i-2} \end{array}\right), \text{ for } i = 2, \dots, n-2.$$

The *standard representation* $\gamma_n(t) : B_n \to GL_n(\mathbb{C}[t^{\pm 1}])$ is defined by

$$\gamma_n(t)(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & \multicolumn{2}{c|}{0} & 0 \\ \hline \multirow{2}{*}{0} & 0 & t & \multirow{2}{*}{0} \\ & 1 & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_{n-i-1} \end{array}\right), \text{ for } i = 1, \dots, n-1.$$

The *Hecke algebra* (of type A_{n-1}) is the group algebra of B_n over $\mathbb{C}[t^{\pm 1}]$ modulo the two-sided ideal generated by $\sigma_1^2 + (1-t)\sigma_1 - t$. It is denoted $H_n[t^{\pm 1}]$.

Specializing t to a nonzero complex number z gives complex representations $\chi(z)$, $\beta_n(z)$, $\gamma_n(z)$ and a Hecke algebra $H_n(z)$. The representation $\beta_n(z)$ is irreducible unless z is a root of $1+t+t^2+\cdots+t^{n-1}$, in which case it has two composition factors, one trivial and one of degree $n-2$ [F, Lemma 6]. For $n \ge 3$, the representation $\gamma_n(z)$ is irreducible unless $z = 1$ [S, Lemmas 5.3 and 5.4]. The algebra $H_n(z)$ is semisimple unless z is a primitive i-th root of unity, where $2 \le i \le n$ [M, Corollary 3.44, p. 48].

Definition. Let ρ be an irreducible complex representation of B_n.

It is of *Burau type* if it is not of degree one and it is equivalent to the tensor product of a one-dimensional representation and a composition factor of a specialization of the reduced Burau representation.

It is of *standard type* if it is equivalent to the tensor product of a one-dimensional representation and a specialization $\gamma_n(z), z \ne 1$, of the standard representation.

It is of *Hecke type* if it is equivalent to the tensor product of a one-dimensional representation and a representation which factors through the Hecke algebra.

Note that one-dimensional representations and representations of Burau type are also of Hecke type, but representations of standard type are never of Hecke type.

3. Preliminaries.

In this section we review and slightly recast parts of [F,S]. Lemma 1 is based on results originally due to W.-L. Chow [C].

Lemma 1. *Set* $\tau_n = \sigma_1 \cdots \sigma_{n-1}$.
(a) [C, p. 655, F, Lemma 1(d)]. *For* $1 \le i \le n-2$, $\tau_n \sigma_i \tau_n^{-1} = \sigma_{i+1}$.
(b) [C, Theorem III, F, Lemma 1(e)]. *For* $n \ge 3$, *the center of* B_n *is infinite cyclic, with generator* τ_n^n.
(c) [F, Lemma 1(a)]. *For* $n \ge 2$, $B_n/B_n' \cong \mathbb{Z}$. *Hence the characters* $\{\chi(z) \mid z \in \mathbb{C}^*\}$ *are a complete set of one-dimensional complex representations of* B_n.
(d) [F, Corollary 2(c)]. *If* $\rho : B_n \to G$ *is a group homomorphism and* $\rho(\langle \sigma_1, \sigma_2 \rangle)$ *is nilpotent, then* $\rho(B_n)$ *is abelian.*
(e) (Consequence of (d)). *If* $\rho : B_n \to GL_r(\mathbb{C})$ *is an irreducible representation and* $\rho \mid \langle \sigma_1, \sigma_2 \rangle$ *is unitriangularizable, then* $r = 1$. ■

Much of the classification of the irreducible complex representations of B_n depends on the Jordan canonical form of the image of σ_1 (or of any σ_i, since all the σ_i are conjugate).

There is a one-to-one correspondence between partitions of m and conjugacy classes of nilpotent $m \times m$ matrices. If λ is a partition of m, express λ in Frobenius notation, $\lambda = \{1^{\lambda_1} 2^{\lambda_2} \cdots s^{\lambda_s}\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are nonnegative integers and, by convention, $\lambda_s \ne 0$. This notation means that s is the largest part of λ, and that for $i = 1, 2, ..., s$, λ has λ_i parts equal to i. Thus $\lambda_1 + 2\lambda_2 + \cdots + s\lambda_s = m$, and we write $|\lambda| = m$. Corresponding to λ is the nilpotent $m \times m$ Jordan matrix $J(\lambda)$, which is a block diagonal matrix with λ_i elementary upper triangular $i \times i$ nilpotent Jordan blocks for $i = 1, 2, ..., s$.

For $x \in \mathbb{C}$, set $J(x, \lambda) = xI + J(\lambda)$. If X is an $r \times r$ matrix with k distinct eigenvalues $x_1, x_2, ..., x_k$, its Jordan canonical form is the block diagonal matrix whose diagonal blocks are $J(x_1, \lambda_1), ..., J(x_k, \lambda_k)$, for various partitions $\lambda_1, ..., \lambda_k$, where $|\lambda_1| + ... + |\lambda_k| = r$. We will then say that X has *Jordan form*

$$J(x_1, \lambda_1) \oplus \cdots \oplus J(x_k, \lambda_k), \quad \text{or} \quad \lambda_1 \oplus \cdots \oplus \lambda_k.$$

We use the abbreviated notation on the right when the actual values of the eigenvalues are irrelevant. The direct sum notation is intended to suggest the decomposition of $\mathbb{C}^n$ into a direct sum of X-invariant subspaces $V_1 \oplus V_2 \oplus ... \oplus V_k$, where V_k is $|\lambda_i|$-dimensional and X acts like $J(x_i, \lambda_i)$ on each V_i. Each V_i is also invariant under the centralizer of X. More generally, if $f(t) \in \mathbb{C}[t]$ is a polynomial in one variable, then $f(X)(\mathbb{C}^r)$ is invariant under the centralizer of X. Note also that the minimal polynomial of X is $m(t) = (t - x_1)^{s_1} ... (t - x_k)^{s_k}$, where s_i is the largest part of λ_i. Furthermore, if $f_i(t) = m(t)/(t - x_i)$, then $f_i(X)(\mathbb{C}^r)$ has dimension μ_i, where μ_i is the multiplicity of s_i in λ_i.

