Irreducible Modules over the Affine Hecke Algebra: a Strong Multiplicity One Result

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Abstract

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Let H_n denote the affine Hecke algebra of type A, and let M denote an irreducible H_n-module. The main new result of this thesis is that the irreducible submodules of $\mathrm{Res}^{H_n}_{H_{n-1}} M$ occur with multiplicity one. This holds regardless of the characteristic of the field R or the multiplicative order of $q \in R$. The main tools of the proof are the functor e_i, which is a slight refinement of restriction, and the functor $\mathrm{Ind}^{H_n}_{H_P}$, which denotes induction within the affine Hecke algebra.

Professor Tsit Y. Lam
Dissertation Committee Chair

Contents

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Words are not sufficient to express my gratitude to Ian Grojnowski for taking me under his wing, teaching me about Hecke algebras (and lots of other math), spending countless hours talking to me, and the list goes on. So, I'll express that gratitude instead by proving theorems.

Chapter 1

Introduction

Let S_n denote the symmetric group on n letters and M denote an irreducible S_n-module over a field R. The decomposition of $\mathrm{Res}^{S_n}_{S_{n-1}} M$ into irreducible S_n-modules is referred to as the branching rule. When R has characteristic 0, it is a classical result that $\mathrm{Res}^{S_n}_{S_{n-1}} M$ is a direct sum of irreducible modules occurring with multiplicity one.

However, when the characteristic of R is prime, the decomposition of $\mathrm{Res}^{S_n}_{S_{n-1}} M$ is still a mystery. It is only in the last decade that Kleshchev described the socle (largest semisimple submodule) of restriction and showed it is multiplicity free. We extend this result to the irreducibles of a larger category of modules.

One can consider the group algebra RS_n a special case of a family of algebras called Hecke algebras. The finite Hecke algebra H_n^{fin}, the cyclotomic (or Ariki-Koike) Hecke algebra H_n^λ, and the affine Hecke algebra H_n are deformations of S_n, and of the wreath products $S_n \wr \mathbb{Z}/r\mathbb{Z}$, and $S_n \wr \mathbb{Z}$, respectively. One may search for a partial branching rule for the Hecke algebra.

In section 5.4, corollary 5.9, we prove that the socle of restriction of any irreducible H_n^λ-module is multiplicity free by proving it for a large class of modules for the affine Hecke algebra. This is a corollary to the strong multiplicity one result, theorem 5.7, that the socle of a certain refinement of restriction is either 0 or irreducible.

This result, theorem 5.7, is the main new result of this thesis.

A major tool in proving the main theorem is the idea of induction and restriction within the affine Hecke algebra H_n as opposed to within the cyclotomic Hecke algebra H_n^λ. This viewpoint was taught to the author by Ian Grojnowski. The structure and the ideas comprising Chapter 5 sprung from conversations with Grojnowski and the results of this paper are our joint work. Further, the organization of this paper is modeled after [G].

Chapter 2 is self-explanatory. Chapter 4 lists examples of several irreducible H_n-modules. Chapter 3 lays out some main tools we will use in the proof of the main theorem, in particular the functor $\mathrm{Ind}^{H_n}_{H_P}$. The other key tool, the functor e_i, is defined in section 5.1. Section 5.3 is also crucial to the proof—it proves the main theorem for the modules $q^i K_n$ which we later use to prove the theorem in general by employing the tools $\mathrm{Ind}^{H_n}_{H_P}$ and e_i. We conclude with a few consequences, most notably the independence of characters.

Chapter 6 is a slight departure from the main thrust of the paper. Although a given irreducible occurs once in the socle, it can occur many times as a composition factor of

$\mathrm{Res}_{H_{n-1}}^{H_n} M$. In Chapter 6 we investigate how these other copies sit in $\mathrm{Res}_{H_{n-1}}^{H_n} M$. Further, we use Grojnowski's functor f_i which is adjoint to e_i, and some results from [G], to describe the copies of M that sit as composition factors of $e_i f_i M$ and $f_i e_i M$. The description is in terms of the action of the center, whose Jordan block structure we give. We conclude Chapter 6 with the result that one of $e_i f_i M$ and $f_i e_i M$ is a direct summand of the other, and that the cyclotomic Mackey map induces the decomposition.

The results of Chapter 5 will appear in a joint paper with Grojnowski and those of Chapter 6 in a separate paper.

The viewpoint of the thesis is only a small piece of the much larger puzzle that investigates the rigid structure on the representation theory of H_n^λ. For further investigation, we urge the reader to read [G].

Chapter 2

Notation and Definitions

Throughout the paper, we fix a field R, and an invertible element $q \in R$, i.e., a homomorphism $\mathbb{Z}[q, q^{-1}] \to R$. We write $\mu_q = \{q^i \mid i \in \mathbb{Z}\}$ for the powers of q in R, and $\ell = |\mu_q|$, so $\ell \in \mathbb{N} \cup \{\infty\}$, and ℓ is the order of $q \in R^\times$.

We further *assume that* $q \neq 1$. This is for clarity of exposition; the changes in the statements of definitions and theorems that must be made when $q = 1$ are sketched in the Appendix.

If A is an R-algebra, we write A-mod for the category of *all* left A-modules, and $\operatorname{Rep} A$ for the category of left A-modules which are finite dimensional as R-modules. (If R is an arbitrary ring, we would also need to define various subcategories, such as the category of A-modules which are projective as R-modules, and so on.)

We recall that the socle of a module M, denoted $\operatorname{soc}(M)$, is the largest semisimple submodule of M, and that the cosocle of M, denoted $\operatorname{cosoc}(M)$, is its largest semisimple quotient.

2.0.1 Grothendieck Group

If $\mathfrak{C}$ is an abelian category, we write $K(\mathfrak{C})$ for the Grothendieck group of $\mathfrak{C}$. This is the quotient of the free abelian group with generators the objects $M \in \mathfrak{C}$ by the ideal generated by the elements $M_1 - M_2 + M_3$ for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

in $\mathfrak{C}$. If objects in $\mathfrak{C}$ have finite length and unique composition factors, we write $K(\mathfrak{C})^*$ for the topological dual of $K(\mathfrak{C})$; i.e. for the linear functions $f{:}K(\mathfrak{C}) \to \mathbb{Z}$ such that $f(M) = 0$ for all but finitely many isomorphism classes of irreducible objects $M \in \mathfrak{C}$.

If $M \in \mathfrak{C}$, let's write $[M]$ for its image in $K(\mathfrak{C})$. Then as M runs through the irreducible objects in $\mathfrak{C}$, the elements $[M]$ form a basis of $K(\mathfrak{C})$, and the functions

$$\delta_M{:}K(\mathfrak{C}) \to \mathbb{Z} \qquad \delta_M(N) = \begin{cases} 0 & \text{if } M \not\simeq N,\ N \text{ irreducible} \\ 1 & \text{if } M \simeq N \end{cases}$$

form a basis of $K(\mathfrak{C})^*$.

Now, if $F{:}\mathfrak{C} \to \mathfrak{C}'$ is an exact functor of abelian categories, we get an induced $\mathbb{Z}$-linear map $F{:}K(\mathfrak{C}) \to K(\mathfrak{C}')$, and we can define its transpose $F^*{:}K(\mathfrak{C}')^* \to K(\mathfrak{C})^*$ by $F^*f = f \circ F$.

We will apply all this to the category $\operatorname{Rep} A$ of finite dimensional representations of an algebra A (over R). Recall that if A and A' are two such algebras, the *absolutely irreducible* representations of $A \otimes A'$ are of the form $M \boxtimes M'$, where M is an absolutely irreducible A-module and M' is an absolutely irreducible A'-module. This is certainly *not* true for general representations of $A \otimes A'$. Throughout the paper, $M \boxtimes M'$ denotes the $A \otimes A'$-module which is isomorphic to $M \otimes_R M'$ as an R-module and has $A \otimes A'$ action given by

$$(a \otimes a') \cdot (m \otimes m') = am \otimes a'm'.$$

2.0.2 The symmetric group

We write S_n for the symmetric group on n letters, $\tau_i = (i \quad i+1)$, $1 \le i \le n-1$, for the simple transpositions. The simple transpositions satisfy the braid relations and the relation $\tau_i^2 = 1$, which we may write in RS_n as

$$\begin{array}{lcc} \textit{braid relations} & \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, & 1 \le i < n-1 \\ & \tau_i\tau_j = \tau_j\tau_i,\ |i-j| > 1 & 1 \le i, j \le n-1 \\ \textit{quadratic relations} & (\tau_i + 1)(\tau_i - 1) = 0. & \end{array}$$

Any $w \in S_n$ can be written as a product of simple transpositions $w = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}$, and if this expression is of minimal length we call it a reduced word expression of w and denote its length $k = \ell(w)$. Length is well-defined, although a permutation can have many different reduced word expressions.

Define a partial order, Bruhat order, on S_n via $u \le w$ if some reduced word expression of u is a subword of some reduced word expression of w. A subword of $w_1w_2\cdots w_k$ is a string $w_{i_1}w_{i_2}\cdots w_{i_m}$ where $1 \le i_1 < i_2 < \cdots < i_m \le k$. There is a unique longest word w_0 of S_n defined by $w_0(i) = n - i + 1$. It has length $\binom{n}{2}$ and $w_0 \ge w$ for all $w \in S_n$.

If $P = (a_1, \ldots, a_k), a_i \in \mathbb{Z}_{>0}, \sum a_i = n$, it is convenient to denote $S_P = S_{a_1} \times \cdots \times S_{a_k} \subseteq S_n$. For example, $S_{(n)} = S_n$, $S_{(1,\ldots,1)} = \{1\}$.

For later use, we establish the notation $W_P = \{w\}$ for the set of minimal length left coset representatives of $S_P \subseteq S_n$, $W_P^{-1} = \{w^{-1}\}_{w \in W_P}$ for the set of minimal length right coset representatives, and x_0 as the longest element of W_P, i.e. the representative of w_0S_P.

Also write $P' = \operatorname{rev} P = (a_k, \ldots, a_2, a_1)$, noting $x_0S_Px_0^{-1} = w_0S_Pw_0^{-1} = S_{P'}$. Then $W_{P'} = \{wx_0^{-1}\}_{w \in W_P}$, $W_{P'}^{-1} = \{x_0w^{-1}\}_{w \in W_P}$.

2.1 Definitions and first properties of Hecke algebras

2.1.1 H_n^{fin}, H_n and H_n^λ

The *finite Hecke algebra*, H_n^{fin} is the R-algebra with generators

$$T_1, \ldots, T_{n-1}$$

and relations

$$\textit{braid relations} \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i, \ |i-j| > 1 \tag{2.1}$$
$$\textit{quadratic relations} \quad (T_i+1)(T_i-q) = 0. \tag{2.2}$$

The braid relations imply that if $w = \tau_{i_1}\cdots\tau_{i_k}$ and $\ell(w) = k$, then $T_{i_1}\cdots T_{i_k}$ depends only on $w \in S_n$. It is denoted T_w, and the $\{T_w\}_{w\in S_n}$ form a basis of H_n^{fin} over R.

The *affine Hecke algebra* H_n (or H_n^{aff}) is the R-algebra, which as an R-module is isomorphic to
$$H_n^{\mathrm{fin}} \otimes_R R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}].$$
The algebra structure is given by requiring that H_n^{fin} and $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ are subalgebras, and that

$$T_iX_iT_i = qX_{i+1}. \tag{2.3}$$

Equivalently, if $f \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, then

$$T_if - {}^{\tau_i}fT_i = (q-1)\frac{f - {}^{\tau_i}f}{1 - X_iX_{i+1}^{-1}} \tag{2.4}$$

where $\tau_i \in S_n$ acts on $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ by permuting X_i and X_{i+1}.

We justify (2.4) by first reducing to the case $f = X_j$. If (2.4) holds for $f, g \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, it holds for fg because ${}^{\tau_i}f{}^{\tau_i}g = {}^{\tau_i}(fg)$ and

$$\begin{aligned}
T_ifg &= \left({}^{\tau_i}fT_i + (q-1)\frac{f - {}^{\tau_i}f}{1 - X_iX_{i+1}^{-1}}\right) g \\
&= {}^{\tau_i}f{}^{\tau_i}gT_i + (q-1){}^{\tau_i}f\frac{g - {}^{\tau_i}g}{1 - X_iX_{i+1}^{-1}} + (q-1)\frac{f - {}^{\tau_i}f}{1 - X_iX_{i+1}^{-1}}g \\
&= {}^{\tau_i}(fg)T_i + (q-1)\frac{fg - {}^{\tau_i}(fg)}{1 - X_iX_{i+1}^{-1}}.
\end{aligned}$$

It also holds for $f+g$ as ${}^{\tau_i}(f+g) = {}^{\tau_i}f + {}^{\tau_i}g$.

If $j \neq i, i+1$, then ${}^{\tau_i}X_j^{\pm 1} = X_j^{\pm 1}$ and $T_iX_j^{\pm 1} = X_j^{\pm 1}T_i$, showing (2.4) holds for $f = X_j^{\pm 1}$. To show that (2.4) holds for $f = X_i$, multiply both sides of (2.3) on the right by $T_i^{-1} = q^{-1}(T_i + 1 - q)$, yielding

$$T_iX_i = X_{i+1}(T_i + 1 - q) = X_{i+1}T_i + (q-1)\frac{X_i - X_{i+1}}{1 - X_i/X_{i+1}}.$$

Multiplying on the left yields (2.4) for $f = X_{i+1}$. For X_{i+1}^{-1}, we invert (2.3) and left multiply by qT_i to get

$$T_iX_{i+1}^{-1} = qT_iT_i^{-1}X_i^{-1}T_i^{-1} = X_i^{-1}(T_i + 1 - q) = X_i^{-1}T_i + (q-1)\frac{X_i^{-1} - X_{i+1}^{-1}}{1 - X_i/X_{i+1}}.$$

Similarly, right multiplication gives the relation for $f = X_i^{-1}$. This verifies the relation (2.4) for any $f \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

The *cyclotomic Hecke algebra* or *Ariki-Koike algebra* H_n^λ is the quotient H_n/I_λ, where I_λ is the ideal generated by $\prod_{q^i \in \mu_q}(X_1 - q^i)^{\lambda_i}$, and $\lambda{:}\mu_q \to \mathbb{Z}_+$ is a function such that $\sum_{i\geq 0}\lambda_i = \sum_{i\geq 0}\lambda(q^i) = r < \infty$. In other words, H_n^λ-modules are those H_n-modules on which X_1 has eigenvalues in μ_q with Jordan block size bounded by λ.

We warn the reader that given an exact sequence $0 \to A \to B \to C \to 0$, where A and C are H_n^λ-modules, B need not be an H_n^λ-module.

Remark 2.1. Observe that any irreducible H_n-module (being finite-dimensional by proposition 2.3 below) is an irreducible H_n^λ-module if we take λ large enough. Conversely, any irreducible H_n^λ-module is an irreducible H_n-module (on which I_λ acts as 0).

The statements and proofs throughout this paper will be made for $\operatorname{Rep} H_n$. However, by the remark above, all results on irreducible modules will extend to $\operatorname{Rep} H_n^\lambda$ as well. This viewpoint, along with proposition 2.6 and remark 4.5 to follow, motivate the following definition.

2.1.2 $\operatorname{Rep} H_n$ and $\operatorname{Rep}_q H_n$

We define $\operatorname{Rep}_q H_n$ to be the full subcategory of $\operatorname{Rep} H_n$ consisting of those modules M in $\operatorname{Rep} H_n$ such that the only eigenvalues of the X_j on M are in μ_q, $1 \leq j \leq n$.

We will focus our attention on $\operatorname{Rep}_q H_n$, although below present some definitions and statements that will hold for $M \in \operatorname{Rep} H_n$ as long as the X_i have all eigenvalues in R (e.g. if $M \in \operatorname{Rep}_q H_n$ or if R is algebraically closed).

Proposition 2.2. *(Bernstein) The center of H_n, $Z(H_n)$, is isomorphic to symmetric Laurent polynomials.*

$$Z(H_n) \simeq R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n}$$

Proof. The relation (2.4) makes it clear that $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n}$ is contained in the center. Suppose $h = \sum f_w T_w \in Z(H_n)$ where $f_w \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Let $u \in S_n$ be maximal with respect to Bruhat order such that $f_u \neq 0$. If $u \neq 1$ then there exists some i such that $u(i) \neq i$. As $T_w X_i = X_{w(i)} T_w + \sum_{w' < w} g_{w'} T_{w'}$, for some $g_{w'} \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, comparing leading terms of $X_i h$ and $h X_i$ we reach a contradiction: the coefficient of T_u in $X_i h - h X_i$ is $f_u(X_i - X_{u(i)}) \neq 0$. Hence $h = f_1 \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. By the relation (2.4), we see $h \in R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n}$. □

Proposition 2.3. *If $M \in H_n$-mod is irreducible, then M is finite dimensional. If the X_i act on M with eigenvalues in R, then $\dim_R M \leq n!$.*

Proof. Write $S = R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Let N be an irreducible submodule of $\operatorname{Res}_S^{H_n} M$. Then N is finite dimensional. By the adjointness of induction and restriction, discussed in section 3 below, M is a quotient of $\operatorname{Ind}_S^{H_n} N$ which has dimension less than or equal to $n! \dim N$. If the X_i take eigenvalues in R, then N will be one-dimensional so that $\dim M \leq n!$. □

2.1.3 Characters

For the following definitions, and throughout the rest of the paper, suppose the X_i act on the module M with eigenvalues in R. Write $S = R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$.

Define the generalized S-eigenspace

$$M[s] = \{v \in M \mid (X_i - s_i)^m v = 0,\ m \gg 0,\ \text{for all } 1 \le i \le n\}$$

where $s = (s_1, \dots, s_n) \in R^n$.

Because the X_i commute, we can decompose M into a direct sum of generalized eigenspaces

$$\operatorname{Res}_S^{H_n} M = \bigoplus_{s \in (R^\times)^n} M[s].$$

We define the character of M to be

$$\operatorname{ch} M = \sum_{s \in (R^\times)^n} (\dim M[s])[s] = [\operatorname{Res}_S M] \in K(\operatorname{Rep} S).$$

As $S = R[X_1^{\pm 1}, \dots, X_n^{\pm 1}] = R[X_1^{\pm 1}] \otimes_R \cdots \otimes_R R[X_n^{\pm 1}]$, we will also write $s = s_1 \boxtimes s_2 \boxtimes \cdots \boxtimes s_n$ for the 1-dimensional S-module with character $s = (s_1, \dots, s_n)$ (omitting the square brackets).

Write $M_s = \{v \in M \mid X_i v = s_i v, 1 \le i \le n\}$ for the *actual* simultaneous eigenspace of the elements X_i. Then

$$M_s \neq 0 \iff M[s] \neq 0.$$

Proposition 2.4. *If M is indecomposable and $M[s] \neq 0$, $M[t] \neq 0$, then there exists $w \in S_n$ such that $w \cdot s = (s_{w^{-1}(1)}, \dots, s_{w^{-1}(n)}) = t$.*

Proof. If $f = f(X_1, \dots, X_n) \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n} = Z(H_n)$ then M indecomposable implies f acts with the same eigenvalue on all of M (perhaps not semisimply). In other words, $\operatorname{Res}_{Z(H_n)} M$ has a filtration with factors all the same 1-dimensional representation (i.e. character). If not, we could decompose M as a direct sum of generalized eigenspaces for $Z(H_n)$, which the H_n action respects.

