Title: A bijection between dominant Shi regions and core partitions

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Proposed running head: Bijection from Shi regions to cores

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A BIJECTION BETWEEN DOMINANT SHI REGIONS AND CORE PARTITIONS

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ABSTRACT. It is well-known that Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of dominant regions in the Shi arrangement of type A, and that they also count partitions which are both n-cores as well as $(n+1)$-cores. These concepts have natural extensions, which we call here the m-Catalan numbers and m-Shi arrangement. In this paper, we construct a bijection between dominant regions of the m-Shi arrangement and partitions which are both n-cores as well as $(mn+1)$-cores. The bijection is natural in the sense that it commutes with the action of the affine symmetric group.

1. INTRODUCTION

In this paper, we build on the work of Anderson [1] to give a direct bijection between dominant regions in the "extended" m-Shi arrangement of type A_{n-1} and partitions that are simultaneously n-cores and $(mn+1)$-cores.

Core partitions and regions in the Shi arrangement are both central objects in algebraic combinatorics. Core partitions of integers (see Definition 5.1) are used in Nakayama's Conjecture to describe when two ordinary irreducible representations of the symmetric group belong to the same block in positive characteristic [14]. They also arise in connection with q-series, for example in [9, 8, 6], symmetric functions [17], and other areas.

The Shi arrangement (see Definition 2.1) has also been extensively studied. See, for example, [20, 23, 5, 4, 2, 3, 19]. The Shi arrangement arose in Shi's study of left cells in affine Weyl groups. The Shi arrangement, and more generally hyperplane arrangements, have been studied extensively within combinatorics, but also arise in many branches of mathematics such as algebraic topology, algebraic geometry, and the theory of hypergeometric functions. The complement of a complexified arrangement is a smooth manifold whose topological invariants, such as its cohomology ring, are of interest. The Shi arrangement can be seen as a deformation of the braid arrangement, whose complement's fundamental group is a braid group. One

Date: September 2, 2009.
Both authors wish to thank AIM and the SQuaREs program where this work was started.

can also define multivariate hypergeometric integrals on the complement, which in special cases satisfy the Kniznik-Zomolodchikov (KZ) equations, and which are related to the representation theory of Lie algebras and quantum groups. Although it can be defined in all types, in this paper we are only concerned with the (extended) Shi arrangement of type A.

Anderson's result, that $\frac{1}{s+t}\binom{s+t}{t}$ counts the number of partitions which are both t-cores and s-cores generalizes the well-known interpretation of Catalan numbers C_n as counting n-cores that are also $(n+1)$-cores. Catalan numbers are also known to count dominant regions in the Shi arrangement, termed here the 1-Shi arrangement. (See [24, 25].) Our main result provides a direct bijective proof between two more general sets of combinatorial objects which are counted by the higher Catalan numbers, called here the m-Catalan numbers. The bijection is interesting first in that it involves two very interesting families of combinatorial objects, and second in that it is very natural. The bijection highlights some extra structure the combinatorial objects carry, which is largely induced from the action of a group, and our bijection commutes with that group action.

Let W be the affine symmetric group, defined in Section 2. Our bijection is W-equivariant in the following sense. In each connected component of the m-Shi hyperplane arrangement of type A_{n-1}, there is exactly one "representative," or m-minimal, alcove closest to the fundamental alcove $\mathcal{A}_0$. Since the affine Weyl group W acts freely and transitively on the set of alcoves, there is a natural way to associate an element $w \in W = \widehat{\mathfrak{S}}_n$ to any alcove $w^{-1}\mathcal{A}_0$, and to this one in particular. There is also a natural action of $\widehat{\mathfrak{S}}_n$ on partitions, whereby the orbit of the empty partition $\emptyset$ is precisely the n-cores. We will show that $w\emptyset$ is also an $(mn+1)$-core and that all such $(mn+1)$-cores that are also n-cores can be obtained this way.

Roughly speaking, to each n-core λ we can associate an integer vector $\vec{n}(\lambda)$ whose entries sum to zero. When λ is also an $(mn+1)$-core, these entries satisfy certain inequalities. On the other hand, these are precisely the inequalities that describe when a dominant alcove is m-minimal.