If we apply the above remarks to $X = \rho(\sigma_{n-1})$, where ρ is a representation of B_n, we get the following useful result. (As remarked earlier, B_{n-2} denotes the subgroup of B_n generated by $\sigma_1, ..., \sigma_{n-3}$.)

Lemma 2. *Let* $\rho : B_n \to GL_r(\mathbb{C})$ *be a representation, and let* $\lambda_1 \oplus \cdots \oplus \lambda_k$ *be the Jordan form of* $\rho(\sigma_{n-1})$. *Suppose that the largest part of* λ_i *has multiplicity* μ_i. *Then* $\rho \mid (B_{n-2} \times \langle \sigma_{n-1} \rangle)$ *has an invariant subspace of dimension* μ_i. ■

Lemma 3 below is proved using Lemma 2 and the classification of [F, Theorem 23], which implies that if $2 \le r \le n-3$, then B_n has no irreducible complex representations of degree r.

Lemma 3. [S, Lemma 6.4]. *Let $\rho : B_n \to GL_n(\mathbb{C})$ be a representation, where $n \ge 6$, and let $\lambda_1 \oplus \cdots \oplus \lambda_k$ be the Jordan form of $\rho(\sigma_{n-1})$. Suppose that the multiplicity of the largest part of some λ_i is less than or equal to $n-5$. Then $\rho \mid (B_{n-2} \times \langle \sigma_{n-1} \rangle)$ has a one-dimensional invariant subspace.* ■

Lemma 4 [S, Theorem 6.2]. *Let $\rho : B_n \to GL_n(\mathbb{C})$ be an irreducible representation, where $n \ge 5$. Suppose that $\rho \mid (B_{n-2} \times \langle \sigma_{n-1} \rangle)$ stabilizes a one-dimensional subspace. Then $\rho(\sigma_1) - yI$ has rank two for some $y \in \mathbb{C}^*$.* ■

Lemma 5 [S, Theorem 5.5]. *Let $\rho : B_n \to GL_n(\mathbb{C})$ be an irreducible representation, where $n \ge 6$. Suppose that $\rho(\sigma_1) - I$ has rank two. Then ρ is equivalent to a specialization of the standard representation.* ■

Combining Lemmas 3, 4 and 5 with the observation that if $\rho(\sigma_1) - yI$ has rank two, then $(\chi(y^{-1}) \otimes \rho)(\sigma_1) - I$ also has rank two gives the next lemma.

Lemma 6. *Let $\rho : B_n \to GL_n(\mathbb{C})$ be a representation, where $n \ge 6$, and let $\lambda_1 \oplus \cdots \oplus \lambda_k$ be the Jordan form of $\rho(\sigma_{n-1})$. Suppose that the multiplicity of the largest part of some λ_i is less than or equal to $n-5$. Then ρ is of standard type.* ■

The next results from [S] are based on the notion of the *friendship graph*, which we briefly review. More details can be found in [S]. Set $\tau = \tau_n = \sigma_1 \cdots \sigma_{n-1}$, and set $\sigma_0 = \tau^{-1}\sigma_1\tau$. Then conjugation by τ permutes $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ cyclically, by Lemma 1(a,b).

Definition. Let $\rho : B_n \to GL_r(\mathbb{C})$ be a representation, where $\rho(\sigma_i) = I + A_i$. The *full friendship graph* of ρ is the simple-edged graph with n vertices $A_0, A_1, \ldots, A_{n-1}$ and an edge joining A_i and A_j $(i \neq j)$ if and only if $im(A_i) \cap im(A_j) \neq 0$.

The friendship graph is useful when 1 is an eigenvalue of $\rho(\sigma_i)$ of high multiplicity, which means that the rank of A_i is small compared to r. If 1 is not an eigenvalue of $\rho(\sigma_i)$, then the full friendship graph of ρ is the complete graph with vertices $A_0, A_1, \ldots A_{n-1}$, and no useful information about ρ is obtained.

Lemma 7. *Let $\rho : B_5 \to GL_5(\mathbb{C})$ be an irreducible representation, with $\rho(\sigma_i) = I + A_i$ $(i = 0, 1, 2, 3, 4)$ and $rank(A_i) = 2$.*
(a) [S, Theorem 4.4(2)] *The full friendship graph of ρ is either the chain graph (the only edges are between A_i and A_{i+1}) or the star graph (the only edges are between A_i and A_{i+2}). (Indices are modulo 5.)*
(b) [S, Theorem 5.1] *If the full friendship graph of ρ is the chain graph, then ρ is of standard type.* ■

Finally, we interpret Hecke type in terms of the Jordan form of $\rho(\sigma_1)$.

Lemma 8. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation, where $r \ge 2$. Then the following are equivalent.*
(a) *The representation ρ is of Hecke type.*

(b) *The minimal polynomial of* $\rho(\sigma_1)$ *has degree two.*
(c) *The Jordan form of* $\rho(\sigma_1)$ *is one of the following:* $(1^\mu 2^\nu)$ *or* $(1^\mu) \oplus (1^\nu)$. ■

4. Reduction to Representations of Hecke Type.

Theorem 9. *Let* $\rho : B_5 \to GL_5(\mathbb{C})$ *be an irreducible representation. Then* ρ *is of standard type or Hecke type.*

Proof. We suppress ρ and write σ instead of $\rho(\sigma)$, for $\sigma \in B_5$, and we let $\tau = \tau_5 = \sigma_1\sigma_2\sigma_3\sigma_4$. We proceed by considering all of the possible Jordan forms for σ_4. The Jordan form (1^5) is impossible, for then $\rho(B_5)$ consists of scalar matrices. The Jordan forms (12^2) and $(1^3) \oplus (1^2)$ correspond to representations of Hecke type, by Lemma 8.