If $v \in M_s$ is nonzero then $fv = f(s_1, \dots, s_n)v$ and likewise $fu = f(t_1, \dots, t_n)u$ for $0 \neq u \in M_t$. Thus $f(s_1, \dots, s_n) = f(t_1, \dots, t_n)$ for any $f \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$ yielding s and t are in the same S_n-orbit. □

For a module M such that $M[t] \neq 0$ implies there exists $w \in S_n$ such that $t = w \cdot s$, we say the central character of M is the orbit $S_n \cdot s$. Thus, by the above proposition, indecomposable modules have a well-defined central character. Our assumptions on M ensure $s \in (R^\times)^n$, so that conversely, if M has central character $S_n \cdot s$ there there exists $w \in S_n$ such that $M[w \cdot s] \neq 0$. Further, Schur's Lemma implies $Z(H_n)$ acts by scalars on irreducible modules (in particular $f(X_1, \dots, X_n)$ acts as $f(s_1, \dots, s_n)$).

Remark 2.5. It is worthwhile to note that for any algebra A, an A-module can be decomposed into a direct sum of isotypic components, i.e. generalized eigenspaces, of $Z(A)$. That an indecomposable module has a well-defined central character is one instance of this. We will see another instance of this in claim 5.1 of Chapter 5.

Proposition 2.6. *If M an H_n-module and $\{\alpha\} \subseteq R^{\times}$ are all of the eigenvalues for some X_i on M, then for any j, the only eigenvalues of X_j on M lie in $\{q^i\alpha\}_{q^i \in \mu_q}$.*

Proof. It suffices to prove this for H_2.

Let α be an eigenvalue of X_1. There exists a simultaneous eigenvector $v \in M$ for X_1 and X_2 such that $X_1 v = \alpha v$. Suppose $X_2 v = \beta v$.

Case 1. $\{v, T_1 v\}$ are linearly dependent. The relation $(T_1+1)(T_1-q)=0$ implies $T_1 v \in \{-v, qv\}$. Then $\beta v = X_2 v = q^{-1}T_1X_1T_1 v \in \{q\alpha v, q^{-1}\alpha v\}$ by relation (2.3), yielding $\beta \in \{q\alpha, q^{-1}\alpha\}$.

Case 2. $\{v, T_1 v\}$ are linearly independent. Then their span forms an H_2 submodule, and with respect to this basis

$$X_1 = \begin{bmatrix} \alpha & (1-q)\beta \\ 0 & \beta \end{bmatrix} \quad X_2 = \begin{bmatrix} \beta & (q-1)\beta \\ 0 & \alpha \end{bmatrix} \quad T_1 = \begin{bmatrix} 0 & q \\ 1 & q-1 \end{bmatrix}.$$

Thus β is also an eigenvalue of X_1. □

Corollary 2.7. *A finite dimensional H_n-module M is in $\operatorname{Rep}_q H_n$ if and only if all eigenvalues for X_1 on M lie in μ_q. In particular, any module in $\operatorname{Rep} H_n^{\lambda}$ can be realized as being in $\operatorname{Rep}_q H_n$.*

In the next chapter we will compute the characters of induced modules and of several irreducible modules. In theorem 5.11 we show an irreducible module is determined by its character.

Chapter 3

Induction and Restriction

Recall that if $A \subset B$ are R-algebras, the exact functor of restriction

$$\operatorname{Res}_A^B : B\text{-mod} \to A\text{-mod}$$

has left and right adjoints, Ind and $\check{\operatorname{Ind}}$ defined by

$$\operatorname{Ind}_A^B : A\text{-mod} \to B\text{-mod} \qquad M \mapsto B \otimes_A M$$

$$\check{\operatorname{Ind}}_A^B : A\text{-mod} \to B\text{-mod} \qquad M \mapsto \operatorname{Hom}_A(B, M);$$

i.e.

$$\operatorname{Hom}_B(\operatorname{Ind}_A^B M, N) = \operatorname{Hom}_A(M, \operatorname{Res} N) \qquad \operatorname{Hom}_B(N, \check{\operatorname{Ind}} M) = \operatorname{Hom}_A(\operatorname{Res} N, M).$$

We will also refer to the adjointness of Ind (respectively $\check{\operatorname{Ind}}$) and Res as Frobenius reciprocity. Explicit isomorphisms are given by the maps

$$\alpha \mapsto (v \mapsto \alpha(1 \otimes v)) \qquad \beta \mapsto (w \mapsto \beta(w)(1)).$$

We will repeatedly make use of this when discussing B-homomorphisms $\operatorname{Ind}_A^B M \xrightarrow{\alpha} N$. To define such a map, we need only define an A-homomorphism $M \to \operatorname{Res}_A^B N$; in other words, define $\alpha(1 \otimes M)$ and extend by $\alpha(b \otimes M) = b\alpha(1 \otimes M)$.

If B is a free A-module, then Ind and $\check{\operatorname{Ind}}$ are exact functors also. Further, if $A \subset B \subset C$ are inclusions of R-algebras, we have transitivity of induction and restriction:

$$\operatorname{Res}_A^B \operatorname{Res}_B^C = \operatorname{Res}_A^C, \quad \operatorname{Ind}_B^C \operatorname{Ind}_A^B = \operatorname{Ind}_A^C, \quad \check{\operatorname{Ind}}_B^C \check{\operatorname{Ind}}_A^B = \check{\operatorname{Ind}}_A^C$$

Now apply these remarks to the affine Hecke algebra. Given a sequence $P = (a_1, \ldots, a_k)$ of non-negative integers summing to n, write $H_P = H_{a_1} \otimes \cdots \otimes H_{a_k}$. We have an obvious embedding

$$H_P = H_{a_1} \otimes \cdots \otimes H_{a_k} \hookrightarrow H_n$$

which makes H_n a free H_P-module. Applying the previous remarks we get exact functors Res, Ind, $\check{\operatorname{Ind}}$. These functors depend on the order $(a_1, \ldots, a_k)$ and not just on the underlying set! When unambiguous from context, we may just write Res_{H_P}, Res_P or simply Res for $\operatorname{Res}_{H_P}^{H_n}$, and similarly for Ind to lighten notation.

3.1 Induced modules and their characters

3.1.1 $\mathrm{Ind}_S^{H_n}$

Claim 3.1. *Let $s = (s_1, \ldots, s_n) \in (R^\times)^n$ (or $\in \mu_q^n$). We also write $s = s_1 \boxtimes \cdots \boxtimes s_n$. Then*

$$\mathrm{ch}(\mathrm{Ind}_S^{H_n} s) = \sum_{w \in S_n} w \cdot s.$$

Proof. First, $\mathrm{Ind}_S^{H_n} s = H_n \otimes_S s$, which as an R-module is $H_n^{\mathrm{fin}} \otimes_R S \otimes_S s = H_n^{\mathrm{fin}} \otimes_R s$ and so $\dim \mathrm{Ind}_S^{H_n} s = \dim H_n^{\mathrm{fin}} = n!$. Further, if we write $Rv = s = s_1 \boxtimes \cdots \boxtimes s_n$, then a basis of $\mathrm{Ind}_S^{H_n} s$ is given by $\{T_w \otimes v\}_{w \in S_n}$. If we order the basis respecting Bruhat order, then by the relation (2.4)

$$\begin{aligned} X_i T_w \otimes v &= T_w \otimes X_{w^{-1}(i)} v + \sum_{\substack{u<w \\ g_u \in S}} T_u \otimes g_u v \\ &= s_{w^{-1}(i)} T_w \otimes v + \text{lower terms} \\ &= (w \cdot s)_i T_w \otimes v + \text{lower terms.} \end{aligned}$$

Thus the action of all the X_i is upper triangular with respect to this basis, and the diagonal entries give the simultaneous eigenvalues, i.e. the character of $\mathrm{Ind}_S^{H_n} s$, which is thus $\sum\limits_{w \in S_n} w \cdot s$. ☐

Claim 3.2. *1. $X_1 (T_1 \left(\frac{1 - X_1/X_2}{1-q}\right) + 1) = (T_1 \left(\frac{1 - X_1/X_2}{1-q}\right) + 1) X_2$.*

2. For any S-module N there is an H_2-homomorphism $\mathrm{Ind}_S^{H_2} N \xrightarrow{\sigma} \mathrm{Ind}_S^{H_2} \mathrm{rev}^* N$ given by $1 \otimes u \longmapsto (T_1 \left(\frac{1-X_1/X_2}{1-q}\right) + 1) \otimes u$.

3. The map $\sigma^2 : \mathrm{Ind}_S^{H_2} s_1 \boxtimes s_2 \longrightarrow \mathrm{Ind}_S^{H_2} s_1 \boxtimes s_2$ is multiplication by the scalar

$$\left(\frac{1 - q\frac{s_1}{s_2}}{1-q}\right)\left(\frac{1 - q\frac{s_2}{s_1}}{1-q}\right).$$

4. When $s_2 \neq q^{\pm 1} s_1$, σ is an isomorphism. When $s_2 = q^{\pm 1} s_1$, the sequence

$$\mathrm{Ind}_S^{H_2} s_1 \boxtimes s_2 \xrightarrow{\sigma} \mathrm{Ind}_S^{H_2} s_2 \boxtimes s_1 \xrightarrow{\sigma} \mathrm{Ind}_S^{H_2} s_1 \boxtimes s_2 \tag{3.1}$$

is exact, and so $\mathrm{Ind}_S^{H_2} s_1 \boxtimes s_2$ and $\mathrm{Ind}_S^{H_2} s_2 \boxtimes s_1$ have the same composition factors.

Proof. Write $p = (T_1 \left(\frac{1-X_1/X_2}{1-q}\right) + 1)$. The proof of (1) is a straightforward computation.

(2) follows from (1). By Frobenius reciprocity it suffices to give σ as an S-map $N \to \mathrm{Res}_S^{H_2} \mathrm{Ind}_S^{H_2} \mathrm{rev}^* N$, where $\mathrm{rev}^* N$ denotes the module with underlying set N and S-action induced by the automorphism $\mathrm{rev}_S : S \to S$, $X_1 \mapsto X_2$, $X_2 \mapsto X_1$.

One may compute $p^2 = \left(\frac{1-qX_1/X_2}{1-q}\right)\left(\frac{1-qX_2/X_1}{1-q}\right)$, which in particular is central, yielding (3) when $N = s_1 \boxtimes s_2$.

For (4), because σ^2 is multiplication by a nonzero scalar when $s_2 \neq q^{\pm 1}s_1$, σ itself must be an isomorphism. When $s_2 = s_1$, σ is just the identity.

In the case $s_2 = q^{\pm 1}s_1$, write Rv and Rw for $s_1 \boxtimes s_2$ and $s_2 \boxtimes s_1$ (not necessarily in that order). One application of σ has image $R(T_1 - q) \otimes v$ and kernel $R(T_1 + 1) \otimes w$ while the next has image $R(T_1 + 1) \otimes w$ and kernel $R(T_1 - q) \otimes v$, yielding the exactness of the sequence above (even when $q = -1$). □

Corollary 3.3. *All irreducible modules with central character $S_n \cdot s = S_n \cdot (s_1, \ldots, s_n)$ occur as composition factors of $\operatorname{Ind}_S^{H_n} s$. In particular, all the $\operatorname{Ind}_S^{H_n} w \cdot s$, $w \in S_n$, have the same composition factors.*

Proof. First we prove the second statement. We can extend the map of claim 3.2 to

$$\operatorname{Ind}_S^{H_n} s_1 \boxtimes \cdots \boxtimes s_i \boxtimes s_{i+1} \boxtimes \cdots \boxtimes s_n \xrightarrow{\sigma_i} \operatorname{Ind}_S^{H_n} s_1 \boxtimes \cdots \boxtimes s_{i+1} \boxtimes s_i \boxtimes \cdots \boxtimes s_n.$$

By claim 3.2 statement 4, σ_i is either an isomorphism or gives an exact sequence as in (3.1); in either case, the above induced modules have the same composition factors. As the simple transpositions generate S_n, we see all the $\operatorname{Ind}_S^{H_n} w \cdot s$ have the same composition factors.

If M is irreducible with central character $S_n \cdot s$ then $M[w \cdot s] \neq 0$ for some $w \in S_n$ by the discussion following proposition 2.4. In other words, there is a nonzero S-map $w \cdot s \to \operatorname{Res}_S^{H_n} M$. By Frobenius reciprocity there is a nonzero, hence surjective, map $\operatorname{Ind}_S^{H_n} w \cdot s \to M$ and so M is a composition factor of $\operatorname{Ind}_S^{H_n} s$. □

3.1.2 Shuffles

To describe the character of a module induced from H_P to H_n we introduce the notion of a shuffle. Then we proceed as in claim 3.1.

We say a string $s = s_1 s_2 \cdots s_k$ is a shuffle of t and u if t is a subword of s and u is its complementary subword. The shuffle of t and u, denoted $t \sqcup\!\sqcup u$, is then the formal sum of all shuffles of t and u, with multiplicity. We can extend $\sqcup\!\sqcup$ linearly to sums of words. If we write tu for the concatenation of t and u, which have length m and n respectively, then a useful description of shuffling is

$$t \sqcup\!\sqcup u = \sum_{w \in W_{(m,n)}} w \cdot tu.$$

In other words the permutations of $12 \cdots n+m$ that keep $1 \cdots m$ in order and $m+1 \cdots m+n$ in order are exactly the minimal length left coset representatives $W_{(m,n)}$.

Example 1. $1q \sqcup\!\sqcup qq^2 = 1qqq^2 + 1qqq^2 + q1qq^2 + 1qq^2q + q1q^2q + qq^21q$

Example 2. $1 \sqcup\!\sqcup 23 = 123 + 213 + 231 = (1 + s_1 + s_2 s_1) \cdot 123$

Lemma 3.4 (shuffle lemma). *Let $M \in \operatorname{Rep}_q H_m$, $N \in \operatorname{Rep}_q H_n$. Then*

$$\operatorname{ch} \operatorname{Ind}_{H_m \otimes H_n}^{H_{m+n}} M \boxtimes N = \operatorname{ch} M \sqcup\!\sqcup \operatorname{ch} N.$$

Proof. Write $\operatorname{ch} M = \sum s^{(j)}, \operatorname{ch} N = \sum t^{(k)}$. Let $\{u_j\}, \{v_k\}$ be bases of M, N respectively on which S acts upper triangularly, i.e. $u_j \in M[s^{(j)}]$ and $X_i u_j = s_i^{(j)} u_j + \text{lower terms}$, $v_k \in N[t^{(k)}]$ and $X_i v_k = t_i^{(k)} v_k + \text{lower terms}$. Write $s^{(j,k)}$ for the concatenation $s^{(j)} t^{(k)}$ so that $X_i(u_j \otimes v_k) = s_i^{(j,k)}(u_j \otimes v_k) + \text{lower terms}$. Then

$$\operatorname{ch} M \sqcup\!\sqcup \operatorname{ch} N = \sum_{w \in W_{(m,n)}} \sum_{j,k} w \cdot s^{(j,k)}.$$

Let $W = W_{(m,n)} = \{w\}$ denote minimal length left coset representatives as above. Then $\{T_w \otimes (u_j \otimes v_k)\}_{w \in W}$ is a basis of $\operatorname{Ind}_{H_m \otimes H_n}^{H_{m+n}} M \boxtimes N$, ordered respecting Bruhat order and the order on $\{u_j\}, \{v_k\}$. As in claim 3.1 we compute

$$\begin{aligned} X_i T_w \otimes (u_j \otimes v_k) =& T_w X_{w^{-1}(i)} \otimes (u_j \otimes v_k) + \text{lower terms} \\ =& s^{(j,k)}_{w^{-1}(i)} T_w \otimes (u_j \otimes v_k) + \text{lower terms} \\ =& (w \cdot s^{(j,k)})_i T_w \otimes (u_j \otimes v_k) + \text{lower terms.} \end{aligned}$$

This shows all X_i act upper triangularly with respect to this basis, yielding

$$\operatorname{ch} \operatorname{Ind}_{H_m \otimes H_n}^{H_{m+n}} M \boxtimes N = \sum_{w \in W_{(m,n)}} \sum_{j,k} w \cdot s^{(j,k)} = \operatorname{ch} M \sqcup\!\sqcup \operatorname{ch} N.$$

□

3.1.3 Ind and $\check{\operatorname{Ind}}$

To describe one of the relationships between Ind and $\check{\operatorname{Ind}}$, we introduce the involution $\operatorname{rev}: H_n \to H_n$ defined by

$$X_i \mapsto X_{n-i+1} \qquad T_i \mapsto -(T_{n-i} + 1 - q).$$

In other words, rev is induced by w_0, as $X_i \mapsto X_{w_0(i)}$, $T_i = T_{s_i} \mapsto -qT^{-1}_{w_0 s_i w_0^{-1}}$, and hence $T_u \mapsto (-q)^{\ell(u)} T^{-1}_{w_0 u w_0^{-1}}$, where w_0 is the longest permutation in S_n.

For $P = (a_1, \ldots, a_k), \sum a_i = n$, write $\operatorname{rev}(P) = P' = (a_k, \ldots, a_1)$, so that $\operatorname{rev}(H_P) = H_{P'}$. Let $\operatorname{rev}_P = \otimes \operatorname{rev}: H_P \to H_P$. So the rev_S of claim 3.2 was $\operatorname{rev}_{1,1}$.

For x_0 the shortest element of the coset $w_0 S_P$, there is also a map $H_P \to H_{P'}$, $\operatorname{rev} \circ \operatorname{rev}_P$, which is in fact induced by x_0. All of the maps above induce functors on H_P-modules via $h \cdot \operatorname{rev}_P^*(M) = \operatorname{rev}_P(h) \cdot M$. Write $\check{M}$ for $\operatorname{rev}^* \operatorname{rev}_P^*(M)$, so that

$$T_x \check{M} = T_{x_0 x x_0^{-1}} M, \qquad X_i \check{M} = X_{x_0(i)} M. \tag{3.2}$$

Proposition 3.5. *Let M be an H_P-module and M_i be H_{a_i}-modules. Then*

1. $\operatorname{Ind}_{H_P}^{H_n} M \simeq \check{\operatorname{Ind}}_{H_{P'}}^{H_n} \check{M}$.

2. $\operatorname{Ind}_{H_P}^{H_n} M \simeq \operatorname{rev}^* \check{\operatorname{Ind}}_{H_P}^{H_n} \operatorname{rev}_P^* M$.

3. $\mathrm{Ind}_{H_P}^{H_n} M_1 \boxtimes \cdots \boxtimes M_k \simeq \check{\mathrm{Ind}}_{H_{P'}}^{H_n} M_k \boxtimes \cdots \boxtimes M_1$.

Proof. We will construct an explicit isomorphism. It would suffice to give the proof for $k = 2$, but as the general case is no more complicated notationally, we present that.

Recall $W = W_P = \{w\}$ = minimal length left coset representatives for $S_P \subseteq S_n$. Let $W' = \{x_0 w^{-1}\}_{w \in W} = \{w'\}$ = minimal length right coset representatives for $S_{P'} = S_{\mathrm{rev}\, P} \subseteq S_n$, where x_0 is the longest element of W. If $\{v_j\}_{j \in J}$ is a basis of M, then $\{\check{v}_j\}_{j \in J}$ is a basis of $\check{M}$, $\{T_w \otimes v_j\}_{\substack{j \in J \\ w \in W}}$ is a basis of $\mathrm{Ind}_{H_P}^{H_n} M$, and $\{\phi_{w',v_j}\}_{\substack{j \in J \\ w' \in W'}}$ is a basis of $\check{\mathrm{Ind}}_{H_{P'}}^{H_n} \check{M}$, where $\phi_{w',v_j}(T_{u'}) = \delta_{w',u'} \check{v}_j$ for $u' \in W'$.