The article is organized as follows. In Section 2 we review facts about Coxeter groups and root systems of type A. Sections 3 and 4 explain how the position of $w^{-1}\mathcal{A}_0$ relative to our system of affine hyperplanes is captured by the action of w on affine roots and that m-minimality can be expressed by certain inequalities on the entries of $w(0, 0, \ldots, 0)$. In Section 5 we review facts about core partitions and in particular remind the reader how to associate an element of the root lattice to each core. Our main theorem, the bijection between dominant regions of the m-Shi arrangement and special cores, is in Section 6. Section 7 describes a related bijection on alcoves. In Section 8, we derive further results refining our bijection between alcoves and cores that involve Narayana numbers. Narayana numbers give a refinement of the Catalan numbers, and will be defined in section 8.

2. THE TYPE A ROOT SYSTEM, SHI ARRANGEMENT, AND WEYL GROUP

Our main result is a bijection between two sets of objects. In this section, we define one these sets: dominant regions in the m-Shi arrangement. We define the affine symmetric group as well and review the properties we will need.

2.1. **The type A root system.** Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the standard basis of $\mathbb{R}^n$ and $\langle \, | \, \rangle$ be the bilinear form for which this is an orthonormal basis. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is a basis of

$$V = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i = 0\}.$$

We let $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$ be identified with the root lattice of type A_{n-1}. The elements of $\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ are called roots and we say a root α is positive, written $\alpha > 0$, if $\alpha \in \Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. We let $\Delta^- = -\Delta^+$ and say $\alpha < 0$ if $\alpha \in \Delta^-$. Then Π is the set of simple roots.

2.2. **Extended Shi arrangements.** A *(real) hyperplane arrangement* is a discrete set of hyperplanes, possibly affine hyperplanes, in $\mathbb{R}^n$. We first define a system of affine hyperplanes

$$H_{\alpha,k} = \{v \in V \mid \langle v \mid \alpha \rangle = k\}$$

given $\alpha \in \Delta$ and $k \in \mathbb{Z}$. Note $H_{-\alpha,-k} = H_{\alpha,k}$ so we usually take $k \in \mathbb{Z}_{\geq 0}$.

Definition 2.1. The (type A) extended Shi arrangement $\mathcal{S}_n^m$, here called the *m-Shi arrangement*, is the collection of hyperplanes

$$\mathcal{H}_m = \{H_{\alpha,k} \mid \alpha \in \Delta^+, -m < k \leq m\}.$$

This arrangement can be defined for root systems of all finite types; here we are concerned with type A.

The extended Shi arrangement was defined by Stanley in [23]. The arrangement $\mathcal{S}_n^1$ is known as the Shi arrangement and was first considered by Shi [20, 22] and later by Headley [11, 12]. The extended Shi arrangement was further studied in, for example, [5, 4, 2, 3, 19]. In [3], Athanasiades gave formulas for the number of dominant regions of the m-Shi arrangement $\mathcal{S}_n^m$ in any type. We recommend the reader see [3, Section 5.1] for an excellent discussion of the m-Shi arrangement in type A.

In this paper, we are primarily concerned with the connected components of the hyperplane arrangement complement $V \setminus \bigcup_{H \in \mathcal{H}_m} H$. For ease of notation we will refer to these as *regions* of the m-Shi arrangement.

Each connected component of $V \setminus \bigcup_{\substack{\alpha \in \Delta^+ \\ k \in \mathbb{Z}}} H_{\alpha,k}$ is called an *alcove*.

We denote the (closed) half-spaces ${H_{\alpha,k}}^+ = \{v \in V \mid \langle v \mid \alpha \rangle \geq k\}$ and ${H_{\alpha,k}}^- = \{v \in V \mid \langle v \mid \alpha \rangle \leq k\}$.

Definition 2.2. The *dominant* chamber of V is $V \cap \bigcap_{i=1}^{n-1} {H_{\alpha_i,0}}^+$ (also referred to as the fundamental chamber in the literature).

A *dominant region* of the m-Shi arrangement is a region that is contained in the dominant chamber. A *dominant alcove* is one contained in the dominant chamber.

Definition 2.3. The *fundamental alcove* is denoted $\mathcal{A}_0$, and is the interior of $V \cap {H_{\theta,1}}^- \cap \bigcap_{i=1}^{n-1} {H_{\alpha_i,0}}^+$, where $\theta = \alpha_1 + \cdots + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n$.

Definition 2.4. An alcove is *m-minimal* if it is the unique alcove of all those contained in one region of the m-Shi arrangement that is separated by the fewest number of affine hyperplanes in $\mathcal{H}_m$ from $\mathcal{A}_0$.