All other Jordan forms for 5×5 matrices include a partition whose largest part has multiplicity one. Then Lemma 2 implies that $\rho \mid (B_3 \times \langle\sigma_4\rangle)$ has an invariant subspace of dimension one, after which Lemma 4 implies that $\sigma_1 - yI$ has rank two for some $y \in \mathbb{C}^*$.

It therefore suffices to show that ρ is of standard or Hecke type if $\sigma_1 - yI$ has rank two. Replacing ρ by $\chi(y^{-1}) \otimes \rho$ (which does not affect whether ρ has standard or Hecke type), we may suppose that $y = 1$. By Lemma 7(a), the full friendship graph of ρ is either the chain or the star graph, and by Lemma 7(b), ρ is of standard type if it is the chain graph.

It remains to show that if the full friendship graph of ρ is the star graph, then ρ is of Hecke type.

Let $A_i = \sigma_i - I$, for $i = 0, 1, 2, 3, 4$, with indices taken modulo 5. Set $V_i = im(A_{i-1}) \cap im(A_{i+1})$, which is nonzero by the definition of the star graph. Conjugation by $\tau = \sigma_1\sigma_2\sigma_3\sigma_4$ permutes V_0, V_1, V_2, V_3, V_4 cyclically. The dimension of V_i is one or two, but if it is two, then the action of τ shows that all the $im(A_i)$ are equal and invariant under B_5, contradicting the irreducibility of ρ. Hence the V_i are one-dimensional.

Let v_i be a basis vector for V_i. Then span$\{v_0, v_1, v_2, v_3, v_4\}$ is invariant under B_5, so it must be all of $\mathbb{C}^5$, since ρ is irreducible. Moreover, since the v_i are linearly independent, $im(A_i) = span\{v_{i-1}, v_{i+1}\}$. Since A_1 and A_3 commute,

$$im(A_1A_3) = im(A_3A_1) \subseteq im(A_1) \cap im(A_3) = \mathbb{C}v_2\,.$$

Thus v_2 is an eigenvector for both A_1 and A_3, say $A_1v_2 = \lambda v_2$, $A_3v_2 = \mu v_2$.

We now follow the argument of [S, Lemma 3.7]. The braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ implies that

$$A_1 + A_1^2 + A_1A_2A_1 = A_2 + A_2^2 + A_2A_1A_2,$$

and since $im(A_1) \cap im(A_2) = 0$ (because the friendship graph is the star graph), both sides of the equation equal zero. Equating the left side multiplied by A_2 on the left with the right side multiplied by A_1 on the right gives $A_2A_1^2 = A_2^2A_1$. A similar computation gives $A_2A_3^2 = A_2^2A_3$.

Now let $v_2 = A_1x = A_3y$ for some $x, y \in \mathbb{C}^5$. Then

$$\begin{aligned} \lambda A_2v_2 &= A_2A_1v_2 = A_2A_1^2x = A_2^2A_1x = A_2^2v_2, \\ \mu A_2v_2 &= A_2A_3v_2 = A_2A_3^2t = A_2^2A_3y = A_2^2v_2. \end{aligned}$$

We claim that $A_2 v_2 \neq 0$, and hence that $\lambda = \mu$. Since $v_2 = A_1 x$,

$$A_1 A_2 v_2 = A_1 A_2 A_1 x = -A_1^2 x - A_1 x = -(A_1+1)A_1 x = -\sigma_1 v_2 \neq 0.$$

Since $\lambda = \mu$, the action of τ permuting $\mathbb{C}v_0, \mathbb{C}v_1, \ldots, \mathbb{C}v_4$ shows that $A_1 v_0 = \mu v_0 = \lambda v_0$. But then $A_1(A_1 - \lambda I) = 0$, which means that σ_1 satisfies a quadratic equation over $\mathbb{C}$, so ρ is of Hecke type, by Lemma 8. ∎

Theorem 10. *Let $\rho : B_6 \to GL_6(\mathbb{C})$ be an irreducible representation. Then ρ is of standard type or Hecke type.*

Proof. As above, we suppress ρ and write σ instead of $\rho(\sigma)$, for $\sigma \in B_5$, and we let $\tau = \tau_6 = \sigma_1 \cdots \sigma_5$. In addition, we let $\mathcal{Z}(\sigma)$ denote the centralizer of σ in $M_n(\mathbb{C})$.

Let $\lambda_1 \oplus \cdots \oplus \lambda_k$ be the Jordan form of σ_5. If the multiplicity of the largest part of any λ_i is one $(6-5=1)$, then ρ is of standard type, by Lemma 6. The remaining Jordan forms for 6×6 matrices are

$$(1^6),\ (1^2 2^2),\ (2^3),\ (1^4)\oplus(1^2),\ (1^3)\oplus(1^3),\ (3^2),\ (1^2)\oplus(1^2)\oplus(1^2),\ (2^2)\oplus(1^2).$$

As in Theorem 9, (1^6) is not possible, and the next four Jordan forms correspond to representations of Hecke type, by Lemma 8. That leaves three Jordan forms to eliminate.

$$\text{(I) } (3^2). \qquad \text{(II) } (1^2)\oplus(1^2)\oplus(1^2). \qquad \text{(III) } (2^2)\oplus(1^2).$$

Case I: (3^2). After tensoring ρ with a one-dimensional representation and replacing it with an equivalent representation, we may suppose that

$$\sigma_5 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{Z}(\sigma_5) = \left\{ \begin{pmatrix} A & B & C \\ 0 & A & B \\ 0 & 0 & A \end{pmatrix} \,\middle|\, A, B, C \in M_2(\mathbb{C}) \right\}.$$

Hence there is a representation $\rho_1 : \langle \sigma_1, \sigma_2 \rangle \to GL_2(\mathbb{C})$ such that for all $\sigma \in \langle \sigma_1, \sigma_2 \rangle$

$$\sigma = \begin{pmatrix} \rho_1(\sigma) & * & * \\ 0 & \rho_1(\sigma) & * \\ 0 & 0 & \rho_1(\sigma) \end{pmatrix}.$$

If ρ_1 is reducible, then $\rho \mid \langle \sigma_1, \sigma_2 \rangle$ is unitriangularizable, which contradicts Lemma 1(e). Thus ρ_1 is irreducible, so $\rho \mid \langle \sigma_1, \sigma_2 \rangle$ has three composition factors, each equivalent to ρ_1. Similarly (or via the action of τ^3), $\rho \mid \langle \sigma_4, \sigma_5 \rangle$ has three composition factors, each equivalent to an irreducible representation $\rho_2 : \langle \sigma_4, \sigma_5 \rangle \to GL_2(\mathbb{C})$.