Write ϕ_v for $\phi_{x_0,v}$. Define the isomorphism

$$\psi \colon \mathrm{Ind}_{H_P}^{H_n} M \longrightarrow \check{\mathrm{Ind}}_{H_{P'}}^{H_n} \check{M}$$

$$h \otimes v \overset{\psi}{\longmapsto} h\phi_v \qquad \text{for } h \in H_n.$$

To check that ψ is well-defined, it suffices by Frobenius reciprocity to check that the map $v \mapsto \phi_v$ is an H_P-map. In other words, that $T_x \cdot \phi_v = \phi_{T_{x_0 x x_0^{-1}} v}$ for $x \in S_P$ and $X_i \cdot \phi_v = \phi_{X_{x_0(i)} v}$ by (3.2). To show this, we first observe that for $x \in S_P$, because x_0 is the shortest element of $x_0 S_P$ and of $S_{P'} x_0$,

$$T_{x_0} T_x = T_{x_0 x} = T_{x_0 x x_0^{-1} x_0} = T_{x_0 x x_0^{-1}} T_{x_0};$$

and second, that if $T_{w'} T_x = \sum_{\substack{y \in S_{P'} \\ u' \in W'}} c_{yu'} T_y T_{u'}$ then $W' \ni w' \neq x_0$, $y \in S_{P'}$ implies $c_{yx_0} = 0$. The second observation follows because no reduced word expression for x_0 can occur as a subword of the concatenation of w' and x. In greater detail: considering the s_i such that $\ell(x_0 s_i) = \ell(x_0) - 1$ we see $s_i \notin S_P$ so s_i is not a subword of x but is a (final) subword of x_0. Of these, there must be some s_j such that $\ell(w' s_j) = \ell(w') + 1$ as $w' = x_0 w^{-1}$ for some $1 \neq w \in W$. Given that, $T_y T_{x_0}$ cannot occur in $T_{w'} T_x$ for $y \in S_{P'}, x \in S_P$.

Then

$$\begin{aligned}
(T_x \cdot \phi_v)(T_{w'}) &= \phi_v(T_{w'} T_x) = \begin{cases} \phi_v(T_{x_0 x x_0^{-1}} T_{x_0}) = T_{x_0 x x_0^{-1}} v & \text{if } w' = x_0 \\ 0 & \text{else} \end{cases} \\
&= \phi_{T_{x_0 x x_0^{-1}} v}(T_{w'}).
\end{aligned}$$

As they agree on $\{T_{w'}\}$, $T_x \phi_v = \phi_{T_{x_0 x x_0^{-1}} v}$.

Similarly

$$\begin{aligned}
(X_i \cdot \phi_v)(T_{w'}) &= \phi_v(T_{w'} X_i) = \phi_v(X_{w'(i)} T_{w'} + \text{ lower terms}) = \begin{cases} X_{x_0(i)} v & \text{if } w' = x_0 \\ 0 & \text{else} \end{cases} \\
&= \phi_{X_{x_0(i)} v}(T_{w'}).
\end{aligned}$$

Next, we show ψ is surjective, yielding it is an isomorphism by a dimension count. Consider

$$T_w \cdot \phi_v(T_{u'}) = \phi_v(T_{u'} T_w) = \phi_v(T_{x_0 u^{-1}} T_w) = \begin{cases} v & \text{if } u = w, \\ 0 & \text{if } u \not\leq w. \end{cases}$$

This shows the $\{T_w \cdot \phi_{v_j}\}_{\substack{j\in J \\ w\in W}}$ have a triangular change of basis matrix with respect to $\{\phi_{w',v_j}\}$, so in particular span $\check{\operatorname{Ind}}_{H_{P'}}^{H_n} \check{M}$, yielding ψ is onto.

Statement 2 of the proposition follows from the first as $\operatorname{rev}^*(\check{\operatorname{Ind}}_{H_{P'}}^{H_n} \operatorname{rev}_P^* M) = \check{\operatorname{Ind}}_{\operatorname{rev} H_P}^{\operatorname{rev} H_n} \operatorname{rev}^* \operatorname{rev}_P^* M = \check{\operatorname{Ind}}_{H_{P'}}^{H_n} \check{M}$.

Finally, since $M = M_1 \boxtimes \cdots \boxtimes M_k$ implies $\check{M} = M_k \boxtimes \cdots \boxtimes M_1$, the third is a special case of the first. □

Given proposition 3.5, we will rarely use the functor $\check{\operatorname{Ind}}$. However, it does imply the following useful observation which allows us to detect whether certain induced modules are irreducible.

Proposition 3.6. *Suppose* $\operatorname{Ind}(M \boxtimes N) \simeq \operatorname{Ind}(N \boxtimes M)$. *If* $M \boxtimes N$ *is an irreducible module which occurs with multiplicity one in a composition series for* $\operatorname{Res}\operatorname{Ind}(M \boxtimes N)$, *then* $\operatorname{Ind}(M \boxtimes N)$ *is irreducible.*

Proof. Let $0 \to K \to \operatorname{Ind}(M \boxtimes N) \to Q \to 0$ be an exact sequence. Then $M \boxtimes N$ is a submodule of $\operatorname{Res} Q$, if $Q \neq 0$, by Frobenius reciprocity, and as $\operatorname{Ind}(M \boxtimes N) \simeq \operatorname{Ind}(N \boxtimes M) \simeq \check{\operatorname{Ind}}(M \boxtimes N)$, $M \boxtimes N$ is also a submodule of $\operatorname{Res} K$ if $K \neq 0$. As $M \boxtimes N$ occurs with multiplicity one, either K or Q is zero, and so $\operatorname{Ind}(M \boxtimes N)$ is irreducible. □

It is well-known that there exists an involution $D \colon \operatorname{Rep}_q H_n \to \operatorname{Rep}_q H_n$ that takes irreducibles to themselves and

$$0 \to A \to B \to C \to 0 \iff 0 \to D(C) \to D(B) \to D(A) \to 0.$$

This is an unpublished result of Bernstein [B] that has since entered the literature.

This functor also has the property that

$$D(\operatorname{Ind} M) = \check{\operatorname{Ind}}\, D(M).$$

In other words, there is an even stronger relationship between Ind and $\check{\operatorname{Ind}}$: for M irreducible, the factors in the socle series of $\operatorname{Ind} M$ coincide with the factors in the cosocle series of $\check{\operatorname{Ind}}\, M$ *in order*.

Chapter 4

Examples and characters of some modules

Remark 4.1. If $M \in \operatorname{Rep}_q H_n$ is irreducible, then so is $\operatorname{rev}^* M$, and so is $q^i M$, meaning the module isomorphic to M as an H_n^{fin}-module, but with S action $X_j \cdot v = q^i X_j v$ for all $1 \leq j \leq n$. If $\operatorname{ch} M = \sum s = \sum(s_1, \dots, s_n)$ then

$$\operatorname{ch} q^i M = \sum q^i s = \sum(q^i s_1, \dots, q^i s_n) \qquad \operatorname{ch} \operatorname{rev}^* M = \sum w_0 \cdot s = \sum(s_n, \dots, s_1).$$

The Hecke algebra H_n has a 1-dimensional module $\mathbf{1}_{(q^i, q^{i+1}, \dots, q^{i+n-1})}$ with character $(q^i, q^{i+1}, \dots, q^{i+n-1})$, which we will call the trivial module, on which $X_1 - q^i$ and $T_j - q$ vanish for all j.

H_n also has a 1-dimensional module, which we will call the Steinberg module, and denote $\operatorname{St}_{(q^i, \dots, q^{i-n+1})}$. The Steinberg module has character $(q^i, q^{i-1}, \dots, q^{i-n+1})$, and $X_1 - q^i$ and $T_j + 1$ vanish on it for all j. Notice that the Steinberg module is distinct from the trivial if $q \neq -1$.

Furthermore, any 1-dimensional module in $\operatorname{Rep}_q H_n$ must be one of the above. If $M = Rv$ is such a module, then as $(T_j - q)(T_j + 1) = 0$, we must have $T_j v = qv$ or $-v$. If $T_j v \neq T_{j\pm 1} v$ (and so $q \neq -1$) then $T_j T_{j\pm 1} T_j v \neq T_{j\pm 1} T_j T_{j\pm 1} v$ contradicting the braid relations. As $M \in \operatorname{Rep}_q H_n$ there is some i such that $X_1 v = q^i v$ and the relation (2.3) $qX_{j+1} = T_j X_j T_j$ yields $\operatorname{ch} M = (q^i, \dots, q^{i+n-1})$ or $(q^i, \dots, q^{i-n+1})$.

Remark 4.2. We make the obvious but useful observation that if A is a composition factor of B and $\operatorname{ch} A = \operatorname{ch} B$, then $A = B$.

4.1 H_0

$H_0 = R$ has just one irreducible R, which we denote by $\mathbf{1}$.

4.2 H_1

$H_1 = R[X_1^{\pm 1}]$ is commutative, so its irreducible modules in $\operatorname{Rep}_q H_1$ are one-dimensional and determined by the X_1 action, i.e. by their characters $s = (q^i)$. In chapter

5, we will see that for any n, an H_n-irreducible is determined by its character.

4.3 H_2

By corollary 3.3, any irreducible in $\mathrm{Rep}_q H_2$ occurs as a composition factor of $\mathrm{Ind}\, q^i \boxtimes q^j$, which is 2-dimensional with character $(q^i, q^j) + (q^j, q^i)$ by claim 3.1. The map τ of claim 3.2 gives an isomorphism $\mathrm{Ind}\, q^i \boxtimes q^j \to \mathrm{Ind}\, q^j \boxtimes q^i$ if $q^j \neq q^{i\pm 1}$. In this case, the irreducibility of $\mathrm{Ind}\, q^i \boxtimes q^j$ follows from the above description of all 1-dimensional modules. However, we directly compute the H_2 action for completeness.

Write $Rv = q^i \boxtimes q^j$. With respect to the basis $\{1 \otimes v, T_1 \otimes v\}$

$$X_1 = \begin{bmatrix} q^i & (1-q)q^j \\ 0 & q^j \end{bmatrix} \quad X_2 = \begin{bmatrix} q^j & (q-1)q^i \\ 0 & q^i \end{bmatrix} \quad T_1 = \begin{bmatrix} 0 & q \\ 1 & q-1 \end{bmatrix}$$

on $\mathrm{Ind}\, q^i \boxtimes q^j$. Notice that when $q^i = q^j$, the S eigenspace is only 1-dimensional.

In the case $q^i \neq q^j$, then with respect to the basis $\{(T_1 + \frac{1-q}{1-q^{i-j}}) \otimes v, 1 \otimes v\}$,

$$X_1 = \begin{bmatrix} q^j & 0 \\ 0 & q^i \end{bmatrix} \quad X_2 = \begin{bmatrix} q^i & 0 \\ 0 & q^j \end{bmatrix} \quad T_1 = \begin{bmatrix} -\frac{1-q}{1-q^{j-i}} & 1 \\ \frac{(1-qq^{i-j})(q-q^{i-j})}{(1-q^{i-j})(1-q^{i-j})} & -\frac{1-q}{1-q^{i-j}} \end{bmatrix}.$$

From this, or from considering images and kernels of τ, we see

$$0 \to \mathrm{St}_{(q^{i+1}, q^i)} \to \mathrm{Ind}\, q^i \boxtimes q^{i+1} \to \mathbf{1}_{(q^i, q^{i+1})} \to 0 \tag{4.1}$$

$$0 \to \mathbf{1}_{(q^{i-1}, q^i)} \to \mathrm{Ind}\, q^i \boxtimes q^{i-1} \to \mathrm{St}_{(q^i, q^{i-1})} \to 0 \tag{4.2}$$

In summary, the 1-dimensional irreducibles of $\mathrm{Rep}_q H_2$ are $\mathbf{1}_{(q^i, q^{i+1})}$ and for $q \neq -1$ $\mathrm{St}_{(q^i, q^{i-1})}$. The 2-dimensional irreducibles are $\mathrm{Ind}\, q^i \boxtimes q^j$ where $q^j \neq q^{i\pm 1}$.

Remark 4.3. We can reinterpret the irreducibility of $\mathrm{Ind}\, q^i \boxtimes q^j$, $q^j \neq q^{i\pm 1}$, as meaning that if M is any module, $\dim M[(s_1, s_2, \ldots, q^i, q^j, \ldots, s_n)] = \dim M[(s_1, s_2, \ldots, q^j, q^i, \ldots, s_n)]$ and that $\dim M[(s_1, s_2, \ldots, q^i, q^i, \ldots, s_n)]$ is even. Another interpretation of this is given in remark 5.13.

Claim 4.4. *Let N, M be irreducible modules of H_n and H_m respectively. Write* $\mathrm{ch}\, M = \sum_j s^{(j)}$, $\mathrm{ch}\, N = \sum_k t^{(k)}$.

1. *If $\prod_{h,i,j,k} (s_i^{(j)} - q t_h^{(k)})(s_i^{(j)} - q^{-1} t_h^{(k)}) \neq 0$, then $\mathrm{Ind}\, M \boxtimes N$ is irreducible.*

2. *If M has central character $S_m \cdot (s_1, \ldots, s_m)$ and (after reordering) there exists $1 \leq k < m$ such that for all j, $\prod_{\substack{1 \leq i \leq k \\ k < h \leq m}} (s_i - q^j s_h) \neq 0$, then M is an induced module.*

Proof. Suppose Q is an irreducible quotient of $\mathrm{Ind}_{H_m \otimes H_n}^{H_{m+n}} M \boxtimes N$. Then the concatenation $s^{(j)} t^{(k)}$ occurs in $\mathrm{ch}\, Q$ by Frobenius reciprocity. However, by the above remark and the hypothesis, if $s^{(j)} t^{(k)}$ occurs then so does $s^{(j)} \sqcup\!\sqcup\, t^{(k)}$. In other words, $\mathrm{ch}\, Q = \mathrm{ch}\, M \sqcup\!\sqcup\, \mathrm{ch}\, N$, yielding $\mathrm{Ind}\, M \boxtimes N$ is irreducible.

Another way to state the hypothesis of part 2 is that the $s_1, \ldots, s_k$ are in different μ_q-orbits from the $s_{k+1}, \ldots, s_m$, where μ_q acts on $R^\times$ by multiplication. By remark 2.5 $\mathrm{Res}^m_{k,m-k} M$ has a direct summand Q' such that $\mathrm{Res}^{k,m-k}_k Q'$ has central character $S_k \cdot (s_1, \ldots, s_k)$. Pick an irreducible submodule $A \boxtimes B \subseteq Q$. Then by Frobenius reciprocity, $\mathrm{Ind}\, A \boxtimes B \twoheadrightarrow M$. $A \boxtimes B$ satisfies the hypotheses of part 1, yielding that both sides are irreducible, so this must be an isomorphism. $\square$

Remark 4.5. One important conclusion we draw from the claim is that to understand the irreducibles of $\mathrm{Rep}\, H_n$ (on which the X_j have eigenvalues in R), it is enough to understand the irreducibles of $\mathrm{Rep}_q H_n$, in other words when all the $\{s_1, \ldots, s_n\}$ lie in a single μ_q-orbit.

4.4 H_3

Let $M \in \mathrm{Rep}_q H_3$ be irreducible with central character $s = (q^{i_1}, q^{i_2}, q^{i_3})$. By corollary 3.3 M is a composition factor of $\mathrm{Ind}_S^{H_3} q^{i_1} \boxtimes q^{i_2} \boxtimes q^{i_3}$.

4.4.1 q^{i_j} all distinct, i.e. $\prod(s_i - s_j) \neq 0$

Case 1. $\prod(s_i - qs_j) \neq 0$

Then $M = \mathrm{Ind}\, q^{i_1} \boxtimes q^{i_2} \boxtimes q^{i_3}$ which has character $\sum_{w \in S_3} w \cdot s$. Applying claim 4.4 twice, by the transitivity of induction, yields the irreducibility of $\mathrm{Ind}\, q^{i_1} \boxtimes q^{i_2} \boxtimes q^{i_3}$.

For the remaining cases by remark 4.1 we may assume $s_1 = 1$ and that it occurs with highest multiplicity.

Case 2. $s = (1, q, q^i)$, $q^i \notin \{q^{-1}, 1, q, q^2\}$.

We will decompose $\mathrm{Ind}\, 1 \boxtimes q \boxtimes q^i$ using (4.1) and the exactness of induction:

$$0 \to \mathrm{Ind}^{H_3}_{H_2 \otimes H_1} \mathrm{St}_{(q,1)} \boxtimes q^i \to \mathrm{Ind}^{H_3}_{H_1 \otimes H_1 \otimes H_1} 1 \boxtimes q \boxtimes q^i \to \mathrm{Ind}^{H_3}_{H_2 \otimes H_1} \mathbf{1}_{(1,q)} \boxtimes q^i \to 0. \quad (4.3)$$

Again, applying claim 4.4, the kernel and image of the above are irreducible. By the shuffle lemma, $\mathrm{ch}\, \mathrm{Ind}\, \mathbf{1}_{(1,q)} \boxtimes q^i = (1, q, q^i) + (1, q^i, q) + (q^i, 1, q)$ and $\mathrm{ch}\, \mathrm{Ind}\, \mathrm{St}_{(q,1)} \boxtimes q^i = (q, 1, q^i) + (q, q^i, 1) + (q^i, q, 1)$. M is one of these modules.

Case 3. $s = (1, q, q^2)$

The sequence (4.3) still holds, but further

$$0 \to N \to \mathrm{Ind}\, \mathbf{1}_{(1,q)} \boxtimes q^2 \xrightarrow{\alpha} \mathbf{1}_{(1,q,q^2)} \to 0 \quad (4.4)$$

$$0 \to \mathrm{St}_{(q^2,q,1)} \xrightarrow{\beta} \mathrm{Ind}\, \mathrm{St}_{(q,1)} \boxtimes q^2 \to \mathrm{rev}^* N \to 0 \quad (4.5)$$

where for instance α and β may be constructed via Frobenius reciprocity along with the observation $\mathrm{Ind}^{H_3}_{H_2 \otimes H_1} \mathrm{St}_{(q,1)} \boxtimes q^2 \simeq \check{\mathrm{I}}\mathrm{nd}^{H_3}_{H_1 \otimes H_2} q^2 \boxtimes \mathrm{St}_{(q,1)}$ of proposition 3.5. N, and thus $\mathrm{rev}^* N$, is irreducible by remark 4.3 as long as $q^3 \neq 1$ since $\mathrm{ch}\, N = (1, q^2, q) + (q^2, 1, q)$. Thus M is one of $\mathbf{1}_{(1,q,q^2)}$, $\mathrm{St}_{(q^2,q,1)}$, N, or $\mathrm{rev}^* N$.

In the case that $q^2 = 1$, so in fact not all the q^{i_j} are distinct, the irreducibility of $\mathrm{Ind}_S^{H_2} 1 \boxtimes 1$ still gives N and $\mathrm{rev}^* N$ are irreducible.

In the case $q^3 = 1$, the six 1-dimensional irreducibles $\mathbf{1}_{(1,q,q^2)}, \mathbf{1}_{(q,q^2,1)}, \mathbf{1}_{(q^2,1,q)}$, $\mathrm{St}_{(q^2,q,1)}, \mathrm{St}_{(q,1,q^2)}, \mathrm{St}_{(1,q^2,q)}$ must all occur as composition factors of $\mathrm{Ind}\, 1 \boxtimes q \boxtimes q^2$ and thus these exhaust all possible M.