A *wall* of a region or alcove is a hyperplane in $\mathcal{H}_m$ which supports a facet of that region or alcove.

2.3. **The affine symmetric group.**

Definition 2.5. The affine symmetric group, denoted $\widehat{\mathfrak{S}}_n$, is defined as

$$\begin{aligned} \widehat{\mathfrak{S}}_n = \langle s_1, \ldots, s_{n-1}, s_0 \mid s_i^2 = 1, \quad & s_i s_j = s_j s_i \text{ if } i \not\equiv j \pm 1 \bmod n, \\ & s_i s_j s_i = s_j s_i s_j \text{ if } i \equiv j \pm 1 \bmod n \rangle \end{aligned}$$

for $n > 2$, but $\widehat{\mathfrak{S}}_2 = \langle s_1, s_0 \mid s_i^2 = 1 \rangle$.

The affine symmetric group contains the symmetric group $\mathfrak{S}_n$ as a subgroup. $\mathfrak{S}_n$ is the subgroup generated by the s_i, $0 < i < n$. We identify $\mathfrak{S}_n$ as permutations of $\{1, \ldots, n\}$ by identifying s_i with the simple transposition $(i, i+1)$.

The affine symmetric group $\widehat{\mathfrak{S}}_n$ acts on V (preserving Q) via affine linear transformations, and acts freely and transitively on the set of alcoves. We thus identify each alcove $\mathcal{A}$ with the unique $w \in \widehat{\mathfrak{S}}_n$ such that $\mathcal{A} = w\mathcal{A}_0$. Each simple generator s_i, $i > 0$ acts by reflection with respect to the simple root α_i. In other words, it acts by reflection over the hyperplane ${H_{\alpha_i,0}}^+$. Whereas the element s_0 acts as reflection with respect to the affine hyperplane ${H_{\theta,1}}^+$.

More specifically, the action on V is given by

$$s_i(a_1,\ldots,a_i,a_{i+1},\ldots,a_n)=(a_1,\ldots,a_{i+1},a_i,\ldots,a_n)\quad \text{for } i\neq 0\text{, and}$$
$$s_0(a_1,\ldots,a_n)=(a_n+1,a_2,\ldots,a_{n-1},a_1-1).$$

Note $\mathfrak{S}_n$ preserves $\langle\,|\,\rangle$, but $\widehat{\mathfrak{S}}_n$ does not.

We note that $\widehat{\mathfrak{S}}_n$ contains a normal subgroup consisting of translations by elements of Q and that $\widehat{\mathfrak{S}}_n=\mathfrak{S}_n\ltimes Q$ is their semidirect product, i.e.

$$1\to Q\to\widehat{\mathfrak{S}}_n\to\mathfrak{S}_n\to 1$$

is exact. We denote by t_γ translation by the vector γ, and in terms of the coordinates above, if $\gamma=(\gamma_1,\ldots,\gamma_n)\in Q\subseteq V$ then $t_\gamma(a_1,\ldots,a_n)=(a_1+\gamma_1,\ldots,a_n+\gamma_n)$. Consequently, we may express any $w\in\widehat{\mathfrak{S}}_n$ as $w=ut_\gamma$ for unique $u\in\mathfrak{S}_n$, $\gamma\in Q$, or equivalently $w=t_{\gamma'}u$ where $\gamma'=u(\gamma)$. Note that $\gamma'=w(0,\ldots,0)$.

We let δ denote the null root. It is the unique (up to scalar) root that corresponds to the null space of the Cartan matrix of type $A_{n-1}^{(1)}$, also known as affine type A. More precisely, its coordinate vector with respect to the basis $\{\alpha_i \mid 0\leq i<n\}$ spans the null space. Hence for type $A_{n-1}^{(1)}$, $\delta=\sum_i\alpha_i$. Note $\alpha_0=\delta-\theta$. The finite root system above extends to an affine root system

$$\widetilde{\Delta}=\{k\delta+\alpha \mid k\in\mathbb{Z},\alpha\in\Delta\}=\widetilde{\Delta^+}\cup\widetilde{\Delta^-}$$

where

$$\widetilde{\Delta^+}=\{k\delta+\alpha \mid k\in\mathbb{Z}_{\geq 0},\alpha\in\Delta^+\}\cup\{k\delta+\alpha \mid k\in\mathbb{Z}_{>0},\alpha\in\Delta^-\},$$

$\widetilde{\Delta^-}=-\widetilde{\Delta^+}$, and the simple roots are now $\widetilde{\Pi}=\Pi\cup\{\alpha_0\}$. See [15, Chapters 5, 6] for details. Again we write $\alpha>0$ if $\alpha\in\widetilde{\Delta^+}$ and $\alpha<0$ if $\alpha\in\widetilde{\Delta^-}$.