Consider $\rho \mid (\langle \sigma_1, \sigma_2 \rangle \times \langle \sigma_4, \sigma_5 \rangle)$. Each of its composition factors is equivalent to $\rho_1 \otimes \rho_2$, and hence is four-dimensional. This is impossible, since six is not a multiple of four. Hence Case I cannot occur.

Case II: $(1^2)\oplus(1^2)\oplus(1^2)$. As above, we may assume that

$$\rho(\sigma_5) = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & z \end{pmatrix}, \quad \mathcal{Z}(\sigma_5) = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \,\middle|\, A, B, C \in M_2(\mathbb{C}) \right\}.$$

where x, y, z are distinct nonzero complex numbers. Since $B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ centralizes σ_5, there are representations $\rho_1, \rho_2, \rho_3 : B_4 \to GL_2(\mathbb{C})$ such that for all $\sigma \in B_4$,

$$\sigma = \begin{pmatrix} \rho_1(\sigma) & 0 & 0 \\ 0 & \rho_2(\sigma) & 0 \\ 0 & 0 & \rho_3(\sigma) \end{pmatrix}.$$

We claim that ρ_1, ρ_2, ρ_3 are irreducible. If not, then for $u = x, y$ or z,

$$W = Ker(\sigma_1 - uI) \cap Ker(\sigma_2 - uI) \cap Ker(\sigma_3 - uI) \neq 0.$$

The dimension of $Ker(\sigma_1 - uI) \cap Ker(\sigma_2 - uI)$ is either one or two. In either case, the action of τ shows that it is equal to W and invariant under $\rho(B_6)$, contradicting the irreducibility of ρ. Hence ρ_1, ρ_2, ρ_3 are irreducible, as claimed.

Since σ_1 is semisimple, each $\rho_i(\sigma_1)$ is semisimple. Hence the eigenvalues of any $\rho_i(\sigma_1)$ cannot be equal, for then $\rho_i(\sigma_1)$ would be a scalar matrix, contradicting the irreducibility of ρ_i. This implies that the eigenvalues of $\rho_1(\sigma_1), \rho_2(\sigma_1)$ and $\rho_3(\sigma_1)$ are $\{x, y\}, \{y, z\}$, and $\{z, x\}$ in some order.

By [F, Theorem 12], an irreducible representation $B_4 \to GL_2(\mathbb{C})$ remains irreducible when restricted to B_3. Thus $\rho_1, \rho_2, \rho_3 \mid B_3$ are irreducible; they are mutually inequivalent since for $i \neq j$, the eigenvalues of $\rho_i(\sigma_1)$ and $\rho_j(\sigma_1)$ are different. This implies that the $\mathbb{C}$-linear span of $\rho(B_3)$ is $\mathcal{Z}(\sigma_5)$, the centralizer of σ_5. Since σ_4 centralizes $\rho(B_3)$, it centralizes $\mathcal{Z}(\sigma_5)$, thus it is a diagonal matrix (in fact, with the same shape as σ_5), so it commutes with σ_5. But then σ_4 and σ_5 commute, which means that $\rho(B_6)$ is abelian, by Lemma 1(d) and the action of τ. This contradicts the irreducibility of ρ, so Case II cannot occur.

Case III: $(2^2) \oplus (1^2)$. As above, we may assume that

$$\sigma_5 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \quad \mathcal{Z}(\sigma_5) = \left\{ \begin{pmatrix} A & B & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix} \,\middle|\, A, B, C \in M_2(\mathbb{C}) \right\}.$$

where $x \in \mathbb{C}^*$, $x \neq 1$.

Hence there are representations $\rho_1, \rho_2 : B_4 \to GL_2(\mathbb{C})$ such that for all $\sigma \in B_4$

$$\sigma = \begin{pmatrix} \rho_1(\sigma) & * & 0 \\ 0 & \rho_1(\sigma) & 0 \\ 0 & 0 & \rho_2(\sigma) \end{pmatrix}.$$

Consider the two-dimensional vector space $K_i = Ker(\sigma_i - uI)$, where $u = 1$ or $u = x$. We claim that $K_1 \cap K_2 \cap K_3 = 0$. First, $K_1 \neq K_2$, since otherwise the action of τ would imply that K_1 is invariant under the action of $\rho(B_6)$, contradicting the irreducibility of ρ. Similarly if

$$(K_1 \cap K_2) \cap (K_2 \cap K_3) = K_1 \cap K_2 \cap K_3 \neq 0,$$

then $K_1 \cap K_2$ is one-dimensional and invariant under $\rho(B_6)$, again contradicting the irreducibility of ρ. Thus $K_1 \cap K_2 \cap K_3 = 0$, as claimed.

This implies that ρ_2 is irreducible, since otherwise σ_1, σ_2, and σ_3 would have either a common 1-eigenvector or a common x-eigenvector. As in Case II, [F, Theorem 12] implies that $\rho_2|B_3$ is irreducible; moreover, $\rho_2|B_3$ cannot be equivalent to $\rho_1|B_3$, since the eigenvalues of $\rho(\sigma_1)$ form the set $\{1,1,1,1,x,x\}$, which cannot be split into three identical subsets. This implies that the $\mathbb{C}$-linear span of $\rho(B_3)$ contains the set of block matrices

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix} \,\middle|\, A \in M_2(\mathbb{C}) \right\}.$$

Since σ_4 centralizes $\rho(B_3)$, it must have the 2×2 block form

$$\sigma_4 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

But then the subspace $\{(0000 * *)^t\}$ is invariant under $\rho(B_6)$, contradicting the irreducibility of ρ ■

Theorem 11. *let $\rho : B_n \to GL_n(\mathbb{C})$ be an irreducible representation, where $n = 7$ or $n = 8$. Then ρ is of standard type or Hecke type.*

Proof. Let $\lambda_1 \oplus \cdots \oplus \lambda_k$ be the Jordan form of $\rho(\sigma_1)$. By Lemma 6, either ρ is of standard type or the largest part of every λ_i has multiplicity greater than $n-5$.