4.4.2 $s = (1, 1, q^i)$, $q^i \neq 1$.

Case 1. $q^i \notin \{1, q, q^{-1}\}$.

Applying claim 4.4 to $\mathrm{Ind}_S^{H_3} 1 \boxtimes 1 \boxtimes q^i = \mathrm{Ind}_{H_2 \otimes H_1}^{H_3} (\mathrm{Ind}_{H_1 \otimes H_1}^{H_2} 1 \boxtimes 1) \boxtimes q^i$ yields its irreducibility.

Case 2. $s = (1, 1, q^{\pm 1})$.

In fact, we'll just show this for $s = (1, 1, q)$, and the other case is similar. First the exact sequence (4.3) still holds. If $q^2 \neq 1$, then $\mathrm{Ind}\, \mathbf{1}_{(1,q)} \boxtimes 1$ is irreducible as direct computation below verifies. With respect to the basis $\{1 \otimes v, (T_2 + 1) \otimes v, T_1(T_2 + 1) \otimes v\}$

$$X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1-q \\ 0 & 0 & 1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} q & 0 & 0 \\ 0 & 1 & q-1 \\ 0 & 0 & 1 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{bmatrix}$$

$$T_1 = \begin{bmatrix} q & 0 & 0 \\ 0 & 0 & q \\ 0 & 1 & q-1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} -1 & 0 & -q(1+q) \\ 1 & q & q \\ 0 & 0 & q \end{bmatrix}$$

on $\mathrm{Ind}\, \mathbf{1}_{(1,q)} \boxtimes 1$. Thus $\mathrm{Ind}\, \mathrm{St}_{(q,1)} \boxtimes 1$ which must be $\mathrm{rev}^*(\mathrm{Ind}\, \mathbf{1}_{(1,q)} \boxtimes 1)$ is also irreducible. Notice their characters are $2(1, 1, q) + (1, q, 1)$ and $2(q, 1, 1) + (1, q, 1)$ respectively. Similarly, $\mathrm{Ind}\, \mathrm{St}_{(1,q^{-1})} \boxtimes 1$ and $\mathrm{Ind}\, \mathbf{1}_{(q^{-1},1)} \boxtimes 1$ are irreducible.

Remark 4.6. As an analogue to remark 4.3 we can reinterpret this as saying for $q^2 \neq 1$: $M[(s_1, s_2, \ldots, 1, q^{\pm 1}, 1 \ldots, s_n)] \neq 0$ if and only if $M[(s_1, s_2, \ldots, 1, 1, q^{\pm 1}, \ldots, s_n)] \neq 0$ *or* $M[(s_1, s_2, \ldots, q^{\pm 1}, 1, 1, \ldots, s_n)] \neq 0$ and further that the dimensions of these spaces satisfy that twice the first is the sum of the next two. This observation will be made more precise in remark 5.13.

However, if $q^2 = 1$, we refer back to Case 3 giving M has character $(1, q, 1)$ (trivial), or $2(1, 1, q)$ or $2(q, 1, 1)$. (Given the case below, the list of characters of all eight irreducibles in $\mathrm{Rep}_q\, H_3$ when $q^2 = 1$ is $(1, q, 1), (q, 1, q), 2(1, 1, q), 2(q, q, 1), 2(q, 1, 1), 2(1, q, q), 6(1, 1, 1)$, $6(q, q, q)$.)

4.4.3 $s = (1, 1, 1)$.

$\mathrm{Ind}_S^{H_3} 1 \boxtimes 1 \boxtimes 1$ is irreducible with character $6(1, 1, 1)$. This will be shown in theorem 5.4 below.

4.5 H_4

We will list the characters of all fourteen irreducible modules in $\mathrm{Rep}_q\, H_4$ for $q = -1$. Based on the list, one could make a statement analogous to remarks 4.3 and 4.6.

By scaling as in remark 4.1 and by corollary 3.3, we need only decompose $\mathrm{Ind}_S^{H_4} 1 \boxtimes 1 \boxtimes 1 \boxtimes 1$, $\mathrm{Ind}_S^{H_4} 1 \boxtimes 1 \boxtimes 1 \boxtimes q$, and $\mathrm{Ind}_S^{H_4} 1 \boxtimes 1 \boxtimes q \boxtimes q$.

The first is irreducible by theorem 5.4 below. Thus there are two 24-dimensional irreducibles with character $24(1,1,1,1)$ and $24(q,q,q,q)$ respectively.

By Case 3, and the exactness of induction $\mathrm{Ind}_S^{H_4} 1 \boxtimes 1 \boxtimes 1 \boxtimes q$ has subquotients $\mathrm{Ind}_{H_1 \otimes H_3}^{H_4} 1 \boxtimes \mathbf{1}_{(1,q,1)}$ with multiplicity two, $\mathrm{Ind}_{H_1 \otimes H_3}^{H_4} 1 \boxtimes N$, and $\mathrm{Ind}_{H_1 \otimes H_3}^{H_4} 1 \boxtimes \mathrm{rev}^* N$. These are each irreducible with characters $2(1,1,q,1) + 2(1,q,1,1)$, $6(1,1,1,q) + 2(1,1,q,1)$ and $2(1,q,1,1) + 6(q,1,1,1)$ respectively. Similarly, the irreducibles with central character $s = (q,q,q,1)$ have characters $2(q,q,1,q) + 2(q,1,q,q)$, $6(q,q,q,1) + 2(q,q,1,q)$ and $2(q,1,q,q) + 6(1,q,q,q)$.

The decomposition of $\mathrm{Ind}_S^{H_4} 1 \boxtimes 1 \boxtimes q \boxtimes q$ is more complicated (it has factors that are not induced modules). We merely list the characters of its composition factors (six without multiplicity): $(1,q,1,q)$, $(q,1,q,1)$, $2(1,q,q,1)$, $2(q,1,1,q)$, $4(1,1,q,q)$, $4(q,q,1,1)$.

Chapter 5

The Strong Multiplicity One Result

This chapter contains the main theorem 5.7 and its corollary 5.9 which state that if M is irreducible, then the irreducibles in $\operatorname{soc}\operatorname{Res}^{H_n}_{H_{n-1}} M$ occur with multiplicity one. In order to prove this, we introduce a functor e_i which is a slight refinement of restriction, and show in the main theorem 5.7 that the socle of $e_i M$ is in fact irreducible or 0. This theorem is a strengthening of the statement that the socle is multiplicity-free, and so is our "strong" multiplicity one result. We conclude by showing that a module is determined by its character.

5.1 Definitions

Throughout this chapter, assume all modules lie in $\operatorname{Rep}_q H_n$.

First let i-Res M denote the generalized eigenspace of $X_n - q^i$ in $\operatorname{Res}^{H_n}_{H_{n-1}\otimes H_1} M$. In other words, because $X_n - q^i \subseteq Z(H_{n-1}\otimes H_1)$, left multiplication by $(X_n - q^i)^m$ induces an endomorphism of $\operatorname{Res}^{H_n}_{H_{n-1}\otimes H_1} M$ and i-Res M is the kernel of this map for $m \gg 0$. We can take $m \leq \dim M$. In particular, i-Res M is an $H_{n-1}\otimes H_1$-module. As restriction and taking generalized eigenspaces are exact functors, i-Res is exact. Furthermore $\operatorname{Res}^{H_n}_{H_{n-1}\otimes H_1} M = \bigoplus_{q^i\in\mu_q}$ i-Res M as X_n acts on $M \in \operatorname{Rep}_q H_n$ with eigenvalues in μ_q. This is an instance of remark 2.5.

If $s = S_n \cdot (s_1, s_2, \dots, s_n)$ write $s + q^i = S_{n+1}\cdot(s_1, \dots, s_n, q^i)$, and if some $s_j = q^i$ we may assume it is s_n and write $s \setminus q^i = S_{n-1}\cdot(s_1,\dots,s_{n-1})$. Let $|s|$ stand for $\sum s_j$.

If M is finite dimensional and indecomposable with central character s, then for $m \gg 0$, $(\sum_1^n X_j - |s|)^m M = 0$. If i-Res $M \neq 0$, the fact $\sum_1^n (X_j - s_j) = (X_n - q^i) + \sum_1^{n-1}(X_j - s_j)$ yields

$$\ker(X_n - q^i)^m = \ker\left(\sum_1^{n-1} X_j - |s\setminus q^i|\right)^m \qquad \text{for } m \gg 0.$$

Thus, i-Res M is also the generalized eigenspace of $\sum_1^{n-1} X_j - |s\setminus q^i| \in Z(H_{n-1})$.

Define

$$e_i M = \mathrm{Res}^{H_{n-1}\otimes H_1}_{H_{n-1}} (\text{i-Res}\, M).$$

Alternately, $e_i M$ is the generalized eigenspace of $\sum_1^{n-1} X_j - |s \setminus q^i|$ on $\mathrm{Res}^{H_n}_{H_{n-1}} M$. The above discussion makes the following claim clear.

Claim 5.1. *e_i is an exact functor* $\mathrm{Rep}_q H_n \to \mathrm{Rep}_q H_{n-1}$ *and* $\mathrm{Res}^{H_n}_{H_{n-1}} M = \bigoplus_i e_i M$.

As e_i is an exact functor, we can also view it as an operator on $\bigoplus_n K(\mathrm{Rep}_q H_n)$.

Notice that if M has central character s, each $e_i M$ has distinct central character $s \setminus q^i$.

On the level of characters, if $\mathrm{ch}\, M = \sum_j s^{(j)}$ then

$$\mathrm{ch}\, e_i M = \sum_{j : s_n^{(j)} = q^i} (s_1^{(j)}, \dots, s_{n-1}^{(j)}) = \sum_{j : s_n^{(j)} = q^i} (s \setminus q^i)$$

and so, iterating,

$$\mathrm{ch}\, e_i^k M = \sum_{j : (s_{n-k+1}^{(j)}, \dots, s_n^{(j)}) = (q^i, \dots, q^i)} (s_1^{(j)}, \dots, s_{n-k}^{(j)}).$$

Notice that $e_i^n M \neq 0$ if and only if the central character of M is $S_n \cdot (q^i, \dots, q^i) = (q^i, \dots, q^i)$, and in this case $e_i^n M = \oplus_{(\dim M)} \mathbf{1}$.

Remark 5.2. We also observe that one can describe e_i^k as follows. Let Q be the largest submodule of $\mathrm{Res}^{H_n}_{H_{n-k}\otimes H_k} M$ such that $\mathrm{Res}^{H_{n-k}\otimes H_k}_{H_k} Q$ has central character $(q^i, \dots, q^i)$, which, by remark 2.5, is a direct summand. Then $e_i^k M = \mathrm{Res}^{H_{n-k}\otimes H_k}_{H_{n-k}} Q$. If Q has a filtration by irreducibles $N_a \boxtimes N'_a$, then $e_i^k M$ has a filtration by the $N_a \boxtimes e_i^k N'_a = \oplus_{\dim N'_a} N_a$.

If M, N are irreducible, then

$$N \subseteq e_i M \iff N \boxtimes q^i \subseteq \mathrm{Res}^{H_n}_{H_{n-1}\otimes H_1} M. \tag{5.1}$$

One direction is clear. For the other, let $\tilde{N} \subset \mathrm{Res}\, M$ be such that $\mathrm{Res}\, \tilde{N} = N$ (which a priori might not be a submodule). Because N is irreducible with central character $s \setminus q^i$, it is killed by $\sum_{j=1}^{n-1} X_j - |s \setminus q^i|$, and hence $\tilde{N}$ is killed by $X_n - q^i$. In other words, $\tilde{N} = N \boxtimes q^i$. Notice we do require N irreducible for (5.1) to hold; consider for example $M = \mathrm{Ind}_S^{H_2} q^i \boxtimes q^i$.

Define

$$\varepsilon_i(M) = \max\{m \geq 0 \mid \widetilde{e}_i^m M \neq 0\}.$$

We also define

$$\widetilde{e}_i M = \mathrm{soc}\, e_i M.$$

Then (5.1) implies that for M, N irreducible $N \subseteq \widetilde{e}_i^k M \iff N \boxtimes q^i \boxtimes \cdots \boxtimes q^i \subseteq \mathrm{Res}^{H_n}_{H_{n-k}} M$. Later we shall also see that $\widetilde{e}_i M \simeq \mathrm{cosoc}\, e_i M$.

We remark that

$$\widetilde{e}_i M = 0 \iff e_i M = 0 \iff \varepsilon_i(M) = 0.$$

The major result of this chapter is that if M is irreducible, then $\widetilde{e}_i M$ is either 0 or also irreducible.

5.2 The affine Mackey formula, a special case

The following is a special case of the affine Mackey formula. For the general statement and for cyclotomic Hecke algebras, see [G].

Theorem 5.3. *Let $A \in \operatorname{Rep}_q H_a$, $B \in \operatorname{Rep}_q H_b$. Then the following are exact sequences of H_{n-1}-modules, where $n = a+b$.*

$$0 \to \operatorname{Ind}^{n-1}_{a,b-1} A \boxtimes \operatorname{Res}^{b}_{b-1} B \xrightarrow{\alpha} \operatorname{Res}^{n}_{n-1}\left(\operatorname{Ind}^{n}_{a,b} A \boxtimes B\right) \xrightarrow{\beta} \operatorname{Ind}^{n-1}_{a-1,b}(\operatorname{Res}^{a}_{a-1} A) \boxtimes B \to 0$$
$$0 \to \operatorname{Ind}^{n-1}_{a,b-1} A \boxtimes e_i B \to e_i\left(\operatorname{Ind}^{n}_{a,b} A \boxtimes B\right) \to \operatorname{Ind}^{n-1}_{a-1,b} e_i A \boxtimes B \to 0.$$

Proof. The general Mackey formula gives a filtration of $\operatorname{Res}^{H_n}_{H_Q} \operatorname{Ind}^{H_n}_{H_P} M$, which has subquotients roughly of the form $\operatorname{Ind}^{H_Q} \operatorname{Res}^{H_P} \tilde{M}$. Underlying the proof is really an exercise in double cosets $S_Q \backslash S_n / S_P$.

The map α is built from the identity map $\operatorname{Res}^{b}_{b-1} B \to \operatorname{Res}^{b}_{b-1} B$ via adjunction and composing with $\operatorname{Ind} A\boxtimes -$. Similarly for β if we make use of proposition 3.5 relating Ind and $\check{\operatorname{Ind}}$. However, we describe the maps on a basis to check the exactness of the sequence.

The map $\operatorname{Ind}^{n-1}_{a,b-1} A \boxtimes \operatorname{Res}^{b}_{b-1} B \xrightarrow{\alpha} \operatorname{Res}^{n}_{n-1}\left(\operatorname{Ind}^{n}_{a,b} A \boxtimes B\right)$ is defined (by Frobenius reciprocity) by

$$1 \otimes (u \otimes v) \overset{\alpha}{\mapsto} 1 \otimes (u \otimes v),$$

so $\alpha(T_w \otimes (u \otimes v)) = T_w \otimes (u \otimes v)$, where $\{w\} = W_{(a,b-1)}$, which is a subset of $W_{(a,b)}$ (minimal length left coset representatives of $S_a \times S_{b-1} \subseteq S_{n-1}$ and $S_a \times S_b \subseteq S_n$ respectively). Since the $\{T_w \otimes (u_j \otimes v_k)\}_{w \in W_{(a,b)}}$ are linearly independent as $\{u_j\}, \{v_k\}$ run over bases of A and B, we see α is injective.

Before describing β, notice that $\{T_w\}_{w \in W_{(a,b)}} = \{T_x T_y\}_{x \in W_{(a-1,b)}} \bigcup \{T_w\}_{w \in W_{(a,b-1)}}$, where $T_y = T_{n-1} T_{n-2} \cdots T_a$. In other words, the shuffles of $12 \cdots a$ and $a+1 \cdots n$ consist of those fixing n, $W_{(a-1,b)}$, and those that do not and hence send a to the n^{th} position. Equivalently, $\{1, s_{n-1} \cdots s_a\}$ are minimal length double coset representatives of $S_{n-1} \backslash S_n / S_a \times S_b$.

Then

$$T_x T_y \otimes (u \otimes v) \overset{\beta}{\mapsto} T_x \otimes (\bar{u} \otimes v) \qquad T_w \otimes (u \otimes v) \overset{\beta}{\mapsto} 0$$

where $\bar{u} = u \in \operatorname{Res} A$. As $\{T_x \otimes (\bar{u}_j \otimes v_k)\}$ runs over a basis of $\operatorname{Ind}^{n-1}_{a-1,b}(\operatorname{Res}^{a}_{a-1} A) \boxtimes B$ for $x \in W_{(a-1,b)}$, both the surjectivity of β and the fact $\ker \beta = \operatorname{Im} \alpha$ are clear.

As $T_w \in H_{n-1}$, $T_y \notin H_{n-1}$, for $T_j, X_j \in H_{n-1}$, $\beta(T_j T_w) = 0$ and $\beta(X_j T_w) = 0$. Thus to show β is well-defined as an H_{n-1}-map, it suffices to check $\beta(T_j T_y \otimes (u \otimes v)) = T_j \beta(T_y \otimes (u \otimes v)) = T_j \otimes (\bar{u} \otimes v)$ and $\beta(X_j T_y \otimes (u \otimes v)) = X_j \otimes (\bar{u} \otimes v)$.

We first observe there is a shift of indices

$$T_j(\operatorname{Res} A \boxtimes B) = \operatorname{Res} A \boxtimes T_{j+1} B \qquad X_j(\operatorname{Res} A \boxtimes B) = \operatorname{Res} A \boxtimes X_{j+1} B$$

as B is acted on by $H_b = H_{\{a+1,\ldots a+b\}}$ in $A \boxtimes B$, but by $H_b = H_{\{a,\ldots,n-1\}}$ in $\operatorname{Res} A \boxtimes B$.

Next, as $s_{a-1} \in W_{(a-1,b)}$, $\beta(T_{a-1} T_y) = T_{a-1} \otimes (\bar{u} \otimes v)$ by definition.

$$T_j T_y = \begin{cases} T_y T_j & 1 \le j \le a-2 \\ T_y T_{j+1} & a \le j \le n-2 \end{cases}$$

so that

$$\beta(T_jT_y \otimes (u \otimes v)) = \begin{Bmatrix} 1 \otimes (T_j\bar{u} \otimes v) & 1 \le j \le a-2 \\ 1 \otimes (\bar{u} \otimes T_{j+1}v) & a \le j \le n-2 \end{Bmatrix} = T_j \otimes (\bar{u} \otimes v)$$

by the indices shift.

Observe if $j \ge a$, $X_jT_y = T_yX_{j+1} + (1-q)T_{j-1}\cdots T_aX_{j+1}T_{n-1}\cdots T_{j+1}$ so that

$$X_jT_y \otimes (u \otimes v) = \begin{cases} T_y \otimes (X_ju \otimes v) & j < a, \\ T_y \otimes (u \otimes X_{j+1}v) + \ker\beta \text{ terms} & j \ge a. \end{cases}$$

As above, this gives $\beta(X_jT_y \otimes (u \otimes v)) = X_j \otimes (\bar{u} \otimes v)$ and completes the proof.

As homomorphisms preserve central character, by claim 5.1 and its subsequent comments, the second sequence is also exact, via the same maps α and β.