We can picture V sitting in the span of $\widetilde{\Pi}$ as an affine subspace; whereas $\widehat{\mathfrak{S}}_n$ acts on the larger space linearly, it acts on V by affine linear transformations. In most of this paper, we found it more useful to work with V, its coordinate system, and affine hyperplanes. However, we could have chosen to express everything in terms of the span of $\widetilde{\Pi}$, as we found useful to do in Section 3. In terms of affine roots, the action of the simple reflection $s_i\in\widehat{\mathfrak{S}}_n$, $0\leq i<n$, is given by

$$s_i(\gamma)=\gamma-\langle\gamma\mid\alpha_i\rangle\alpha_i, \tag{2.1}$$

where $\gamma\in\operatorname{span}\widetilde{\Pi}$ and we have extended $\langle\,|\,\rangle$ as appropriate (given by the Cartan matrix of type $A_{n-1}^{(1)}$). In this setting, we see s_0 is a reflection with respect to the simple root α_0. We can also re-express the action of translations as

$$t_\gamma(\alpha) = \alpha - \langle \gamma \mid \alpha \rangle \delta \tag{2.2}$$

for $\alpha \in \widetilde{\Delta}$, γ in the integer span of $\widetilde{\Pi}$ or indeed γ in the affine weight lattice. See [15]. Note, in the case $\gamma \in Q$, $\alpha \in \Delta$, $u \in \mathfrak{S}_n$, expressions like $\langle \gamma \mid u(\alpha) \rangle$ are unambiguous as they agree in either setting.

3. Inversion sets

This section reminds the reader of the exact correspondence between the inversions of w and the affine hyperplanes that separate the fundamental alcove $\mathcal{A}_0$ from $w^{-1}\mathcal{A}_0$. The inversions of $w \in \widehat{\mathfrak{S}}_n$ are defined to be

$$\mathrm{Inv}(w) = \{\alpha > 0 \mid w(\alpha) < 0\}.$$

The length of w is defined to be $\ell(w) = |\mathrm{Inv}(w)|$, and this quantity counts the number of affine hyperplanes $H_{\alpha,k}$ separating $\mathcal{A}_0$ from $w^{-1}\mathcal{A}_0$.

While we could have defined $\mathrm{Inv}(w)$ in terms of the $\widehat{\mathfrak{S}}_n$ action on V, it is more conventional to define it in terms of the action on $\widetilde{\Delta}$.

Example 3.1. For example, note for $\gamma \in Q$ that

$$\begin{aligned}\mathrm{Inv}(t_\gamma) =& \{\alpha + k\delta \mid \alpha \in \Delta^+,\, 0 \le k \le \langle \gamma \mid \alpha \rangle\} \cup \\ & \{-\alpha + k\delta \mid \alpha \in \Delta^+,\, 0 < k < -\langle \gamma \mid \alpha \rangle\}.\end{aligned}$$

Proposition 3.2. Let $w \in \widehat{\mathfrak{S}}_n$ and $\alpha + k\delta \in \widetilde{\Delta^+}$, i.e. either $k > 0$ and $\alpha \in \Delta$ or $k = 0$ and $\alpha \in \Delta^+$. Then $\alpha + k\delta \in \mathrm{Inv}(w)$ iff $w^{-1}\mathcal{A}_0 \subseteq {H_{-\alpha,k}}^+$

Proof. We first write $w = t_\gamma u$ for $\gamma \in Q$, $u \in \mathfrak{S}_n$. Note, in terms of affine roots, $w(\alpha + k\delta) = u(\alpha) + (k - \langle \gamma \mid u(\alpha)\rangle)\delta$. Assume $\alpha + k\delta \in \mathrm{Inv}(w)$ which implies either $k - \langle \gamma \mid u(\alpha) \rangle = 0$ and $u(\alpha) \in \Delta^-$ or that $k - \langle \gamma \mid u(\alpha) \rangle < 0$.