If $n = 7$, $n - 5 = 2$, and the only partitions not eliminated by Lemma 6 are (1^7), (12^3), and $(1^4) \oplus (1^3)$. If $n = 8$, $n - 5 = 3$, and the only partitions not eliminated by Lemma 6 are (1^8), (2^4), and $(1^4) \oplus (1^4)$.

The Jordan forms (1^7) and (1^8) are not possible for an irreducible representation, and the remaining Jordan forms correspond to representations of Hecke type, by Lemma 8. ■

5. Representations of Hecke Type.

In Section 2 the Hecke algebra $H_n[t^{\pm 1}]$ and its specializations $H_n(z)$ for $z \in \mathbb{C}^*$ were defined. Because of the surjection $\mathbb{C}[B_n] \to H_n(z)$, irreducible representations of $H_n(z)$ give irreducible representations of B_n. We introduced the term *Hecke type* for irreducible representations of B_n which are the tensor product of a one-dimensional representation and a representation which factors through $H_n(z)$. The object of this section is to list the irreducible representations of $H_n(z)$ of degree n for $n = 5, 6, 7, 8$.

In recent years, algorithms have been discovered which give all of the irreducible representations of $H_n(z)$. In order to state the answer in an intelligible way, it is necessary to outline some of this theory. However, we will not include actual calculations, which used A. Mathas's package, "Specht 2.4, Decomposition matrices for the Hecke algebras of type A", which is included as a share package in GAP, version 3.4.4 [S+ 95]. The results we use are due to R. Dipper - G. James [DJ], A.

Lascoux - B. Leclerc - J.-Y. Thibon [LLT], S. Ariki [A] and I. Grojnowski [G]. Two good sources for the remarks in the next paragraph are the books of A. Mathas [M] and M. Geck and G. Pfeiffer [GP], and we follow [M]. For the representation theory of the symmetric group, the book of G. James and A. Kerber [JK] is the standard reference.

The most basic fact is that $H_n(z)$ is of finite dimension $n!$ over $\mathbb{C}$, and for $z = 1$ or z not a primitive r-th root of unity, where $2 \leq r \leq n$, it is isomorphic as a $\mathbb{C}$-algebra to $\mathbb{C}S_n$, the group algebra of the symmetric group on n letters. Thus for such z the irreducible $H_n(z)$-modules are in one-to-one correspondence with the partitions of n, and their dimensions are given by the hook formula [M, p.53]. This shows, for example, that for $n = 5, 6, 7, 8$ and such z, $H_n(z)$ has irreducible representations of degree n only for $n = 5$.

Associated with each partition λ of n is a *Specht module* $S^\lambda(z)$ over $H_n(z)$ [M, p. 38] (most other sources call this a *dual Specht module*). When $H_n(z)$ is isomorphic to $\mathbb{C}[S_n]$, the Specht modules are irreducible and form a complete set of isomorphism classes of irreducible $H_n(z)$-modules. This is a basic fact in the classical theory of representations of the symmetric group over a field of characteristic zero. On the other hand, if z is a primitive r-th root of unity, where $2 \leq r \leq n$, then $H_n(z)$ is not semisimple, and the modules $S^\lambda(z)$ are not irreducible in general. Then modules $D^\lambda(z)$ are defined by $D^\lambda(z) = S^\lambda(z)/rad(S^\lambda(z))$ (here the radical is defined in terms of a certain bilinear form on $S^\lambda(z)$ - see [M, p. 18 and p. 38]), and each module $D^\lambda(z)$ is either irreducible or zero [M, p. 38, Theorem 3.23].

One further definition is needed in order to state the fundamental theorem on representations of $H_n(z)$. Here we employ the usual notation for partitions of n. Thus $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ means that $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$. By convention, $\lambda_{k+1} = \lambda_{k+2} = \cdots = 0$.

Definition. Let e be a positive integer or ∞. A partition $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ is *e-restricted* (or *column e-regular*) if $\lambda_i - \lambda_{i+1} < e$ for $i = 1, 2, \ldots$.
For $z \in \mathbb{C}$, set $e(z) = r$ if z is a primitive r-th root of unity, where $r > 1$, and $e(z) = \infty$ otherwise.

Theorem 12. (R. Dipper and G. James [DJ, Theorem 7.6]).
(a) [M, p. 38, Theorem 3.23] *The set*

$$\{D^\lambda(z) \mid \lambda \text{ is a partition of } n \text{ such that } D^\lambda(z) \neq 0\ \}$$

is a complete set of non-isomorphic irreducible $H_n(z)$-modules.
(b) [M, p. 48, Theorem 3.43] *The set*

$$\{D^\lambda(z) \mid \lambda \text{ is a partition of } n \text{ such that } \lambda \text{ is an } e(z)\text{-restricted partition of } n\ \}$$

is a complete set of non-isomorphic irreducible $H_n(z)$-modules. ■

It is remarked [M, p. 48] that there is no closed formula for the dimension of $D^\lambda(z)$, but it can be calculated as a by-product of the LLT algorithm [M, Chapter 6, p. 95], which was conjectured by Lascoux, Leclerc and Thibon [LLT] and proved by Ariki [A] and by Grojnowski, who announced a proof in [G].