For the general Mackey formula, the same thing happens. We partition the $\{T_w \otimes u\}_{w\in W_P}$ according to $S_Q\backslash S_n/S_P$ double cosets, so H_Q^{fin} preserves the span of each set, which is isomorphic to an induced module. Ordered respecting Bruhat order on minimal length double coset representatives, the X_j action may contribute lower order terms, yielding a filtration as described above. $\square$

5.3 Kato's Theorem

Denote by q^iK_n the module $\operatorname{Ind}_S^{H_n} q^i \boxtimes \cdots \boxtimes q^i$.

Theorem 5.4. *1. (Kato's Theorem) $q^iK_n = \operatorname{Ind}_S^{H_n} q^i \boxtimes \cdots \boxtimes q^i$ is irreducible. It is the unique H_n-irreducible with central character $s = (q^i, \ldots, q^i)$.*

2. $e_i(q^iK_n)$ has an n step socle filtration which coincides with its cosocle filtration and each factor is q^iK_{n-1}.

Proof. Any submodule of q^iK_n would have to contain an S eigenvector. We will show there is only one such and that it generates all of q^iK_n, yielding the irreducibility.

Write Rv for the S-module $q^i \boxtimes \cdots \boxtimes q^i$. q^iK_n has basis $\{T_w \otimes v\}_{w\in S_n}$. Order the basis respecting Bruhat order.

First, we compute the eigenspace of X_1. Any T_w may be written $T_uT_1T_2\cdots T_j$ for some $u \in S_{\{2,\ldots,n\}} = S_{n-1}$. Observe $X_1T_u = T_uX_1$ by relation 2.4. Consider

$$(X_1 - q^i)T_1T_2\cdots T_j \otimes v = (1-q)q^i(T_1\cdots T_{j-1} + qT_j^{-1}T_1\cdots T_{j-2} + \cdots + \\ q^{k-2}T_3^{-1}\cdots T_j^{-1}T_1 + q^{k-1}T_2^{-1}\cdots T_j^{-1}) \otimes v. \quad (5.2)$$

A length argument now shows that if $(X_1 - q^i)\sum_{u,j}\alpha_{u,j}T_uT_1T_2\cdots T_j \otimes v = 0$ then $\alpha_{u,j} = 0$ for $j \ge 1$. Thus the $\{T_u \otimes v\}_{u\in S_{\{2,\ldots,n\}}}$ span the $X_1 - q^i$ eigenspace. The above computation shows the Jordan block structure of X_1 as well.

The same computation with indices shifted shows the intersection of the $X_1 - q^i$ and $X_2 - q^i$ eigenspace is the span of $\{T_u \otimes v\}_{u\in S_{\{3,\ldots,n\}}}$. Continuing, the simultaneous eigenspace for $X_1 - q^i, \dots, X_{n-1} - q^i$ is the span of $\{1 \otimes v\}$ which is 1-dimensional and generates all of $q^i K_n$. This is certainly also an $X_n - q^i$ eigenvector. Thus $q^i K_n$ is irreducible. (That there is only one eigenvector for $X_1, \dots, X_{n-1}$ tells us that $\widetilde{e}_i q^i K_n = q^i K_{n-1}$ and is reflected in what we call our alternate viewpoint, explained below.)

Given any irreducible M with central character $(q^i, \dots, q^i)$, corollary 3.3 implies M is a subquotient of $q^i K_n$ which is irreducible, yielding $M = q^i K_n$.

A similar computation to that for X_1 above gives that $X_n - q^i$ acts on $q^i K_n$ with $(n-1)!$ many Jordan blocks of size n. Consider the filtration of $\mathrm{Res}^{H_n}_{H_{n-1}\otimes H_1} q^i K_n$ whose j^{th} term is $\ker(X_n - q^i)^j$. By the analogue to (5.2) above, this is

$$\begin{aligned} 0 \subsetneq \mathrm{span}\{T_u \otimes v\} \subsetneq \mathrm{span}\{T_u \otimes v, T_u T_{n-1} \otimes v\} \subsetneq \cdots \\ \subsetneq \mathrm{span}\{T_u \otimes v, \dots, T_u T_{n-1} \cdots T_1 \otimes v\} = q^i K_n \end{aligned} \tag{5.3}$$

where each set runs over $u \in S_{n-1}$.

We remind the reader that the socle filtration of a module M has first term $\mathrm{soc}\, M$, next term the preimage of $\mathrm{soc}(M/\mathrm{soc}\, M)$ and so on; the cosocle filtration is the dual notion. (5.3) must be the socle series of $\mathrm{Res}^{H_n}_{H_{n-1}\otimes H_1} q^i K_n = \text{i-Res}\, q^i K_n$ because first, the X_n action tells us (5.3) refines the socle series and second, because each factor is the simple module $q^i K_{n-1} \boxtimes q^i$. We remark the j^{th} term of (5.3) is also $\mathrm{Im}(X_n - q^i)^{n-j}$. The same reasoning applies to the cosocle series, and they must coincide.

Our alternate viewpoint of i-Res and e_i allows us to replace $X_n - q^i$ in the above discussion with a central element of H_{n-1}, and so restricting the filtration (5.3) to H_{n-1} gives the socle (and cosocle) filtration of $e_i q^i K_n$. □

Corollary 5.5. $\mathrm{soc}\,\mathrm{Res}_{a_1,\dots,a_k} q^i K_n = q^i K_{a_1} \boxtimes \cdots \boxtimes q^i K_{a_k}$ *is irreducible.*

Proof. If N is in the socle, then it must contain an S eigenvector, but there is only one such in $q^i K_n$, and it generates $q^i K_a$ over H_a. □

From the corollary above, for M irreducible, $\mathrm{soc}((\text{i-Res})^k M) = (\mathrm{soc} \circ \text{i-Res})^k M$. Our alternate view of e_i and i-Res as the generalized eigenspace of an element in $Z(H_{n-1})$ shows that $\widetilde{e}_i M = \mathrm{Res}^{n-1,1}_{n-1} \mathrm{soc}\ \text{i-Res}\, M$. Iterating this,

$$\widetilde{e}_i^k M = \mathrm{Res}^{n-k,1,\dots,1}_{n-k} \mathrm{soc}(\text{i-Res})^k M. \tag{5.4}$$

We will use this observation in the proof of the main theorem.

From theorem 5.4 and remark 5.2, we see if $[e_i^k M] = \sum_{N \text{ irred}} a_N [N] \in K(\mathrm{Rep}_q H_{n-k})$ then $k! \mid a_N$ and so the divided power $\frac{e_i^k}{k!}$ operates on $\bigoplus_n K(\mathrm{Rep}_q H_n)$. We refer the reader to [G] for the definition of an exact functor $e_i^{(k)}$ on $\bigoplus_n \mathrm{Rep}_q H_n$ that coincides with $\frac{e_i^k}{k!}$ on $\bigoplus_n K(\mathrm{Rep}_q H_n)$. In the notation of remark 5.2, if $\mathrm{Res}^{H_n}_{H_{n-k}\otimes H_k} M$ has a filtration with factors $N_a \boxtimes N'_a$, then $e_i^{(k)} M$ has a filtration with factors the N_a such that $N'_a = q^i K_k$.

5.4 Strong Multiplicity One Result

Lemma 5.6. *Let N be irreducible.*

1. *Suppose $\varepsilon_i(N) = 0$. Then $M = \operatorname{cosoc}(\operatorname{Ind} N \boxtimes q^i K_k)$ is irreducible. Further, $\varepsilon_i(M) = k$ and all other composition factors of $\operatorname{Ind} N \boxtimes q^i K_k$ have $\varepsilon_i < k$. Likewise for $\check{\operatorname{Ind}}\, N \boxtimes q^i K_k$ and $M = \operatorname{soc}(\check{\operatorname{Ind}}\, N \boxtimes q^i K_k)$.*

2. *For any irreducible M, there exists irreducible N with $\varepsilon_i(N) = 0$ such that $M = \operatorname{cosoc}(\operatorname{Ind} N \boxtimes q^i K_{\varepsilon_i(M)})$. Further, we may take $N \subseteq \widetilde{e}_i^{\varepsilon_i(M)} M$.*

3. *$M = \operatorname{cosoc}(\operatorname{Ind} N \boxtimes q^i K_k)$ is irreducible with $\varepsilon_i(M) = \varepsilon_i(N) + k$, and all other composition factors of $\operatorname{Ind} N \boxtimes q^i K_k$ have strictly smaller ε_i.*

Proof. By Frobenius Reciprocity, $N \boxtimes q^i K_k$ must occur in the restriction of any quotient of $\operatorname{Ind}_{H_n \otimes H_k}^{H_{n+k}} N \boxtimes q^i K_k$, in particular in M. On the other hand, on the level of characters, by the shuffle lemma 3.4 and the fact $\varepsilon_i(N) = 0$, we see $N \boxtimes q^i K_k$ occurs in $\operatorname{Res}_{H_n \otimes H_k}^{H_{n+k}} \operatorname{Ind}_{H_n \otimes H_k}^{H_{n+k}} N \boxtimes q^i K_k$ with multiplicity one, yielding the cosocle is irreducible.

By the shuffle lemma, we see $\varepsilon_i(M) = k$ and any other composition factor has strictly smaller ε_i.

By the remark 2.5 and the discussion above,

$$\operatorname{Res} M = N \boxtimes q^i K_k \bigoplus A \tag{5.5}$$

where $e_i^k A = 0$. and in particular $N \boxtimes q^i K_k$ is a quotient of $\operatorname{Res} M$, yielding $M \hookrightarrow \check{\operatorname{Ind}}\, N \boxtimes q^i K_k$. Arguing as above, M is the socle, and all other composition factors have smaller ε_i.

For (2), given an irreducible M, if $\varepsilon_i(M) = 0$ we're done. If not, write $\varepsilon = \varepsilon_i(M)$ and pick irreducible $N \subseteq \widetilde{e}_i^{\varepsilon} M$. Then $\varepsilon_i(N) = 0$. By (5.1), $N \boxtimes q^i \boxtimes \cdots \boxtimes q^i \subseteq \operatorname{Res}^n_{n-\varepsilon,1,\ldots,1} M$ and so by Frobenius reciprocity and the irreducibility of M, there is a surjection $\operatorname{Ind} N \boxtimes q^i \boxtimes \cdots \boxtimes q^i \to M$. By (1) of the lemma, $M = \operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i \boxtimes \cdots \boxtimes q^i = \operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_\varepsilon$.

For a given N, by the exactness of induction and by part (2), there exists irreducible L with $\varepsilon_i(L) = 0$ such that

$$\operatorname{Ind} L \boxtimes q^i K_{\varepsilon_i(N)+k} = \operatorname{Ind} L \boxtimes q^i K_{\varepsilon_i(N)} \boxtimes q^i K_k \twoheadrightarrow \operatorname{Ind} N \boxtimes q^i K_k.$$

As the above map is a surjection, $\operatorname{cosoc} \operatorname{Ind} L \boxtimes q^i K_{\varepsilon_i(N)+k}$ surjects onto $\operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_k = M$, which is thus irreducible by part (1). Again applying (1) to L yields the rest of (3). $\square$

Theorem 5.7 (Main Theorem). *Let M be irreducible. Then $\widetilde{e}_i M = 0$ or is irreducible.*

Proof. Recall $\widetilde{e}_i M = 0 \iff e_i M = 0 \iff \varepsilon_i(M) = 0$, so we may assume $\varepsilon = \varepsilon_i(M) \neq 0$.

Note that if $N \subseteq \operatorname{soc} e_i M$ then $\varepsilon_i(N) = \varepsilon - 1$. As $e_i^{\varepsilon} N \subseteq e_i^{\varepsilon} e_i M = 0$, $\varepsilon_i(N) \leq \varepsilon - 1$. By (5.1) we have a map $N \boxtimes q^i \to \operatorname{Res}_{H_{n-1} \otimes H_1}^{H_n} M$ and thus $\operatorname{Ind}_{H_{n-1} \otimes H_1}^{H_n} N \boxtimes q^i \twoheadrightarrow M$. By the shuffle lemma, $\varepsilon_i(\operatorname{Ind} N \boxtimes q^i) = \varepsilon_i(N) + 1$. By the exactness of e_i, $e_i^{\varepsilon} \operatorname{Ind} N \boxtimes q^i \twoheadrightarrow e_i^{\varepsilon} M \neq 0$, so $\varepsilon_i(N) + 1 = \varepsilon_i(\operatorname{Ind} N \boxtimes q^i) \geq \varepsilon_i(M)$. In particular, this shows $\widetilde{e}_i^{\varepsilon-1} N \neq 0$.

Thus it suffices to show $\widetilde{e}_i^{\varepsilon} M$ is irreducible.

Pick irreducible $N \subseteq \widetilde{e}_i^{\varepsilon} M$. We'll show $N = \widetilde{e}_i^{\varepsilon} M$. As in the proof of lemma 5.6, $\varepsilon_i(N) = 0$ and we have a surjection

$$\operatorname{Ind} N \boxtimes q^i K_\varepsilon \twoheadrightarrow M.$$

From equation (5.5), $(\text{i-Res})^\varepsilon(M) = N \boxtimes \operatorname{Res}_S q^i K_\varepsilon$. By theorem 5.4, $\operatorname{soc} N \boxtimes \operatorname{Res}_S q^i K_\varepsilon = N \boxtimes q^i \boxtimes \cdots \boxtimes q^i$. Then by (5.4), $\widetilde{e}_i^{\varepsilon} M = \operatorname{Res}_{H_{n-\varepsilon}} \operatorname{soc}(\text{i-Res})^\varepsilon(M) = \operatorname{Res}_{H_{n-\varepsilon}} N \boxtimes q^i \boxtimes \cdots \boxtimes q^i = N$. □

Note that in the proof above, $\operatorname{Res}_{H_{n-\varepsilon}} \operatorname{cosoc}(\text{i-Res})^\varepsilon(M) = N$ as well, so that we could repeat the argument above with Ind replaced by $\check{\operatorname{Ind}}$ and reversing arrows. Thus $\widetilde{e}_i^{\varepsilon} M = (\operatorname{cosoc} \circ e_i)^\varepsilon M$. We will use this observation below to conclude $\widetilde{e}_i M = \operatorname{cosoc} e_i M$.

Corollary 5.8. *Let M be irreducible. Then*

1. *Letting $\varepsilon = \varepsilon_i(M)$, $N = \widetilde{e}_i^{\varepsilon} M$, for $0 \le k \le \varepsilon$*
$$\widetilde{e}_i^{k} M = \operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_{\varepsilon-k}. \tag{5.6}$$

2. $\widetilde{e}_i M \simeq \operatorname{cosoc} e_i M$.

3. *If L is also irreducible and $\widetilde{e}_i M \simeq \widetilde{e}_i L \neq 0$ then $M \simeq L$.*

Proof. By theorem 5.7, $N = \widetilde{e}_i^{\varepsilon} M$ is irreducible. Further $\varepsilon_i(N) = 0$ and $\operatorname{Ind} N \boxtimes q^i K_\varepsilon \twoheadrightarrow M$, which by lemma 5.6 is the cosocle. Replacing M by $\widetilde{e}_i^{k} M$, which is again irreducible, finishes the proof of (1) because $N = \widetilde{e}_i^{\varepsilon-k}(\widetilde{e}_i^{k} M)$.

By lemma 5.6, $M = \operatorname{soc} \check{\operatorname{Ind}} N \boxtimes q^i K_\varepsilon$, so that dualizing the proof of theorem 5.7, $N = (\operatorname{cosoc} \circ e_i)^\varepsilon M$; and then by the dual of 1 above, $\operatorname{cosoc} e_i M = \operatorname{soc} \check{\operatorname{Ind}} N \boxtimes q^i K_{\varepsilon-1}$. By lemma 5.6 this is isomorphic to $\operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_{\varepsilon-1} = \widetilde{e}_i M$.

If $\widetilde{e}_i M \simeq \widetilde{e}_i L$, then $\widetilde{e}_i^{\varepsilon} M \simeq \widetilde{e}_i^{\varepsilon} L = N$ and so both M and L are $\operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_{\varepsilon-k}$. □

Theorem 5.7 and corollary 5.8 have the following simple consequence, which if $\lambda = \Lambda_0$ was first shown by Kleshchev [Kv], but which otherwise is new. (We remind the reader here of the Appendix, which covers the case $q = 1$, i.e. $H_n = RS_n$.)

Corollary 5.9 (Multiplicity One Result). 1. *Let M be a simple H_n-module. Then* $\operatorname{soc} \operatorname{Res}_{H_{n-1}}^{H_n} M$ *is multiplicity free and is isomorphic to the* $\operatorname{cosoc} \operatorname{Res}_{H_{n-1}}^{H_n} M$.

2. *Let M be a simple H_n^λ-module, λ cyclotomic. Then* $\operatorname{soc} \operatorname{Res}_{H_n^\lambda}^{H_{n+1}^\lambda} M \simeq \operatorname{cosoc} \operatorname{Res}_{H_n^\lambda}^{H_{n+1}^\lambda} M$ *is multiplicity free. Also the cosocle of* $\operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda} M$ *is multiplicity free.*

Proof. The first follows from the theorem, claim 5.1, and the observation that the central characters of $e_i M$ and $e_j M$ differ if $i \neq j$.

The next statement follows from the first, by remark 2.1 in Chapter 2 and because the restriction of an H_n^λ-module is an H_{n-1}^λ-module. As $\operatorname{soc} \circ \operatorname{Res}$ is adjoint to cosoc $\circ$ Ind when acting on semisimple modules, the final statement follows. In fact, by adjointness, soc $\check{\operatorname{Ind}}_{H_n^\lambda}^{H_{n+1}^\lambda} M$ is also multiplicity free. Although we omit the proof, $\operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda} M \simeq \check{\operatorname{Ind}}_{H_n^\lambda}^{H_{n+1}^\lambda} M$, yielding soc $\operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda} M$ is multiplicity free as well.

We point out that $\operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda}$ is quite different from the induction used throughout the paper. □

Proposition 5.10. *1. $\varepsilon_i(M)$ is the largest size of a Jordan block of $X_n - q^i$ acting on M.*

2. $[e_iM] = \varepsilon_i(M)[\widetilde{e}_iM] + [$lower terms$]$.

3. $[e_i^{\varepsilon_i(M)}M] = \varepsilon_i(M)![\widetilde{e}_i^{\varepsilon_i(M)}M]$.

4. $\varepsilon_i(M) = \dim \operatorname{Hom}_{H_{n-1}}(e_iM, e_iM)$.

Proof. We may assume $\varepsilon = \varepsilon_i(M) \neq 0$. Let $N = \widetilde{e}_i^{\varepsilon} M$.

From the proof of theorem 5.4, $X_n - q^i$ acts on q^iK_ε with $(\varepsilon-1)!$ many Jordan blocks of size ε. That $N \boxtimes q^iK_\varepsilon \hookrightarrow \operatorname{Res} M$ says the block size of $X_n - q^i$ on M is at least ε. To see it is no larger, it suffices to show the block size is no larger on $\operatorname{Ind} N \boxtimes q^iK_\varepsilon$. The generalized $X_n - q^i$ eigenspace is $e_i(\operatorname{Ind} N \boxtimes q^iK_\varepsilon) = \operatorname{Ind} N \boxtimes e_i(q^iK_\varepsilon)$ by the Mackey formula 5.3, and thus is the span of $\{T_w \otimes u\}_{\substack{w \in S_{n-1} \\ u \in N \boxtimes q^iK_\varepsilon}}$. As $X_n - q^i$ commutes with T_w for $w \in S_{n-1}$, its blocksize is no larger than it is acting on $u \in N \boxtimes q^iK_\varepsilon$, namely ε.