Consider $v \in w^{-1}\mathcal{A}_0$, so that $w(v) \in \mathcal{A}_0$. Then

$$0 \le \langle w(v) \mid \eta \rangle \le 1 \quad \text{for all } \eta \in \Delta^+$$

and so translating by $-\gamma$ we obtain $-\langle \gamma \mid \eta \rangle \le \langle u(v) \mid \eta \rangle \le 1 - \langle \gamma \mid \eta \rangle$.

First suppose $u(\alpha) \in \Delta^-$ and consider $\eta = -u(\alpha)$ for k and α as above. Then in particular

$$k \le \langle \gamma \mid u(\alpha) \rangle = -\langle \gamma \mid u(-\alpha) \rangle \le \langle u(v) \mid u(-\alpha) \rangle = \langle v \mid -\alpha \rangle$$

so $v \in {H_{-\alpha,k}}^+$.

If instead $u(\alpha) \in \Delta^+$, we must have $k - \langle \gamma \mid u(\alpha) \rangle \le -1$. Therefore, taking $\eta = u(\alpha)$,

$$k \le -1 + \langle \gamma \mid u(\alpha) \rangle \le -\langle u(v) \mid u(\alpha) \rangle = \langle v \mid -\alpha \rangle$$

so again $v \in {H_{-\alpha,k}}^+$.

The converse is straightforward. □

Similar methods show that if $w(\alpha + k\delta) \in \widetilde{\Delta^+}$ then $w^{-1}\mathcal{A}_0 \subseteq {H_{-\alpha,k}}^-$.

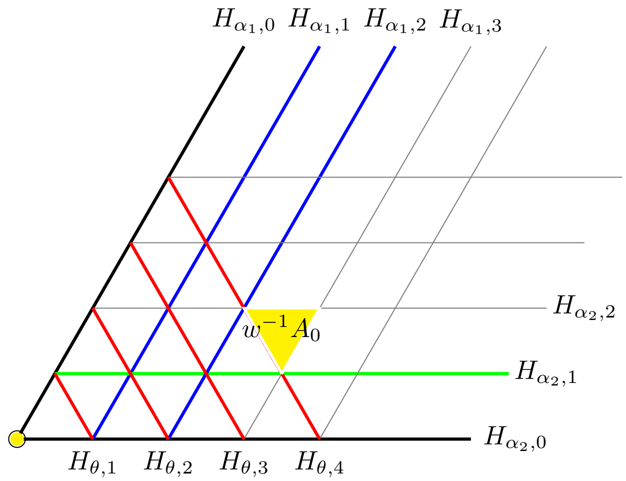


FIGURE 1. When $w^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,k}}^+$ for $\alpha \in \Delta^+$

Example 3.3. Consider $w = s_1 s_2 s_0 s_1 s_2 s_1 s_0$, and let $\mathcal{A} = w^{-1}\mathcal{A}_0$, which is pictured in Figure 1. Observe $w = s_2 t_{(-2,0,2)}$ and so $w^{-1} = t_{(2,0,-2)} s_2$. Then

$$\mathcal{A} \subseteq {H_{\alpha_1,3}}^- \cap {H_{\theta,4}}^+ \cap {H_{\alpha_2,2}}^-,$$

and the three hyperplanes that bound these half-spaces describe the walls of $\mathcal{A}$. Note how the half-spaces above correspond to the action of w^{-1} on the simple roots: $w^{-1}(\alpha_0) = -\alpha_1 + 3\delta$, $w^{-1}(\alpha_1) = \theta - 4\delta$, $w^{-1}(\alpha_2) = -\alpha_2 + 2\delta$. (This phenomenon is explained in Proposition 4.1 below.)

The information in the following chart exhibits the correspondence between $\mathrm{Inv}(w)$ and the half-spaces ${H_{-\alpha,k}}^+$ containing $w^{-1}\mathcal{A}_0$.