The following tables give all the dimensions of the modules $D^\lambda(z)$, for $n \leq 8$. In the table for n, the rows are indexed by partitions of n, and the columns are indexed by $e(z)$, for $2 \leq e(z) \leq n$ and $e(z) = \infty$. For example, for $n = 5$ and $\lambda = (2^2 1)$ (we return to the Frobenius notation for partitions), the entry 4 in the column headed by 4 means that $D^\lambda(z)$ has dimension 4, where z is a primitive fourth root of unity.

We also list $max(\lambda_i - \lambda_{i+1})$ for a partition $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ and λ', the partition conjugate to λ, since much of the literature uses $D^{\lambda'}$ to denote the module we call D^λ (our notation follows [M]).

$n = 1$

λ	$max(\lambda_i - \lambda_{i+1})$	λ'	∞
1	1	1	1

$n = 2$

λ	$max(\lambda_i - \lambda_{i+1})$	λ'	2	∞
1^2	1	2	1	1
2	2	1^2	0	1

$n = 3$

λ	$max(\lambda_i - \lambda_{i+1})$	λ'	2	3	∞
1^3	1	3	1	1	1
21	1	21	2	1	2
3	3	1^3	0	0	1

$n = 4$

λ	$max(\lambda_i - \lambda_{i+1})$	λ'	2	3	4	∞
1^4	1	4	1	1	1	1
21^2	1	31	2	3	2	3
2^2	2	2^2	0	1	2	2
31	2	21^2	0	3	1	3
4	4	1^4	0	0	0	1

$n = 5$

λ	$max(\lambda_i - \lambda_{i+1})$	λ'	2	3	4	5	∞
1^5	1	5	1	1	1	1	1
21^3	1	41	4	4	4	3	4
2^21	1	32	5	1	4	5	5
31^2	2	31^2	0	6	6	3	6
32	2	2^21	0	4	1	5	5
41	3	21^3	0	0	4	1	4
5	5	1^5	0	0	0	0	1

$n = 6$

λ	$max(\lambda_i - \lambda_{i+1})$	λ'	2	3	4	5	6	∞
1^6	1	6	1	1	1	1	1	1
21^4	1	51	4	4	5	5	4	5
2^21^2	1	42	5	9	4	8	9	9
2^3	2	3^2	0	1	4	5	5	5
31^3	2	41^2	0	6	10	10	6	10
321	1	321	16	4	16	8	16	16
3^2	3	2^3	0	0	1	5	5	5
41^2	3	31^3	0	0	10	10	4	10
42	2	2^21^2	0	9	5	1	9	9
51	4	21^4	0	0	0	5	1	5
6	6	1^6	0	0	0	0	0	1

$n = 7$

λ	$max(\lambda_i - \lambda_{i+1})$	λ'	2	3	4	5	6	7	∞
1^7	1	7	1	1	1	1	1	1	1
21^5	1	61	6	6	6	6	6	5	6
2^21^3	1	52	14	13	14	8	13	14	14
2^31	1	43	14	1	8	13	14	14	14
31^4	2	51^2	0	15	15	15	15	10	15
321^2	1	421	21	20	20	35	22	35	35
32^2	2	3^21	0	6	20	8	21	21	21
3^21	2	32^2	0	15	1	13	21	21	21
41^3	3	41^3	0	0	20	20	20	10	20
421	2	321^2	0	13	15	35	13	35	35
43	3	2^31	0	0	6	1	14	14	14
51^2	4	31^4	0	0	0	15	15	5	15
52	3	2^21^3	0	0	14	6	1	14	14
61	5	21^5	0	0	0	0	6	1	6
7	7	1^7	0	0	0	0	0	0	1

$n = 8$

λ	$max(\lambda_i - \lambda_{i+1})$	λ'	2	3	4	5	6	7	8	∞
1^8	1	8	1	1	1	1	1	1	1	1
21^6	1	71	6	7	6	7	7	7	6	7
2^21^4	1	62	14	13	20	20	13	19	20	20
2^31^2	1	53	14	28	8	21	27	28	28	28
2^4	2	4^2	0	1	8	13	14	14	14	14
31^5	2	61^2	0	21	15	21	21	21	15	21
321^3	1	521	64	35	64	43	64	45	64	64
32^21	1	431	42	7	40	70	43	70	70	70
3^21^2	2	42^2	0	35	56	13	43	56	56	56
3^22	2	3^22	0	21	1	21	42	42	42	42
41^4	3	51^3	0	0	20	35	35	35	20	35
421^2	2	421^2	0	90	15	90	90	45	90	90
42^2	2	3^21^2	0	13	56	43	13	56	56	56
431	2	32^21	0	28	6	70	27	70	70	70
4^2	4	2^4	0	0	0	1	14	14	14	14
51^3	4	41^4	0	0	0	35	35	35	15	35
521	3	321^3	0	0	64	21	64	19	64	64
53	3	2^31^2	0	0	20	7	1	28	28	28
61^2	5	31^5	0	0	0	0	21	21	6	21
62	4	2^21^4	0	0	0	20	7	1	20	20
71	6	21^6	0	0	0	0	0	7	1	7
8	8	1^8	0	0	0	0	0	0	0	1

6. The Main Theorem.

Theorem 14 below gives an irredundant list of the irreducible complex representations of B_n of degree $\le n$ for $n \ge 3$. Except for the n-dimensional representations of B_n, $n = 5, 6, 7, 8$, they had been classified in [F,L,S]. In [F], the irreducible representations of B_5 of degree ≤ 4 and those of B_6 of degree ≤ 5 were listed in an ad hoc manner. They are listed here in terms of the Hecke algebra notation of Section 5.

The characters $\chi(y) : B_n \to \mathbb{C}^*$, the Burau representation $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$, and the Hecke algebra $H_n(z)$ were defined in Section 2. In Section 5, modules $S^\lambda(z)$ and $D^\lambda(z)$ over $H_n(z)$ were defined. By abuse of notation, we regard them as representations of B_n via the canonical map $B_n \to H_n(z)$.

We need a few more definitions in order to state Theorem 14. The next lemma is a consequence of the Dipper-James Theorem (Theorem 12), and also is proved in [F].