By lemma 5.6, $[\operatorname{Ind} N \boxtimes q^iK_\varepsilon] = [M] + [\text{lower terms}]$, where lower refers to ε_i. So that $[e_i \operatorname{Ind} N \boxtimes q^iK_\varepsilon] = [e_iM] + [\text{lower terms}]$. By the Mackey formula, theorem 5.4, exactness of induction, and corollary 5.8, $[e_i \operatorname{Ind} N \boxtimes q^iK_\varepsilon] = [\operatorname{Ind} N \boxtimes e_iq^iK_\varepsilon] = \varepsilon[\operatorname{Ind} N \boxtimes K_{\varepsilon-1}] = \varepsilon[\widetilde{e}_iM] + [\text{lower terms}]$. Comparing both equations yields 2.

Applying (2) iteratively yields (3).

We defer the proof of 3 to Chapter 6, as we use the notation introduced there. □

In light of theorem 5.7, we could have defined $\widetilde{e}_iM$ as the socle of the *actual* eigenspace of $(X_n - q^i)$. Dual to this are the statements of lemma 5.6, which motivates the definition

$$\widetilde{f}_iM = \operatorname{cosoc} \operatorname{Ind} M \boxtimes q^i,$$

which we have already seen takes irreducibles to irreducibles. By lemma 5.6, $\widetilde{f}_iM = \operatorname{soc} \check{\operatorname{Ind}} M \boxtimes q^i$. Although $\widetilde{f}_i$ is adjoint to $\widetilde{e}_i$ acting on semisimple modules, we cannot define an adjoint to e_i on $\bigoplus_n K(\operatorname{Rep}_q H_n)$. However, in [G], Grojnowski defines a functor f_i on $\bigoplus_n K(\operatorname{Rep} H_n^\lambda)$ which depends on λ, is adjoint to e_i, and has the property $\widetilde{f}_i = \operatorname{cosoc} f_i$.

5.5 Independence of Characters

Theorem 5.11. *1. An irreducible module $M \in \operatorname{Rep}_q H_n$ is determined by its character $\operatorname{ch} M$.*

2. *If A is any module in $\operatorname{Rep}_q H_n$, then $[A] \in K(\operatorname{Rep}_q H_n)$ is determined by $\operatorname{ch} A \in K(\operatorname{Rep} S)$; in other words, we have linear independence of irreducible characters over $\mathbb{Q}$.*

Proof. We'll induct on n. If M is an H_1-module, $\operatorname{ch} M = [M]$. For $M \in \operatorname{Rep}_q H_n$, pick i such that $\varepsilon_i(M) > 0$, noting that $\varepsilon_i(M)$ can be read off of $\operatorname{ch} M$. By theorem 5.7, $\widetilde{e}_i^{\varepsilon_i(M)} M = N$ is irreducible and $M = \operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_{\varepsilon_i(M)}$. Further, by the shuffle lemma, $\operatorname{ch} e_i^{\varepsilon_i(M)} M = \varepsilon_i(M)! \operatorname{ch} N$ and so we can also read $\operatorname{ch} N$ off of $\operatorname{ch} M$. By induction $\operatorname{ch} N$ determines N, and thus we have determined $M = \operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_{\varepsilon_i(M)}$.

We warn the reader that for M irreducible, M is determined in the sense that

$$M \simeq N \iff \operatorname{ch} M = \operatorname{ch} N.$$

Actually finding the H_n structure of M or determining $\operatorname{ch} \operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_k$ can be quite difficult.

Now for any A, write $[A] = \sum_{M_j \text{ irred}} a_j [M_j]$. Given $\operatorname{ch} A$, we can pick i such that $\varepsilon = \varepsilon_i(A)$ is maximal. Then

$$\operatorname{ch}(e_i^{\varepsilon} A) = \sum_{M_j \text{ irred}} a_j \operatorname{ch}(e_i^{\varepsilon} M_j) = \sum_{M_j : \varepsilon_i(M_j) = \varepsilon} a_j \varepsilon! \operatorname{ch}(\widetilde{e}_i^{\varepsilon} M_j).$$

By induction, knowing $\sum a_j \varepsilon! \operatorname{ch}(\widetilde{e}_i^{\varepsilon} M_j)$ determines $\sum a_j \varepsilon! [\widetilde{e}_i^{\varepsilon} M_j]$ and further we know that $M_j = \operatorname{cosoc} \operatorname{Ind} \widetilde{e}_i^{\varepsilon} M_j \boxtimes q^i K_{\varepsilon}$. Thus we in fact "know" $\sum_{M_j : \varepsilon_i(M_j) = \varepsilon} a_j [M_j]$ and in principle may subtract off $\sum_{M_j : \varepsilon_i(M_j) = \varepsilon} a_j \operatorname{ch} M_j$ from $\operatorname{ch} A$. Continuing, we will eventually determine $[A]$. In other words,

$$\sum_{M_j \text{ irred}} a_j \operatorname{ch} M_j = \sum_{M_k \text{ irred}} b_k \operatorname{ch} M_k, \quad a_j, b_k \in \mathbb{N} \implies a_j = b_j.$$

The linear independence of the characters of irreducibles follows as the $\{[M]\}_{M \text{ irred}}$ form a basis of $K(\operatorname{Rep}_q H_n) \otimes_{\mathbb{Z}} \mathbb{Q}$. □

Corollary 5.12. *The module $\operatorname{Ind}_{H_P}^{H_n} A_1 \boxtimes \cdots \boxtimes A_k$ has the same composition factors as $\operatorname{Ind}_{H_{\sigma(P)}^{H_n}} A_{\sigma^{-1}(1)} \boxtimes \cdots \boxtimes A_{\sigma^{-1}(k)}$ for any $\sigma \in S_k$.*

Proof. By the shuffle lemma 3.4, the above modules have the same character. □

Remark 5.13. Another interpretation of the examples listed in Chapter 4 is that the operators e_i satisfy the following relations, called the Serre relations.

Proposition 5.14. *The operators $e_i \colon \bigoplus_n K(\operatorname{Rep}_q H_n) \to \bigoplus_n K(\operatorname{Rep}_q H_n)$ satisfy the Serre relations; i.e. if $q^i, q^j \in \mu_q$, $q^j \neq q^{i \pm 1}$, then as operators on the Grothendieck group*

$$e_i e_j = e_j e_i.$$

If $q \neq q^{-1}$, then

$$e_i^2 e_{i\pm1} + e_{i\pm1} e_i^2 = 2 e_i e_{i\pm1} e_i.$$

If $q = q^{-1} = -1$, so $q^2 = 1$,

$$e_i^3 e_{i+1} + 3 e_i e_{i+1} e_i^2 = e_{i+1} e_i^3 + 3 e_i^2 e_{i+1} e_i.$$

The proof is a consequence of the fact that, as we are working in the Grothendieck group, we need only consider irreducible modules, and that the irreducible $H_n \otimes H_k$ modules have the form $M \boxtimes N$. Our list of examples shows the proposition is true for H_k, $k = 2, 3$, (4 for $q = -1$), so we are done.

The proof of independence of characters can be restated to show that the associative algebra U defined to have generators the symbols $\{e_i\}_{q^i \in \mu_q}$ and relations the Serre relations above, surjects to $\bigoplus_n K(\operatorname{Rep}_q H_n)^*$ via the map $e_i \mapsto e_i^*(\delta_{\mathbf{1}})$. We refer the reader to section 2.0.1 for the definition of δ_M and $*$.

For more on this viewpoint, the reader is referred to [G]. Regardless, we recommend [G] to the reader as it explains the rigid structure on the representation theory of H_n and H_n^λ.

Chapter 6

The action of the center and a Mackey map result

In this chapter, we use Grojnowski's functor $f_i : \operatorname{Rep} H_n^\lambda \to \operatorname{Rep} H_{n+1}^\lambda$. These pages give a description of how copies of $\widetilde{e}_i M$ sit in $e_i M$, $\widetilde{f}_i M$ sit in $f_i M$, and M in $e_i f_i M$ and $f_i e_i M$, and relate the answers to Jordan blocks. We differentiate between these copies via the action of certain central elements. One result of this description is that $e_i f_i M$ is a direct summand of $f_i e_i M$ when $\varphi_i(M) \leq \varepsilon_i(M)$ and $f_i e_i M$ is a direct summand of $e_i f_i M$ when $\varepsilon_i(M) \leq \varphi_i(M)$.

6.1 Notation

Let M be an irreducible H_n-module with central character $S_n \cdot s = S_n \cdot (s_1, \dots, s_n)$. Let $|s| = s_1 + s_2 + \cdots + s_n$. Write

$$c(n, |s|) = \sum_{j=1}^{n} (X_j - s_j) = \left(\sum_{j=1}^{n} X_j \right) - |s|.$$

Then let $c_n = c(n, |s|)$, $c_{n-1} = c(n-1, |s \setminus q^i|)$, $c_{n+1} = c(n+1, |s + q^i|)$, noting that $c_k \in Z(H_k)$ and

$$c_{n-1} + (X_n - q^i) = c_n \tag{6.1}$$

$$c_n + (X_{n+1} - q^i) = c_{n+1}. \tag{6.2}$$

Write $\varepsilon = \varepsilon_i(M)$.

6.2 $e_i M$ and c_{n-1}

Claim 6.1. *$c_n M = 0$;* $\qquad$ *$c_{n-1} \widetilde{e}_i M = 0$.*

Proof. Because M is irreducible, the center of H_n acts by scalars on M, so by definition of central character, $c_n M = 0$. As the central character of $\widetilde{e}_i M$, if it is nonzero, is $s \setminus q^i$, the same argument gives $c_{n-1} \widetilde{e}_i M = 0$. $\square$

Claim 6.2. $c_{n-1}^{\varepsilon}e_iM = 0$.

Proof. As discussed in the alternate viewpoint for e_iM, $c_{n-1}M = \big(c_n - (X_n - q^i)\big)M = -(X_n - q^i)M$. By proposition 5.10, X_n has maximal q^i-Jordan-block size $\varepsilon = \varepsilon_i(M)$ on M, or equivalently on i-Res M, and therefore so does c_{n-1} on e_iM. The proof of the proposition also showed that it attains that blocksize, i.e. $c_{n-1}^{\varepsilon-1}e_iM \neq 0$. □

As c_k is central in H_k, left multiplication by c_k on any module induces an endomorphism which we'll also denote c_k.

The images of c_{n-1}^j give the following filtration of e_iM

$$e_iM = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_\varepsilon = 0 \tag{6.3}$$

where $A_j = c_{n-1}^j e_iM = \mathrm{Im}\, c_{n-1}^j$.

Claim 6.3. cosoc $A_j/A_{j+1} \simeq \widetilde{e}_iM$ *and* $\widetilde{e}_iM$ *occurs with multiplicity one as a composition factor of* A_j/A_{j+1}.

Proof. First, notice that $A_j \neq A_{j+1}$ as c_{n-1} is nilpotent on e_iM and $c_{n-1}^{\varepsilon-1}e_iM \neq 0$. Consider the map

$$e_iM \xrightarrow{c_{n-1}^j} A_j \twoheadrightarrow A_j/A_{j+1}.$$

As it is surjective and $\widetilde{e}_iM \simeq \mathrm{cosoc}\, M$, we must have cosoc $A_j/A_{j+1} \simeq \widetilde{e}_iM$. Hence there is at least one copy of $\widetilde{e}_iM$ in each of the ε steps of the filtration (6.3). However, by proposition 5.10, there are exactly ε copies of $\widetilde{e}_iM$ in e_iM (which is our informal way of saying $\widetilde{e}_iM$ occurs with multiplicity ε as a composition factor of e_iM). □

It is important to note that this means the copies of $\widetilde{e}_iM$ sit in e_iM uniserially.

Claim 6.4. $A_{\varepsilon-1} = \widetilde{e}_iM$.

Proof. As $\widetilde{e}_iM = \mathrm{soc}\, e_iM$, we must have $\widetilde{e}_iM \subseteq A_{\varepsilon-1}$ and so soc $A_{\varepsilon-1} = \widetilde{e}_iM$. On the other hand, by claim 6.3 above, cosoc $A_{\varepsilon-1} \simeq \widetilde{e}_iM$ and $\widetilde{e}_iM$ occurs with multiplicity one. We must have $A_{\varepsilon-1} = \widetilde{e}_iM$. □

Now we may complete the proof of part (3) of proposition 5.10, and in fact, we prove something slightly stronger.

Claim 6.5. $\mathrm{Hom}_{H_{n-1}}(e_iM, e_iM) \simeq R[x]/(x^\varepsilon)$ *via the map* $c_{n-1} \mapsto \bar{x}$. *In particular, this shows* $\dim \mathrm{Hom}_{H_{n-1}}(e_iM, e_iM) = \varepsilon$.

Proof. Consider $\mathrm{Hom}_{H_{n-1}}(e_iM, A_j)$ and induct on $\varepsilon - j$. In the case $j = \varepsilon - 1$, as $\widetilde{e}_iM \simeq \mathrm{cosoc}\, e_iM$, $\mathrm{Hom}_{H_{n-1}}(e_iM, A_{\varepsilon-1}) = \mathrm{Hom}_{H_{n-1}}(e_iM, \widetilde{e}_iM) = R$. As $c_{n-1}^{\varepsilon-1}$ is a nonzero map $e_iM \to \widetilde{e}_iM$, this map generates the above space, i.e. $\{c_{n-1}^k\}_{\varepsilon > k \geq \varepsilon-1}$ is an R-basis.

Now let $0 \neq \alpha \in \mathrm{Hom}_{H_{n-1}}(e_iM, A_j)$. We may assume $\alpha(e_iM) \not\subseteq A_{j+1}$. Thus $e_iM \xrightarrow{\alpha} A_j \to A_j/A_{j+1}$ is a nonzero map and so its image contains a copy of $\widetilde{e}_iM \simeq \mathrm{cosoc}\, e_iM$. Since $\widetilde{e}_iM$ occurs with multiplicity one in A_j/A_{j+1}, the composition

$$e_iM \xrightarrow{\alpha} A_j \twoheadrightarrow A_j/A_{j+1} \twoheadrightarrow \widetilde{e}_iM$$

is a nonzero map, and as such is given by $ac_{n-1}^{\varepsilon-1}$. By similar reasoning, the composition $e_iM \xrightarrow{\alpha - ac_{n-1}^{\varepsilon-1}} A_j \to A_j/A_{j+1}$ must be the 0 map, and so $(\alpha - ac_{n-1}^{\varepsilon-1})(e_iM) \subseteq A_{j+1}$. By the inductive hypothesis $\alpha - ac_{n-1}^{\varepsilon-1} = \sum_{k>j} a_k c_{n-1}^k$ and so $\{c_{n-1}^k\}_{\varepsilon>k\geq j}$ forms an R-basis of $\mathrm{Hom}_{H_{n-1}}(e_iM, A_j)$. The case $j = 0$ gives the claim since $A_0 = e_iM$. □

Remark 6.6. c_{n-1} acts on the ε copies of $\widetilde{e}_iM$ in e_iM as x does on $R[x]/(x^\varepsilon)$, in other words, like a Jordan block of size ε.

Corollary 6.7. $\dim \mathrm{Hom}_{H_n}(\mathrm{Ind}\, e_iM \boxtimes q^iJ_m, M) = \min(\varepsilon, m)$.

Proof. By Frobenius reciprocity

$$\begin{aligned}\dim \mathrm{Hom}_{H_n}(\mathrm{Ind}\, e_iM \boxtimes q^iJ_m, M) &= \dim \mathrm{Hom}_{H_{n-1}\otimes H_1}(e_iM \boxtimes q^iJ_m, \mathrm{Res}^{H_n}_{H_{n-1}\otimes H_1} M) \\ &= \dim \mathrm{Hom}_{H_{n-1}\otimes H_1}(e_iM \boxtimes q^iJ_m, \text{i-Res}\, M).\end{aligned}$$

If $\beta \in \mathrm{Hom}_{H_{n-1}\otimes H_1}(e_iM \boxtimes q^iJ_m, \text{i-Res}\, M)$, then we know $\beta_{|e_iM} = \sum a_k c_{n-1}^k$ from claim 6.5. We also know that c_{n-1} acts on e_iM as $-(X_n - q^i)$ does on i-Res M, so the action of β on q^iJ_m is determined by its action on e_iM. In particular,

$$c_{n-1}^k\beta(e_iM \boxtimes q^iJ_m) = 0 \iff (X_n - q^i)^k\beta(e_iM \boxtimes q^iJ_m) = 0 = \beta(e_iM \boxtimes q^iJ_{m-k}).$$

As $\dim \mathrm{Hom}_R(q^iJ_m, q^iJ_\varepsilon) = \min(\varepsilon, m)$, the corollary follows. □

We can also dualize the above discussion considering the filtration

$$0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\varepsilon = e_iM \tag{6.4}$$

where $B_j = \ker c_{n-1}^j$.

Observe $B_j \supseteq A_{\varepsilon-j}$ as $c_{n-1}^j(c_{n-1}^{\varepsilon-j}e_iM) = 0$.

Claim 6.8. *$\mathrm{soc}\, B_j/B_{j-1} \simeq \widetilde{e}_iM$ and $\widetilde{e}_iM$ occurs with multiplicity one as a composition factor of B_j/B_{j-1}.*

Proof. First, $B_j/B_{j-1} \neq 0$ for $0 < j \leq \varepsilon$ as $c_{n-1}^{\varepsilon-1}e_iM \neq 0$. Consider $B_j \xrightarrow{c_{n-1}^{j-1}} B_1$ which in nonzero and well-defined as $c_{n-1}^{j-1}B_j \neq 0$, but $c_{n-1}(c_{n-1}^{j-1}B_j) = 0$. The kernel of the above map is exactly B_{j-1} by definition, and so we have an injection

$$B_j/B_{j-1} \hookrightarrow B_1.$$

Thus $\mathrm{soc}\, B_j/B_{j-1} \hookrightarrow \mathrm{soc}\, B_1 \subseteq \mathrm{soc}\, e_iM = \widetilde{e}_iM$, giving $\mathrm{soc}\, B_j/B_{j-1} \simeq \widetilde{e}_iM$. Again, as there are only ε copies of $\widetilde{e}_iM$ in e_iM, there can be only one copy in each B_j/B_{j-1}. □

Claim 6.9. *$e_iM/B_{\varepsilon-1} \simeq \widetilde{e}_iM$.*

Proof. $\widetilde{e}_iM \simeq \mathrm{cosoc}\, e_iM = \mathrm{cosoc}\, e_iM/B_{\varepsilon-1}$ but also $\widetilde{e}_iM \simeq \mathrm{soc}\, e_iM/B_{\varepsilon-1}$ which occurs with multiplicity one yielding $e_iM/B_{\varepsilon-1} \simeq \widetilde{e}_iM$ as before. □

Remark 6.10. As c_{n-1} kills semisimple H_{n-1}-modules with central character $s \setminus q^i$, we see the cosocle filtration of e_iM must refine the filtration (6.3), while the socle filtration refines (6.4). In any case, the copies of $\widetilde{e}_iM$ must all lie in different steps, although we have little control over the other composition factors.

6.3 $f_i M$ and c_{n+1}

In this section, we fix λ and most of the modules considered live in $\operatorname{Rep} H_n^\lambda$. Write

$$q^i J_m = R[x]/((x-q^i)^m),$$

in other words $q^i J_m$ is the H_1-module on which X_1 acts with q^i-Jordan block of size m, or equivalently $X_1 - q^i$ acts with nilpotent Jordan block of size m. So $q^i J_1$ is what we have been calling q^i. In most of the discussion below, the term Jordan block will refer to a nilpotent Jordan block, especially when discussing the action of c_k.