α_i	$w^{-1}(\alpha_i)$	$\mathrm{Inv}(w)$	$w^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,k}}^+$
α_0	$-\alpha_1 + 3\delta$	$\{-\alpha_1 + \delta, -\alpha_1 + 2\delta,$	${H_{\alpha_1,1}}^+, {H_{\alpha_1,2}}^+,$
α_1	$\theta - 4\delta$	$-\theta + \delta, -\theta + 2\delta,$	${H_{\theta,1}}^+, {H_{\theta,2}}^+,$
		$-\theta + 3\delta, -\theta + 4\delta,$	${H_{\theta,3}}^+, {H_{\theta,4}}^+$
α_2	$-\alpha_2 + 2\delta$	$-\alpha_2 + \delta\}$	${H_{\alpha_2,1}}^+$

Remark 3.4. Let w be a minimal length left coset representative for $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$, and write $w = t_\gamma u^{-1}$ where $\gamma \in Q, u \in \mathfrak{S}_n$. This means $w\mathcal{A}_0$ is the alcove separated by the fewest number of affine hyperplanes from $\mathcal{A}_0$ among all alcoves whose closure contains γ. It is well-known (and not hard to show) that $\gamma' = -u(\gamma)$ is in the dominant chamber. See [13] for details. Hence the minimal length right coset representative $w^{-1} = t_{\gamma'}u$. In particular, $w^{-1}\mathcal{A}_0$ is in the dominant chamber.

Corollary 3.5. Suppose w is a minimal length left coset representative for $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$. Then $\mathrm{Inv}(w)$ consists only of roots of the form $-\alpha + k\delta$, $k \in \mathbb{Z}_{>0}, \alpha \in \Delta^+$. Further, if $-\alpha+k\delta \in \mathrm{Inv}(w)$ and $k > 1$ then $-\alpha+(k-1)\delta \in \mathrm{Inv}(w)$.

Proof. By Remark 3.4 $w^{-1}\mathcal{A}_0$ is in the dominant chamber, and so in ${H_{\alpha,0}}^+$ for any $\alpha \in \Delta^+$, and in particular never in ${H_{-\alpha,k}}^+$ for $k \geq 0$.

For the second statement, note ${H_{\alpha,k}}^+ \subseteq {H_{\alpha,k-1}}^+$ if $k > 0$. □

4. m-MINIMAL ALCOVES

In this section, we characterize the m-minimal alcoves, which we will use to define our bijection.

We can identify each connected component of the complement of the m-Shi arrangement with it unique m-minimal alcove. Recall an alcove $w\mathcal{A}_0$ is m-minimal if it is the unique alcove in its region such that $\ell(w)$ is smallest (since $\ell(w) = |\mathrm{Inv}(w)|$). Such alcoves are termed "representative alcoves" by Athanasiades.

The following proposition is useful. For a given alcove, it characterizes the affine hyperplanes containing its walls and which simple reflections flip it over those walls (by the right action). It can be found in [21] in slightly different notation.

Proposition 4.1. Suppose $w\mathcal{A}_0 \subseteq {H_{\alpha,k}}^+$ but $ws_i\mathcal{A}_0 \subseteq {H_{\alpha,k}}^-$

(1) Then $w(\alpha_i) = \alpha - k\delta$.
(2) Let $\beta = w^{-1}(0,\ldots,0) \in V$. Then $\langle \beta \mid \alpha_i \rangle = -k$.

Proof. We first note that for $i > 0$ $H_{\alpha_i,0}$ is the unique hyperplane such that $\mathcal{A}_0 \subseteq {H_{\alpha_i,0}}^+$, but $s_i\mathcal{A}_0 \subseteq {H_{\alpha_i,0}}^-$. In the case $i = 0$ we rewrite this condition as $\mathcal{A}_0 \subseteq {H_{-\theta,1}}^+, s_0\mathcal{A}_0 \subseteq {H_{-\theta,1}}^-$.

Because w is an isometry, $w(H_{\alpha_i,0})$ must be the unique hyperplane separating $w\mathcal{A}_0$ from $ws_i\mathcal{A}_0$, and by hypothesis, this hyperplane is $H_{\alpha,k}$. Then $w({H_{\alpha_i,0}}^\pm) = {H_{\alpha,k}}^\pm$ which implies $w(\alpha_i) = \alpha - k\delta$.

For the second statement, note that we can uniquely write $w = t_\gamma u$ with $u \in \mathfrak{S}_n, \gamma \in Q$ as $\widehat{\mathfrak{S}}_n = Q \ltimes \mathfrak{S}_n$. Then $\beta = w^{-1}(0,\ldots,0) = u^{-1}(-\gamma)$ and $\langle u^{-1}(-\gamma) \mid \alpha_i \rangle = \langle -\gamma \mid u(\alpha_i) \rangle = -\langle \gamma \mid \alpha \rangle = -k$ since $w(\alpha_i) = t_\gamma(u(\alpha_i)) = u(\alpha_i) - \langle \gamma \mid u(\alpha_i) \rangle \delta = \alpha - k\delta$. □

In terms of the coordinates of $\gamma \in V$ with $u \in \mathfrak{S}_n$ as above, we note $k = \gamma_{u(i)} - \gamma_{u(i+1)}$.