Lemma 13 [F, Lemma 6]. *The Burau representation $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$ is irreducible if and only if $z = 1$ or $z^n \neq 1$ (i.e. $1 + z + \cdots + z^{n-1} \neq 0$). If $\beta_n(z)$ is reducible, it has two composition factors, one trivial and one of degree $n - 2$.* ■

Definition. If $1 + z + \cdots + z^{n-1} = 0$, $\widehat{\beta_n}(z) : B_n \to GL_{n-2}(\mathbb{C})$ denotes the nontrivial composition factor of $B_n(z)$.

Definition. For $z \in \mathbb{C}$, the representation $\varepsilon(z) : B_4 \to GL_3(\mathbb{C})$ is defined by

$$\varepsilon(z)(\sigma_1) = \begin{pmatrix} 0 & 1 & 0 \\ z & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon(z)(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & z & 0 \end{pmatrix},$$

$$\varepsilon(z)(\sigma_3) = \begin{pmatrix} 0 & -1 & 0 \\ -z & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition. $\pi : B_4 \to B_3$ is the homomorphism for which $\pi(\sigma_1) = \sigma_1$, $\pi(\sigma_2) = \sigma_2$, $\pi(\sigma_3) = \sigma_1$.

Theorem 14. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation, where $n \geq 3$ and $2 \leq r \leq n$. Then ρ is equivalent to one of the following, where $y, z \in \mathbb{C}^*$.*

(A) *A representation of Burau type, either*

$$\chi(y) \otimes \beta_n(z) : B_n \to GL_{n-1}(\mathbb{C}), \text{ where } 1 + z + \cdots + z^{n-1} \neq 0; \text{ or}$$
$$\chi(y) \otimes \widehat{\beta_n}(z) : B_n \to GL_{n-2}(\mathbb{C}), \text{ where } 1 + z + \cdots + z^{n-1} = 0.$$

There are no equivalences of representations for distinct parameters y and z except that $\chi(y) \otimes \beta_3(z)$ is equivalent to $\chi(-yz) \otimes \beta_3(z^{-1})$ and $\chi(y) \otimes \widehat{\beta_4}(i)$ is equivalent to $\chi(-iy) \otimes \widehat{\beta_4}(-i)$, where $i, -i$ are the primitive fourth roots of unity.

(B) *A representation of standard type,*

$$\chi(y) \otimes \gamma_n(z) : B_n \to GL_n(\mathbb{C}), \text{ where } z \neq 1.$$

(C) *Representations $\rho \circ \pi : B_4 \to GL_r(\mathbb{C})$, where $2 \leq r \leq 4$ and $\rho : B_3 \to GL_r(\mathbb{C})$ is an irreducible representation* (*the irreducible representations $\rho : B_3 \to GL_4(\mathbb{C})$ are parametrized in* [TW, *Prop. 2.6*]).

(D) $\chi(y) \otimes \varepsilon(z) : B_4 \to GL_3(\mathbb{C})$.

(E) $\chi(y) \otimes (\beta_3(z) \circ \pi) \otimes \widehat{\beta_4}(i) : B_4 \to GL_4(\mathbb{C})$, *where $1 + z + z^2 \neq 0$, and i is a prescribed primitive fourth root of unity.*

(F) $\chi(y) \otimes D^\lambda(i) : B_5 \to GL_4(\mathbb{C})$, *where $\lambda = (2^2 1)$ and i is a prescribed primitive fourth root of unity.*

(G) $\chi(y) \otimes D^\lambda(z) : B_5 \to GL_5(\mathbb{C})$, *where $\lambda = (2^2 1)$ and $(1 + z^2)(1 + z + z^2) \neq 0$.*

(H) *For $\mu = (2^2 1^2)$ and $\nu = (2^3)$, $\chi(y) \otimes D^\mu(i) : B_6 \to GL_4(\mathbb{C})$ and $\chi(y) \otimes D^\nu(i) : B_6 \to GL_4(\mathbb{C})$, where i is a prescribed fourth root of unity.*

(I) *For $\nu = (2^3)$, $\chi(y) \otimes S^\nu(z) : B_6 \to GL_5(\mathbb{C})$, where z is not a root of $(1 + z + z^2)(1 + z^2)$.* (*Note: For $z \neq -1$ and z not a root of $(1 + z + z^2)(1 + z^2)$, $S^\nu(z) = D^\nu(z)$. But $D^\nu(-1) = 0$, while $S^\nu(-1) = D^\mu(-1)$, where $\mu = (2^2 1^2)$.*)

(J) *For $\eta = (31^3)$, $\chi(y) \otimes D^\eta(\pm\omega) : B_6 \to GL_6(\mathbb{C})$, where ω is a prescribed primitive third root of unity, and $-\omega$ is a prescribed primitive sixth root of unity.*

(K) *For $\theta = (2^3 1^2)$ and $\psi = (2^4)$, $\chi(y) \otimes D^\theta(i)$ and $\chi(y) \otimes D^\psi(i)$, where i is a prescribed primitive fourth root of unity.*

There are no equivalences among representations within each of the headings (A)-(K) *except as indicated in* (A). *There are no equivalences between representations under distinct headings* (A)-(K) *except that $\chi(y) \otimes \widehat{\beta_4}(-1)$ in* (A) *is equivalent to $[\chi(y) \otimes \beta_3(-1)] \circ \pi$ in* (C) *and $\chi(y) \otimes \beta_4(1)$ in* (A) *is equivalent to $\chi(y) \otimes \varepsilon(1)$ in* (D).

Proof. For $n = 3$ or $n \geq 9$, all irreducible complex representations of B_n are as in (A) or (B), by [F, Theorem 23] and [S, Theorem 6.5]. All irreducible complex representations of B_4 are as in (A)-(E), by [F, Theorem 23] and [L, Theorem 8.1].