Notice $q^i J_{\lambda_i}$ is in $\operatorname{Rep} H_1^\lambda$ while $q^i J_{\lambda_i+1}$ is not.

We use the functor of [G]

$$f_i M = \varprojlim_m \operatorname{pr}_\lambda \operatorname{Ind} M \boxtimes q^i J_m$$

where $\operatorname{pr}_\lambda N = N/I_\lambda N$, recalling $I_\lambda = H_n \prod_i (X_1 - q^i)^{\lambda_i} H_n$. By the definition of $\operatorname{pr}_\lambda$, if $M \in \operatorname{Rep} H_n^\lambda$ then $f_i M \in \operatorname{Rep} H_n^{\lambda+1}$. The functor $\operatorname{pr}_\lambda$ picks off the largest quotient of N that is an H_n^λ-module. There is also a functor $\check{\operatorname{pr}}_\lambda$ which picks off the largest submodule of N that is an H_n^λ-module, i.e. that is killed by I_λ.

Define

$$\varphi_i(M) = \min\{m \mid \operatorname{pr}_\lambda \operatorname{Ind} M \boxtimes q^i J_m = f_i M\}$$

as the stabilization point of the above limit. A major result of [G] is that

$$\varphi_i(M) = \max\{m \mid f_i{}^m M \neq 0\} = \max\{m \mid \widetilde{f}_i^m M \neq 0\},$$

but we will not need this here. We do however note that f_i is adjoint to e_i.

$$\operatorname{Hom}_{H_{n-1}^\lambda}(e_i M, N) = \operatorname{Hom}_{H_n^\lambda}(M, f_i N); \quad \operatorname{Hom}_{H_{n-1}^\lambda}(N, e_i M) = \operatorname{Hom}_{H_n^\lambda}(f_i N, M).$$

Further, we list without proof the facts:

$$f_i M = \check{\operatorname{pr}}_\lambda \operatorname{\check{I}nd} M \boxtimes q^i J_\varphi$$

$$\widetilde{f}_i M = \operatorname{cosoc} \operatorname{Ind} M \boxtimes q^i = \operatorname{cosoc} f_i M \simeq \operatorname{soc} f_i M$$

$$\bigoplus_i f_i M = \operatorname{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda} M$$

$$[f_i M] = \varphi_i(M)[\widetilde{f}_i M] + \text{lower terms}$$

$$\widetilde{e}_i \widetilde{f}_i M \simeq M \text{ or } 0; \qquad \widetilde{f}_i \widetilde{e}_i M \simeq M \text{ or } 0$$

which mirrors much of what we proved for e_i and ε_i in claim 5.1 and proposition 5.10. This also shows there are $(\varepsilon_i(M)+1)\varphi_i(M)$ and $(\varphi_i(M)+1)\varepsilon_i(M)$ copies of M in $e_i f_i M$ and $f_i e_i M$ respectively.

Remark 6.11. We also state the cyclotomic Mackey formula, that

$$\bigoplus_{j=1}^r M \bigoplus \operatorname{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda} \operatorname{Res}_{H_{n-1}^\lambda}^{H_n^\lambda} M \simeq \operatorname{Res}_{H_n^\lambda}^{H_{n+1}^\lambda} \operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda} M$$

via the map $1 \otimes u \mapsto T_n \otimes u$ and $u' \mapsto X_{n+1}^j \otimes u'$ for u' in the j^{th} copy of M in $\bigoplus_{j=1}^{r} M$, where $r = \sum \lambda_i$. Comparing central characters yields

$$\bigoplus_{j=1}^{r} M \bigoplus_i f_i e_i M \simeq \bigoplus_i e_i f_i M.$$

Below we will give a more explicit description of the images of $f_i e_i M$ and $e_i f_i M$ under the isomorphism above.

For the remainder of this chapter, we write

$$kM = \bigoplus_{j=1}^{k} M$$

for the direct sum of k copies of M.

Claim 6.12. $\operatorname{soc} f_i e_i M \simeq \operatorname{cosoc} f_i e_i M = \varepsilon M$;
$\operatorname{soc} e_i f_i M \simeq \operatorname{cosoc} e_i f_i M = \varphi M$

Proof. By the adjointness of f_i and e_i, $\varepsilon = \dim \operatorname{Hom}(e_i M, e_i M) = \dim \operatorname{Hom}(f_i e_i M, M) = \dim \operatorname{Hom}(M, f_i e_i M)$, so we just need to show no other irreducibles can occur in the socle or cosocle.

Suppose N irreducible and $\gamma \in \operatorname{Hom}(e_i M, e_i N) \simeq \operatorname{Hom}(f_i e_i M, N)$. If $\gamma \neq 0$, it has a copy of $\operatorname{cosoc} e_i M = \widetilde{e}_i M$ in its image. As $e_i N$ consists of modules with $\varepsilon_i \leq \varepsilon_i(\widetilde{e}_i N) = \varepsilon_i(N) - 1$ this shows $\varepsilon_i(\widetilde{e}_i M) \leq \varepsilon_i(\widetilde{e}_i N)$. On the other hand, $\operatorname{soc} e_i N = \widetilde{e}_i N \subseteq \operatorname{Im}\gamma$ so that $e_i M$ has $\widetilde{e}_i N$ as a subquotient, yielding $\varepsilon_i(\widetilde{e}_i N) \leq \varepsilon_i(\widetilde{e}_i M)$. As $\widetilde{e}_i M$ is the only composition factor of $e_i M$ with $\varepsilon_i = \varepsilon_i(M) - 1 = \varepsilon_i(N) - 1$, and similarly for $\widetilde{e}_i N$, γ must map them to each other, yielding $\widetilde{e}_i M \simeq \widetilde{e}_i N$ and so $M \simeq N$.

The proof for $e_i f_i M$ is similar. □

As the central character of $f_i M$, if it is nonzero, is $s + q^i$, c_{n+1} acts nilpotently on $f_i M$. Write $\varphi = \varphi_i(M)$.

Again, we can make filtrations by the images and kernels of c_{n+1}^j:

$$f_i M = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_\varphi = 0, \tag{6.5}$$

$$0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_\varphi = f_i M, \tag{6.6}$$

where $C_j = \operatorname{Im} c_{n+1}^j$ and $D_j = \ker c_{n+1}^j$.

Claim 6.13. $c_{n+1}^{\varphi} f_i M = 0$ *and there is exactly one copy of* $\widetilde{f}_i M$ *in* C_j / C_{j+1} *and* D_j / D_{j-1} *occurring in the cosocle and socle, respectively.*

Proof. First, we consider $\operatorname{Ind} M \boxtimes q^i J_\varphi$. c_{n+1} induces the following filtration

$$\operatorname{Ind} M \boxtimes q^i J_\varphi \supseteq \operatorname{Ind} M \boxtimes q^i J_{\varphi-1} \supseteq \cdots \supseteq \operatorname{Ind} M \boxtimes q^i J_1 \supseteq 0, \tag{6.7}$$

where the j^{th} step is $\mathrm{Im} c_{n+1}^{j} = \ker c_{n+1}^{\varepsilon-j}$. This follows from the calculation

$$\begin{aligned} c_{n+1}H_{n+1} \bigotimes_{H_n\otimes H_1} M \boxtimes q^iJ_m =& H_{n+1} \bigotimes_{H_n\otimes H_1} c_{n+1}M \boxtimes q^iJ_m \\ =& H_{n+1} \bigotimes_{H_n\otimes H_1} (c_n + (X_{n+1} - q^i))M \boxtimes q^iJ_m \\ =& H_{n+1} \bigotimes_{H_n\otimes H_1} \left(c_nM \boxtimes q^iJ_m + M \boxtimes (X_{n+1} - q^i)q^iJ_m\right) \\ =& H_{n+1} \bigotimes_{H_n\otimes H_1} M \boxtimes q^iJ_{m-1} \end{aligned}$$

as $c_nM = 0$. Consequently, $c_{n+1}^{\varphi}\,\mathrm{Ind}\, M \boxtimes q^iJ_{\varphi} = 0$.

As $\mathrm{Ind}\, M \boxtimes q^iJ_m / \mathrm{Ind}\, M \boxtimes q^iJ_{m-1} \simeq \mathrm{Ind}\, M \boxtimes q^i$ and

$$[\mathrm{Ind}\, M \boxtimes q^iJ_1] = [\widetilde{f}_iM] + \text{lower terms}, \tag{6.8}$$

by lemma 5.6, it is clear there is exactly one copy of $\widetilde{f}_iM$ in each step of the above filtration (6.7), which occurs in the cosocle. It is clear there are a total of φ copies of $\widetilde{f}_iM$ occurring in $\mathrm{Ind}\, M \boxtimes q^iJ_{\varphi}$.

Returning to f_iM, as φ is the stabilization point, this means precisely that there are φ copies of $\widetilde{f}_iM$ in f_iM; in other words, pr_λ does not kill any of the φ copies in $\mathrm{Ind}\, M \boxtimes q^iJ_{\varphi}$. One can attain C_j by applying pr_λ to the j^{th} step in the filtration (6.7). Thus everything noted for the filtration (6.7) holds for (6.5).

However, to see that $\mathrm{soc}\, D_j/D_{j-1} = \widetilde{f}_iM$, we appeal to the fact $\mathrm{soc}\, f_iM = \widetilde{f}_iM$ and argue as in claim 6.8. (or we can make a filtration using $\check{\mathrm{I}}\mathrm{nd}$, involving the fact $f_iM = \mathrm{pr}_\lambda \check{\mathrm{I}}\mathrm{nd}\, M \boxtimes q^iJ_{\varphi}$.) ☐

Remark 6.14. We have shown the action of c_{n+1} on the φ copies of $\widetilde{f}_iM$ in f_iM is exactly the action of $X_{n+1} - q^i$ on $M \boxtimes q^iJ_{\varphi}$, in other words of a Jordan block of size φ.

As with claims 6.4 and 6.9, the following claim holds.

Claim 6.15. *$C_{\varphi-1} \simeq \widetilde{f}_iM$; $f_iM/D_{\varphi-1} = \widetilde{f}_iM$.*

Claim 6.16. $\mathrm{Hom}_{H_{n+1}}(f_iM, f_iM) \simeq R[x]/(x^{\varphi})$ *via the map* $c_{n+1} \mapsto \bar{x}$. *In particular,* $\dim \mathrm{Hom}_{H_{n+1}}(f_iM, f_iM) = \varphi$.

We omit the proof, as it is similar to claim 6.5 or can be found in [G].

6.4 e_if_iM and c_n

Before studying e_if_iM, we first consider $\mathrm{Ind}\, e_iM \boxtimes q^iJ_{\varphi}$. This has submodules $G_{j,k} = \mathrm{Ind}\, A_j \boxtimes q^iJ_k$ which contains $(\varepsilon - j)k$ copies of M, but we will not use them to form a filtration. As computed above,

$$\begin{aligned}
c_{n+1}G_{j,k} =& c_{n+1} \operatorname{Ind} A_j \boxtimes q^i J_k \\
=& c_{n+1} H_{n+1} \bigotimes_{H_n \otimes H_1} A_j \boxtimes q^i J_k \\
=& H_{n+1} \bigotimes_{H_n \otimes H_1} c_{n+1} A_j \boxtimes q^i J_k \\
=& H_{n+1} \bigotimes_{H_n \otimes H_1} c_{n-1} A_j \boxtimes q^i J_k + H_{n+1} \bigotimes_{H_n \otimes H_1} A_j \boxtimes (X_{n+1} - q^i) q^i J_k \\
=& H_{n+1} \bigotimes_{H_n \otimes H_1} A_{j+1} \boxtimes q^i J_k + H_{n+1} \bigotimes_{H_n \otimes H_1} A_j \boxtimes q^i J_{k+1} \\
=& G_{j+1,k} + G_{j,k+1}
\end{aligned}$$

Thus the action of c_n on the $\varepsilon\varphi$ copies of M in $\operatorname{Ind} e_i M \boxtimes q^i J_\varphi$ is exactly the action of

$$c_{n-1} \otimes 1 + 1 \otimes c_{n+1}$$

on $e_i M \boxtimes q^i J_\varphi$ (over $H_{n-1} \otimes R[X_{n+1}^{\pm 1}]$). In other words as $x \otimes 1 + 1 \otimes y$ acts on $R[x]/(x^\varepsilon) \boxtimes R[y]/(y^\varphi)$.

This is almost the whole story of what happens for $e_i f_i M$, as by the affine Mackey formula

$$0 \longrightarrow \varphi M \longrightarrow e_i \operatorname{Ind} M \boxtimes q^i J_\varphi \longrightarrow \operatorname{Ind} e_i M \boxtimes q^i J_\varphi \longrightarrow 0 \tag{6.9}$$

where the kernel is exactly the H_{n-1}-submodule $1 \otimes (M \boxtimes q^i J_\varphi) \simeq \varphi M$, which in particular is killed by c_n by claim 6.1. By our previous discussion, the structure of the copies of $\widetilde{f}_i M$ in $\operatorname{Ind} M \boxtimes q^i J_\varphi$ is the same as in $f_i M$ so the structure of the copies of M in $e_i \operatorname{Ind} M \boxtimes q^i J_\varphi$ is exactly the same as in $e_i f_i M$.

Lemma 6.17. *The action of c_n on the $\varepsilon\varphi$ copies of M in $e_i f_i M/\operatorname{soc}(e_i f_i M)$ is*

(i) as ε Jordan blocks of sizes $\varphi+\varepsilon-1, \varphi+\varepsilon-3, \dots, \varphi-\varepsilon+1$, if $\varepsilon \leq \varphi$ and $\operatorname{char} R = 0$;

(ii) as φ Jordan blocks of sizes $\varepsilon+\varphi-1, \varepsilon+\varphi-3, \dots, \varepsilon-\varphi+1$, if $\varphi \leq \varepsilon$ and $\operatorname{char} R = 0$.

(iii) as $\min(\varepsilon, \varphi)$ Jordan blocks, which have sizes ranging from $\varepsilon+\varphi-1$ to $\varepsilon+\varphi+1-2\min(\varepsilon,\varphi)$ if $\operatorname{char} R = p > 0$.

Proof. We have $\operatorname{Ind} M \boxtimes q^i J_\varphi \overset{\mathrm{pr}_\lambda}{\twoheadrightarrow} f_i M$, so $e_i \operatorname{Ind} M \boxtimes q^i J_\varphi \twoheadrightarrow e_i f_i M$. Further, the kernel of this map has no copies of M in it. Thus the kernel of the exact sequence (6.9) is mapped exactly to the socle of $e_i f_i M$. Therefore the sequence induces a surjection

$$\operatorname{Ind} e_i M \boxtimes q^i J_\varphi \twoheadrightarrow e_i f_i M/\operatorname{soc}(e_i f_i M). \tag{6.10}$$

Again (by counting), there are no copies of M in the kernel of this map. Thus the action of c_n on the copies of M in $e_i f_i M/\operatorname{soc}(e_i f_i M)$ is exactly as described above for $\operatorname{Ind} e_i M \boxtimes q^i J_\varphi$.

This description is an easy exercise in linear algebra on the Jordan block decomposition of $x\otimes 1 + 1\otimes y$ acting on $R[x]/(x^\varepsilon)\boxtimes R[y]/(y^\varphi)$.

In the case char$R = p$, it is easy to show there are $\min(\varepsilon,\varphi)$ Jordan blocks and that the largest has size $\varepsilon+\varphi-1$, but harder to describe the sizes of the other blocks, so we omit the description here. $\square$

Next, we will compute the Jordan block structure of c_n acting on the copies on M in e_if_iM. As in the previous sections, we create filtrations

$$e_if_iM = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_{\varphi+\varepsilon} = 0 \tag{6.11}$$

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{\varphi+\varepsilon} = e_if_iM \tag{6.12}$$

where $E_j = c_n^j e_if_iM = \mathrm{Im} c_n^j$ and $F_j = \ker c_n^j$.

The length of each filtration is either $\varphi+\varepsilon$ or $\varphi+\varepsilon-1$ because $c_n \operatorname{soc}(e_if_iM) = 0$ and by lemma 6.17 $c_n^{\varphi+\varepsilon-1}(e_if_iM/\operatorname{soc} e_if_iM) = 0$. Below we will see the length is actually $\varphi+\varepsilon$.

We have a slightly more complicated proof of the analogous statements to claims 6.3, 6.8, and 6.13.

Lemma 6.18. *The only copies of M in E_j/E_{j+1} and F_j/F_{j-1} occur in the cosocle and socle respectively. In particular, c_n kills a copy of M if and only if it lies in the socle; a copy of M is outside of $c_ne_if_iM$ if and only if it occurs in the cosocle.*

Proof. As $e_if_iM/c_ne_if_iM = E_0/E_1 \overset{c_n^j}{\twoheadrightarrow} E_j/E_{j+1}$, it suffices to prove the statement for $e_if_iM/c_ne_if_iM$.

Consider the exact sequence

$$0 \longrightarrow c_ne_if_iM + \operatorname{soc} e_if_iM \Big/ c_ne_if_iM \longrightarrow e_if_iM \Big/ c_ne_if_iM \longrightarrow$$
$$(e_if_iM/\operatorname{soc} e_if_iM)\Big/ c_n(e_if_iM/\operatorname{soc} e_if_iM) \longrightarrow 0. \tag{6.13}$$

The image has $\min(\varepsilon,\varphi)$ copies of M occurring only in its cosocle by lemma 6.17. As the kernel is isomorphic to $c_ne_if_iM/(c_ne_if_iM \cap \operatorname{soc} e_if_iM)$, we will describe the copies of M in $c_ne_if_iM\cap \operatorname{soc} e_if_iM$.

To do this, we use the surjection (6.10) from lemma 6.17, recalling it maps $\varphi M \simeq 1\otimes M\boxtimes q^iJ_\varphi \subseteq e_i \operatorname{Ind} M\boxtimes q^iJ_\varphi$ exactly to $\operatorname{soc} e_if_iM$. We shall show $1\otimes M\boxtimes q^iJ_{\min(\varepsilon,\varphi)} \subseteq c_n(e_i\operatorname{Ind} M\boxtimes q^iJ_\varphi)$ by direct computation.

Pick $u\in M$ such that $c_{n-1}^{\varepsilon-1}u \neq 0$ but $c_{n-1}^{\varepsilon}u = 0$ (i.e. $u\in A_0\setminus A_1$). As M is irreducible, $H_nc_{n-1}^ju = M$ for $0\le j<\varepsilon$.

Write v_j for $(X_{n+1}-q^i)^j \in q^iJ_\varphi$ so that $q^iJ_\varphi = \operatorname{span}\{v_0,\dots,v_{\varphi-1}\}$. For $1\le j\le \min(\varepsilon,\varphi)$ let

$$a_j = c_{n-1}^{\varepsilon-j}u\otimes v_{\varphi-1} - c_{n-1}^{\varepsilon-j+1}u\otimes v_{\varphi-2} + \cdots \pm c_{n-1}^{\varepsilon-1}u\otimes v_{\varphi-j}.$$

Notice $c_{n-1}a_j = a_{j-1}$ and $(X_{n+1}-q^i)a_j = -a_{j-1}$.

Then $c_n(T_n \otimes a_j) = T_n \otimes \left(c_{n-1}a_j + (X_{n+1} - q^i)a_j\right) + 1 \otimes (1-q)X_{n+1}a_j = 1 \otimes (1-q)X_{n+1}a_j \in 1 \otimes M \boxtimes q^i J_\varphi$. Hence $1 \otimes M \otimes v_k \subseteq c_n(e_i \operatorname{Ind} M \boxtimes q^i J_\varphi)$ for $\varphi > k \geq \min(\varepsilon, \varphi)$. In other words $1 \otimes M \boxtimes q^i J_{\min(\varepsilon,\varphi)} \subseteq c_n(e_i \operatorname{Ind} M \boxtimes q^i J_\varphi)$.