Remark 4.2. Note, if $w\mathcal{A}_0$ is m-minimal, then whenever $k \in \mathbb{Z}_{\geq 0}$ and $w\mathcal{A}_0 \subseteq {H_{\alpha,k}}^+$ but $ws_i\mathcal{A}_0 \subseteq {H_{\alpha,k}}^-$ then we must have $k \leq m$ in the case $\alpha > 0$ and $k \leq m-1$ in the case $\alpha < 0$.

It is easy to see that this condition is not only necessary but sufficient to describe when $w\mathcal{A}_0$ is m-minimal. Together with Proposition 3.2, Proposition 4.1 says that when $\alpha_i \in \mathrm{Inv}(w)$ and $w(\alpha_i) = \alpha - k\delta$ then $k \leq m$, and for $\beta = w^{-1}(0,\ldots,0)$ that $\langle \beta \mid \alpha_i \rangle \geq -m$.

Applying Remark 4.2 to positive α and alcoves in the dominant chamber, we get the following corollary.

Corollary 4.3. Suppose $w\mathcal{A}_0$ is in the dominant chamber and m-minimal.

(1) If $w\mathcal{A}_0 \subseteq {H_{\alpha,k}}^+$ but $ws_i\mathcal{A}_0 \subseteq {H_{\alpha,k}}^-$ for some $\alpha \in \Delta^+, k \in \mathbb{Z}_{\geq 0}$, then $k \leq m$.
(2) Let $\beta = w^{-1}(0,\ldots,0)$. Then $\langle \beta \mid \alpha_i \rangle \geq -m$, for all i, and in particular $\langle \beta \mid \theta \rangle \leq m+1$.

Proof. The first statement follows directly from Proposition 4.1 and Remark 4.2. To conclude that the second statement holds for all i, note that if $k \leq 0$ then automatically $k \leq m$. □

5. Core partitions and their abacus diagrams

Here we define n-cores, review some well-known facts about them, and review the abacus construction, which will be a useful tool for us. Details can be found in [14].

5.1. **Core partitions and residues.** We identify an integer partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$, with its Young diagram, the array of boxes centered at coordinates $\{(k,l) \mid 1 \leq l \leq \lambda_k\}$.

We do not fix $|\lambda| = \sum_k \lambda_k$ to be a set integer. Indeed, we are interested in partitions of all sizes throughout the paper, and so will rarely state that λ is a partition of $|\lambda|$.

The (k,l)-*hook* of λ consists of the (k,l)-box of λ, all the boxes to the right of it in row k together with all the nodes below it and in column l. The *hook length* $h^\lambda_{(k,l)}$ of the box (k,l) is the number of boxes in the (k,l)-hook.

Definition 5.1. Let n be a positive integer. An *n-core* is a partition λ such that $n \nmid h^\lambda_{(k,l)}$ for all $(k,l) \in \lambda$.

Example 5.2. Figure 2 illustrates the $(2,1)$-hook for the partition $\lambda = (5,2,1,1,1)$. The hooklengths of all of its boxes are listed in Example 5.9, so one can immediately see that λ is a 4-core, a 6-core, a 7-core, an 8-core, and a n-core for $n \geq 10$ and it is not a 1-, 2-, 3-, 5-, or 9-core.

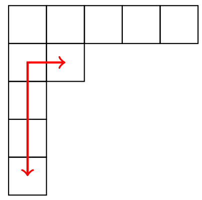


FIGURE 2. The hook length $h^{(5,2,1,1,1)}_{(2,1)}$ is 5.

There is a well-known bijection $\mathcal{C} : \{n\text{-cores}\} \to Q$ described below in Section 5.2, that commutes with the action of $\widehat{\mathfrak{S}}_n$. One can use the $\widehat{\mathfrak{S}}_n$-action to define the bijection, or describe it directly from the combinatorics of partitions via the work of Garvan-Kim-Stanton's $\vec{n}$-vectors [9] or as described in terms of balanced abaci as in [7].

Here, we will recall the description from [7] as well as remind the reader of the $\widehat{\mathfrak{S}}_n$-action on n-cores.