In order to show that the above listing is complete, it must be shown that for $n = 5, 6, 7, 8$, every irreducible representation of B_n which is not of Burau type or standard type, and has degree d, where $2 \leq d \leq n$, occurs in (F)-(K) above. By Theorems 9-11, such representations whose degree is exactly n are of Hecke type. By [F, Theorem 23] those of lower degree are also of Hecke type, although the notation used there differs from the notation of Section 5 which is used here. Thus our list will be complete as soon as we account for all representations of B_n of degree d in the tables of Section 5, for $n = 5, 6, 7, 8$, where $2 \leq d \leq n$.

$n = 5$. Representations of Burau type account for all the representations of degrees 3 and 4 in the table for $n = 5$ except $D^\lambda(z)$, where $\lambda = (2^2 1)$ and z is a primitive fourth root of unity.

Consider $S^\lambda(z)$. Unless z is a primitive third or fourth root of unity (a root of $(1 + z + z^2)(1 + z^2)$), $S^\lambda(z)$ is irreducible and equal to $D^\lambda(z)$. Because $\chi(-z) \otimes S^\lambda(z^{-1})$ and $S^{\lambda'}(z)$ have the same composition factors, we may ignore composition factors of $S^{\lambda'}(z)$, where $\lambda' = (32)$. Thus (G) accounts for all of the five-dimensional irreducible representations of B_5.

This leaves $S^\lambda(z)$ when z is a primitive third or fourth root of unity. If z is a primitive third root of unity, then one composition factor of $S^\lambda(z)$ is one-dimensional and the other is of Burau type (cf. [F, Theorem 19(b)]). If z is a primitive fourth root of unity, then $D^\lambda(z)$ is a four-dimensional irreducible representation of B_5, and $\chi(i) \otimes D^\lambda(i) \cong D^\lambda(-i)$ (cf. [F, Theorem 19(c)]). (No calculations are needed to verify this isomorphism, since there is only one Hecke algebra representation of B_5 in which the eigenvalues of σ_1 are $\{-1, -1, -i, -i\}$.) Thus (F) accounts for all of the four-dimensional irreducible representations of B_5 which are not of Burau type.

$n = 6$. Representations of Burau type account for all representations in the table for $n = 6$ associated with the partition (21^4), plus one further representation in each of the columns headed by $3, 4, 5, 6, \infty$. For $\nu = (2^3)$, $S^\nu(z)$ is five-dimensional and $\chi(-z^{-1}) \otimes S^\nu(z)$ and $S^{\nu'}(z^{-1})$ have the same composition factors, where $\nu' = (3^2)$. Because of this, composition factors of $\chi(y) \otimes S^{\nu'}(z)$ are accounted for by those of $\chi(y) \otimes S^\nu(z)$, for $y, z \in \mathbb{C}^*$.

The module (or representation) $S^\nu(z)$ is irreducible unless z is a primitive second, third or fourth root of unity. For all z which are not second, third or fourth roots of unity, $S^\nu(z) = D^\nu(z)$. These representations $B_6 \to GL_5(\mathbb{C})$ are listed in (I). If $z = -1$, the primitive second root of unity, $D^\nu(-1) = 0$ and $S^\nu(-1) = D^\mu(-1)$, where $\mu = (2^2 1^2)$. If z is a primitive third root of unity, then one composition factor of $S^\nu(z)$ is one-dimensional and the other is of Burau type (cf. [F, Theorem 19(b)]). If z is a primitive fourth root of unity, then there are nonisomorphic four-dimensional modules $D^\nu(z)$ and $D^\mu(z)$, which together give four nonisomorphic modules $D^\nu(\pm i)$ and $D^\mu(\pm i)$ (cf. [F, Theorem 19(c)]). However, since tensoring by $\chi(i)$ carries the unordered pair $\{D^\nu(i), D^\mu(i)\}$ to the unordered pair $\{D^\nu(-i), D^\mu(-i)\}$, the representations in (H) include all $\chi(y) \otimes D^\nu(\pm i)$ and $\chi(y) \otimes D^\mu(\pm i)$, without redundancy.

Lastly, there are the six-dimensional irreducibles $D^\eta(z)$, where $\eta = (31^3)$ and z is a primitive third or sixth root of unity. As above, the fact that $\chi(z^{-1}) \otimes D^\eta(z)$ and $D^\eta(z^{-1})$ have the same composition factors means that only one of $\{\omega, \omega^2\}$ and one of $\{-\omega, -\omega^2\}$ are needed. Thus (J) includes all of these representations without redundancy.

$n = 7$. The table for $n = 7$ shows that there are no irreducible representations of degree ≤ 7 except one-dimensional representations and representations of Burau type.

$n = 8$. The table for $n = 8$ shows that the only representations of degree ≤ 8 other than one-dimensional representations and representations of Burau type are the representations $D^\theta(z)$ and $D^\psi(z)$, where $\theta = (2^3 1^2)$, $\psi = (2^4)$, and z is a primitive fourth root of unity. The eigenvalues of σ_1 in both of these representations are $\{-1, -1, -1, -1, z, z, z, z\}$. Hence tensoring by $\chi(i)$ carries the unordered pair $\{D^\theta(i), D^\psi(i)\}$ to the unordered pair $\{D^\theta(-1), D^\psi(-i)\}$. Thus (K) lists all representations $\chi(y) \otimes D^\theta(\pm i)$ and $\chi(y) \otimes D^\psi(\pm i)$ without redundancy.

Finally, by [F, Theorem 23], the only equivalences between representations in (A),(C),(D) are as claimed. It is easily seen that there are no equivalences among representations within the headings (B)-(K), or any further equivalences between representations under different headings. (In nearly all cases, it suffices to look at the eigenvalues of σ_1.) ■

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Edward Formanek, Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail address: `formanek@math.psu.edu`

Woo Lee, Department of General Education, Kwangju University, Kwangju, 503-703, Korea
E-mail address: woolee@hosim.kwangju.ac.kr

Inna Sysoeva, Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail address: sysoeva@math.psu.edu

Monica Vazirani, Department of Mathematics, The University of California, Berkeley, CA 94720, USA
E-mail address: vazirani@math.berkeley.edu