Applying the surjection (6.10) shows $\min(\varepsilon, \varphi)M \subseteq c_n e_i f_i M$.

Next, we'll show $c_n e_i f_i M \cap \operatorname{soc} e_i f_i M = \min(\varepsilon, \varphi)M$.

Let $Q \subseteq e_i f_i M$ be such that $e_i f_i M/Q = \operatorname{cosoc} e_i f_i M \simeq \varphi M$. The fact $c_n M = 0$ yields $c_n e_i f_i M \subseteq Q$, so that $\min(\varepsilon, \varphi)M \subseteq Q$. On the other hand, as $\operatorname{cosoc} \operatorname{Ind} e_i M \boxtimes q^i J_\varphi = \min(\varepsilon, \varphi)M$ by corollary 6.7, and the kernel of (6.10) has no composition factors isomorphic to M, $\operatorname{cosoc}(e_i f_i M/\operatorname{soc} e_i f_i M) = \min(\varepsilon, \varphi)M$. Recalling $\operatorname{cosoc} e_i f_i M = \varphi M$, we must have that $\varphi - \min(\varepsilon, \varphi)$ copies of M in the socle are also in the cosocle, i.e. have trivial intersection with Q and therefore with $c_n e_i f_i M$.

In the case $\varphi > \varepsilon$, the kernel of (6.13) is isomorphic to $\operatorname{soc} e_i f_i M/(\operatorname{soc} e_i f_i M \cap c_n e_i f_i M) = \varphi M/\min(\varepsilon, \varphi)M = (\varphi - \varepsilon)M$. However, these are the copies of M having trivial intersection with Q, in other words they do occur in $\operatorname{cosoc} e_i f_i M$. In fact, this shows $(\varphi - \varepsilon)M$ is a direct summand of $e_i f_i M$. Later we shall see its direct complement is $f_i e_i M$.

In the case $\varphi \leq \varepsilon$, the kernel of (6.13) is 0, showing that in either case, there are exactly φ copies of M in $e_i f_i M/c_n e_i f_i M$ all of which occur in the cosocle.

The proof for F_j/F_{j-1} is similar. □

Theorem 6.19. *The action of c_n on the $(\varepsilon + 1)\varphi$ copies of M in $e_i f_i M$ is*

(i) as φ Jordan blocks, ε of which have sizes $\varphi + \varepsilon, \varphi + \varepsilon - 2, \ldots, \varphi - \varepsilon + 2$, respectively, and $\varphi - \varepsilon$ of which have size 1, if $\varepsilon \leq \varphi$ and $\operatorname{char} R = 0$;

(ii) as φ Jordan blocks of sizes $\varepsilon + \varphi, \varepsilon + \varphi - 2, \ldots, \varepsilon - \varphi + 2$, respectively, if $\varphi \leq \varepsilon$ and $\operatorname{char} R = 0$.

(iii) as φ Jordan blocks, $\min(\varepsilon, \varphi)$ of which have sizes ranging from $\varphi + \varepsilon$ to $\varphi + \varepsilon + 2 - 2\min(\varepsilon, \varphi)$, and $\varphi - \min(\varepsilon, \varphi)$ of which have size 1, if $\operatorname{char} R = p > 0$.

Proof. We employ the exact sequence

$$0 \longrightarrow \varphi M \longrightarrow e_i f_i M \longrightarrow e_i f_i M/\operatorname{soc} e_i f_i M \longrightarrow 0$$

and lemma 6.17. Lemma 6.18 above tells us exactly how $\operatorname{soc}(e_i f_i M) = \varphi M$ fits into the Jordan block structure of c_n. As $c_n \varphi M = 0$ and $\min(\varepsilon, \varphi)M \subseteq c_n e_i f_i M$, $\min(\varepsilon, \varphi)$ copies of M must lie at the bottom of the $\min(\varepsilon, \varphi)$ Jordan blocks described in lemma 6.17. In other words, we get the same description as for $e_i f_i M/\operatorname{soc} e_i f_i M$ except that each block is size one larger. Furthermore, there are $\varphi - \min(\varepsilon, \varphi)$ singleton Jordan blocks as these are the copies disjoint from $c_n e_i f_i M$ as in the proof above. □

6.5 $f_i e_i M$ and c_n

We use the results of the previous section to prove the analogue to theorem 6.19 for $f_i e_i M$. We also prove the pleasing result that, depending on the relative sizes of $\varphi_i(M)$ and $\varepsilon_i(M)$, one of $e_i f_i M$ and $f_i e_i M$ is a direct summand of the other.

Lemma 6.20. *We have maps*

$$f_i e_i M \xrightarrow{\psi} e_i f_i M \longrightarrow e_i f_i M/\operatorname{soc}(e_i f_i M)$$

where the composition is a surjection, and

$$Q \longrightarrow e_i f_i M \xrightarrow{\hat{\psi}} f_i e_i M$$

where the composition is an injection.

Proof. We will define a map $\operatorname{Ind} e_i M \boxtimes q^i J_m \to e_i f_i M$ which will necessarily factor through $\mathrm{pr}_\lambda \operatorname{Ind} e_i M \boxtimes q^i J_m = f_i e_i M$ for $m \gg 0$, as $e_i f_i M \in \operatorname{Rep} H_n^\lambda$ and pr_λ takes the largest quotient that is an H_n^λ-module. We will also denote the map $f_i e_i M \to e_i f_i M$ by ψ.

Take $m \geq \varepsilon + \varphi$. Define

$$\psi : \operatorname{Ind} e_i M \boxtimes q^i J_m \longrightarrow e_i f_i M \qquad \text{by}$$
$$1 \otimes (u \otimes y_0) \xmapsto{\psi} T_n \otimes (u \otimes v_0)$$

where $y_j = (X_n - q^i)^j \in q^i J_m$, $v_j = (X_{n+1} - q^i)^j \in q^i J_\varphi$. (Really we should write $\overline{T_n \otimes (u \otimes v_0)}$ to denote the image of $T_n \otimes (u \otimes v_0) \in \operatorname{Ind} M \boxtimes q^i J_\varphi$ in $e_i f_i M$, but we reserve the bar to mean modulo $\operatorname{soc} e_i f_i M$ below.)

By Frobenius reciprocity, it suffices to show ψ is a well-defined $H_{n-1} \otimes H_1$-map on $1 \otimes e_i M \boxtimes q^i J_m$. For $h \in H_{n-1}$, $\psi(h \otimes (u \otimes y_0)) = \psi(1 \otimes (hu \otimes y_0)) = T_n \otimes (hu \otimes v_0) = hT_n \otimes (u \otimes v_0)$. By definition, $(X_n - q^i) T_n \otimes (u \otimes v_0) = \psi((X_n - q^i) \otimes (u \otimes y_0)) = \psi(1 \otimes (u \otimes y_1))$, so we need only verify $(X_n - q^i)^m T_n \otimes (u \otimes v_0) = 0$.

This follows as $m \geq \varepsilon + \varphi$, $(X_n - q^i)^\varphi v_0 = 0$, $(X_n - q^i)^\varepsilon u = 0$, and

$$\begin{aligned}(X_n - q^i)^m T_n = T_n (X_{n+1} - q^i)^m + (1-q) X_{n+1} \big(&(X_n - q^i)^{m-1} + \\ &(X_n - q^i)^{m-2}(X_{n+1} - q^i) + \cdots + (X_{n+1} - q^i)^{m-1} \big).\end{aligned}$$

Next, the composition

$$\begin{array}{c} \operatorname{Ind} e_i M \boxtimes q^i J_m \xrightarrow{\psi} e_i f_i M \longrightarrow e_i f_i M/\operatorname{soc}(e_i f_i M) \\ 1 \otimes (u \otimes y_0) \mapsto T_n \otimes (u \otimes v_0) \mapsto \overline{T_n \otimes (u \otimes v_0)} \end{array}$$

is a surjection.

One easy way to see this is to record the discussion about (6.9) and (6.10) in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \varphi M \arrow[r] & e_i \operatorname{Ind} M \boxtimes q^i J_\varphi \arrow[r] \arrow[d, two heads] & \operatorname{Ind} e_i M \boxtimes q^i J_\varphi \arrow[r] \arrow[dl] & 0 \\
& & e_i f_i M \arrow[d, two heads] & & \\
& & e_i f_i M/\operatorname{soc}(e_i f_i M) & &
\end{tikzcd}$$

where the maps in the commuting triangle are

$$\begin{array}{ccc} T_n \otimes (u \otimes v_0) & \longmapsto & 1 \otimes (u \otimes v_0) \\ \downarrow & \swarrow & \\ \overline{T_n \otimes (u \otimes v_0)}. & & \end{array}$$

Then the map

$$\begin{array}{c} \operatorname{Ind} e_i M \boxtimes q^i J_m \twoheadrightarrow \operatorname{Ind} e_i M \boxtimes q^i J_\varphi \twoheadrightarrow e_i f_i M/\operatorname{soc}(e_i f_i M) \\ 1 \otimes (u \otimes y_0) \mapsto 1 \otimes (u \otimes v_0) \mapsto \overline{T_n \otimes (u \otimes v_0)} \end{array}$$

exactly coincides with the composition above and hence is surjective.

For the second statement, recall Q was defined via

$$0 \longrightarrow Q \longrightarrow e_i f_i M \longrightarrow \varphi M \longrightarrow 0.$$

Thus we may use the fact $f_i M = \check{\operatorname{pr}}_\lambda \check{\operatorname{Ind}}\, M \boxtimes q^i J_\varphi$, (where $\check{\operatorname{pr}}_\lambda$ takes the largest submodule that is an H_n^λ-module) and proceed as above, reversing all arrows. Denote the map $e_i f_i M \to f_i e_i M$ by $\hat{\psi}$. ☐

Remark 6.21. The maps above are induced from the cyclotomic Mackey map and the natural inclusion and projection maps

$$\begin{array}{c} f_i e_i M \hookrightarrow rM \bigoplus_i f_i e_i M \xrightarrow{\simeq} \bigoplus_i e_i f_i M \to e_i f_i M \\ e_i f_i M \hookrightarrow \bigoplus_i e_i f_i M \xrightarrow{\simeq} rM \bigoplus_i f_i e_i M \to f_i e_i M. \end{array}$$

We note the obvious fact

Claim 6.22. *If* $A \xrightarrow{\alpha} B \xrightarrow{\beta} \operatorname{cosoc} B$ *and* $\beta \circ \alpha$ *is a surjection, then* α *must also be a surjection.*

If $\operatorname{soc} A \xhookrightarrow{\gamma} A \xrightarrow{\alpha} B$ *and* $\alpha \circ \gamma$ *is an injection, then* α *must also be an injection.*

Theorem 6.23. *If* $\varphi \leq \varepsilon$, *the maps of lemma 6.20 induce*

$$f_i e_i M \twoheadrightarrow e_i f_i M \quad \textit{and} \quad e_i f_i M \hookrightarrow f_i e_i M.$$

If $\varepsilon \leq \varphi$, *then we have*

$$f_i e_i M \hookrightarrow e_i f_i M \quad \textit{and} \quad e_i f_i M \twoheadrightarrow f_i e_i M.$$

Proof. If $\varphi \leq \varepsilon$, we see from (6.10) and claim 6.12 that $\operatorname{cosoc}(e_i f_i M/\operatorname{soc}(e_i f_i M)) \simeq \varphi M \simeq \operatorname{cosoc} e_i f_i M$, so by claim 6.22 the map $f_i e_i M \to e_i f_i M$ of lemma 6.20 is surjective.

Similarly $\operatorname{soc} Q \simeq \operatorname{soc} e_i f_i M$, so the map $e_i f_i M \to f_i e_i M$ of lemma 6.20 is injective.

If $\varepsilon \leq \varphi$, consider

$$f_i e_i M \xrightarrow{\psi} e_i f_i M \longrightarrow e_i f_i M/(\varphi - \varepsilon) M \longrightarrow e_i f_i M/\operatorname{soc}(e_i f_i M)$$

where the submodule $(\varphi-\varepsilon)M$ are those $\varphi-\varepsilon$ singleton Jordan blocks for c_n, i.e. the copies of M in both the socle and cosocle of $e_i f_i M$. Thus $\operatorname{cosoc}(e_i f_i M/(\varphi-\varepsilon)M) \simeq \varepsilon M$, and applying claim 6.22 we have

$$f_i e_i M \longrightarrow e_i f_i M/(\varphi-\varepsilon)M$$

is a surjection.

If that surjection had nontrivial kernel, this kernel would intersect $\operatorname{soc} f_i e_i M \simeq \varepsilon M$. However there are exactly $\varphi\varepsilon+\varepsilon$ copies of M in each module, showing the map is an isomorphism. Thus there is a surjection

$$e_i f_i M \twoheadrightarrow e_i f_i M/(\varphi-\varepsilon)M \simeq f_i e_i M.$$

Further, as $(\varphi-\varepsilon)M$ splits off from $e_i f_i M$, $f_i e_i M$ is isomorphic to a submodule of $e_i f_i M$ yielding

$$f_i e_i M \hookrightarrow e_i f_i M.$$

(In other words, $\ker\psi = 0$ and the above injection is just ψ.) □

Further, the maps of theorem 6.23 are splittings.

Corollary 6.24. *If $\varepsilon \le \varphi$ then $e_i f_i M \simeq f_i e_i M \bigoplus (\varphi-\varepsilon)M$.*
If $\varphi \le \varepsilon$ then $f_i e_i M \simeq e_i f_i M \bigoplus (\varepsilon-\varphi)M$.

Proof. The case $\varepsilon \le \varphi$ is explained in the proof of theorem 6.23 above.

For $\varphi \le \varepsilon$ we apply the same reasoning. From theorem 6.23 we have an injection $e_i f_i M \hookrightarrow f_i e_i M$. As $\operatorname{soc} e_i f_i M \simeq \varphi M$ and $\operatorname{soc} f_i e_i M \simeq \varepsilon M$, by claim 6.12, there is a submodule $(\varepsilon-\varphi)M \subseteq f_i e_i M$ which has trivial intersection with $\operatorname{Im}(e_i f_i M)$. Consider

$$e_i f_i M \longrightarrow f_i e_i M \longrightarrow f_i e_i M/(\varepsilon-\varphi)M$$

is still an injection. Noting $\operatorname{cosoc}(f_i e_i M/(\varepsilon-\varphi)M)$ is a direct sum of copies of M and we count $\varepsilon\varphi+\varphi$ copies of M in each module above, the map is an isomorphism. Thus $\operatorname{cosoc}(f_i e_i M/(\varepsilon-\varphi)M) = \varphi M$ and so the $\varepsilon-\varphi$ copies of M are in the socle and cosocle of $f_i e_i M$, i.e. they split off, yielding

$$e_i f_i M \bigoplus (\varepsilon-\varphi)M \simeq f_i e_i M.$$

□

Theorem 6.25. *The action of c_n on the $\varepsilon(\varphi+1)$ copies of M in $f_i e_i M$ is*

(i) as ε Jordan blocks, φ of which have sizes $\varepsilon+\varphi, \varepsilon+\varphi-2, \dots, \varepsilon-\varphi+2$, respectively, and $\varepsilon-\varphi$ of which have size 1, if $\varphi \le \varepsilon$ and $\operatorname{char} R = 0$;

(ii) as ε Jordan blocks of sizes $\varepsilon+\varphi, \varepsilon+\varphi-2, \dots, \varepsilon-\varphi+2$, respectively, if $\varepsilon \le \varphi$ and $\operatorname{char} R = 0$.

(iii) as ε Jordan blocks, $\min(\varepsilon,\varphi)$ of which have sizes ranging from $\varepsilon+\varphi$ to $\varepsilon+\varphi+2-2\min(\varepsilon,\varphi)$, and $\varepsilon-\min(\varepsilon,\varphi)$ of which have size 1, if $\operatorname{char} R = p > 0$.

Proof. In the case $\varphi \leq \varepsilon$, $f_i e_i M \simeq e_i f_i M \bigoplus (\varepsilon - \varphi) M$, whose Jordan block structure we deduce from theorem 6.19.

In the case $\varepsilon \leq \varphi$, we have from corollary 6.24 $f_i e_i M \simeq e_i f_i M / (\varphi - \varepsilon) M$ whose Jordan block structure we again deduce from theorem 6.19, as the $\varphi - \varepsilon$ copies of M are the singleton Jordan blocks. $\square$

The strong relationship between e_i and f_i and the importance of the quantity $\varphi_i(M) - \varepsilon_i(M)$ are investigated further and explained in [G].

Appendix A

Modifications when $q = 1$.

The following appendix is largely taken, with permission, from [G].

If $q = 1$ all of the above theorems and constructions go through, without change, once we make the appropriate definitions.

Redefine μ_q to be the image of $\mathbb{Z} \to R$ $1 \mapsto 1$, so ℓ is the characteristic of R, $\ell \in \mathbb{N} \cup \{\infty\}$. Instead of the affine Hecke algebra, we work with the *degenerate* or *graded* affine Hecke algebra $\overline{H}_n^{\mathrm{gr}}$, defined by Drinfel'd [D] and Lusztig [L]. This is isomorphic as an R-module to

$$R[S_n] \otimes R[x_1, \ldots, x_n]$$

with algebra structure defined by requiring that $R[S_n]$ and $R[x_1, \ldots, x_n]$ are subalgebras, and that

$$\tau_i \cdot f - {}^{\tau_i}f \cdot \tau_i = \frac{f - {}^{\tau_i}f}{x_{i+1} - x_i}.$$

We motivate the formula above by the following computation: as $1 - q^k = (1 - q)(1 + q + \cdots q^{k-1})$, as q approaches 1, $\frac{1-q}{1-q^k} \rightsquigarrow \frac{1}{k}$. We get

$$(q-1)\frac{f - {}^{\tau_i}f}{1 - q^{x_i}/q^{x_{i+1}}} = (q-1)\frac{f - {}^{\tau_i}f}{1 - q^{x_i - x_{i+1}}} \rightsquigarrow \frac{f - {}^{\tau_i}f}{x_{i+1} - x_i}.$$

Thinking of X_i as q^{x_i}, compare the above to relation to 2.4.

Given $\lambda{:}\mu_q \to \mathbb{Z}_+$ a function such that $\sum \lambda_i < \infty$, Grojnowski defines the *degenerate cyclotomic algebra* $\overline{H}_n^{\lambda} = \overline{H}_n^{\mathrm{gr}}/I_\lambda$, where $I_\lambda = \overline{H}_n^{\mathrm{gr}} \prod_{i \in \mu_q}(x_1 - i)^{\lambda_i}\overline{H}_n^{\mathrm{gr}}$. If $\lambda = \Lambda_0$, $\overline{H}_n^{\Lambda_0} = R[S_n]$; in general one shows that $\dim_R \overline{H}_n^{\lambda} = r^n n!$, where $r = \sum \lambda_i$. We define $\mathrm{Rep}_q\, \overline{H}_n^{\mathrm{gr}}$ to be the subcategory of $\overline{H}_n^{\mathrm{gr}}$-modules on which x_1 acts with eigenvalues in μ_q. All other definitions and theorems are as before, once we agree to write the group law in $\mathrm{Im}\mathbb{Z} = \mu_q$ multiplicatively, so qi denotes what is usually written $i + 1$. No changes are necessary in the proofs.

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