Definition 5.3. We say the (k, l)-box in the partition λ has *residue* equal to $l - k \bmod n$, and letting $i = l - k \bmod n$, we refer to it as a i-box.

Note that the definition of residue depends on n, but by convention and as n is fixed throughout the paper, we make that dependence implicit.

We say a box is *removable* from λ if its removal results in a partition. Equivalently its hook length is 1. A box not in λ is *addable* if its union with λ results in a partition.

Lemma 5.4. Let λ be an n-core. Suppose λ has a removable i-box. Then it has no addable i-boxes. Likewise, if λ has an addable i-box it has no removable i-boxes.

Proof. If λ had both a removable i-box (x, y) and an addable i-box (X, Y), then λ also contains exactly one of (x, Y) or (X, y) and this box has hook length $|X - x + y - Y|$ which is divisible by n, as $y - x \equiv Y - X \equiv i \bmod n$. $\square$

$\widehat{\mathfrak{S}}_n$ acts transitively on the set of n-cores as follows. Let λ be an n-core. Then

$$s_i\lambda = \begin{cases} \lambda \cup \{\text{all addable } i\text{-boxes}\} & \text{if there is an addable } i\text{-box} \\ \lambda \setminus \{\text{all removable } i\text{-boxes}\} & \text{if there is a removable } i\text{-box} \\ \lambda & \text{otherwise.} \end{cases}$$

By Lemma 5.4 $s_i\lambda$ is well-defined and it is easy to check $s_i\lambda$ is an n-core.

Remark 5.5. In fact $\widehat{\mathfrak{S}}_n$ acts on the set of all partitions, but this action is slightly more complicated to describe (see [18] and Section 11 of [16]), and involves the combinatorics of Kleshchev's "good" boxes. For those readers familiar with the realization of the basic crystal $B(\Lambda_0)$ of $\widehat{\mathfrak{s}l}_n$ as having nodes parameterized by n-regular partitions,

$$s_i\lambda = \begin{cases} \tilde{f}_i^{\langle h_i, \mathrm{wt}(\lambda)\rangle}(\lambda) & \langle h_i, \mathrm{wt}(\lambda)\rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, \mathrm{wt}(\lambda)\rangle}(\lambda) & \langle h_i, \mathrm{wt}(\lambda)\rangle \leq 0, \end{cases}$$

where

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{(x,y)\in\lambda} \alpha_{y-x \bmod n}, \tag{5.1}$$

and h_i is the co-root corresponding to α_i. We again refer the reader to [15] for a discussion of the 0-th fundamental weight Λ_0. (Since we need not make a distinction between roots and co-roots in type A, we could have simply substituted α_i for h_i in the expressions above.) Then the n-cores are exactly the $\widehat{\mathfrak{S}}_n$-orbit on the highest weight node, which is the empty partition $\emptyset$.

5.2. **Abacus diagrams.** We can associate to each partition λ its abacus diagram, which we define below. When λ is an n-core, its abacus has a particularly nice form, and can then be used to construct an element $\vec{n}(\lambda)$ of Q. This gives us a bijection $\{n\text{-cores}\} \to Q$ which commutes with the action of $\widehat{\mathfrak{S}}_n$. We follow [7] in describing this bijection, which rests on the work of [14], and we note this bijection agrees with the $\vec{n}$-vector construction of Garvan-Kim-Stanton [9],

Definition 5.6. The *β-numbers* for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ are the hook lengths from the boxes in its first column:

$$\beta_k = h^{\lambda}_{(k,1)}.$$

Each partition is determined by its β-numbers.

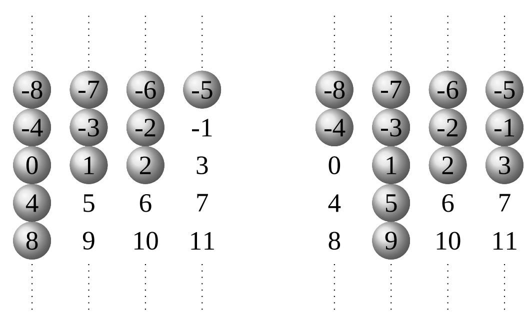
Definition 5.7. An *abacus diagram* is a diagram, with integer entries arranged in n columns labeled $0, 1, \ldots, n-1$, called *runners*. The horizontal cross-sections or rows will be called *levels* and runner k contains the integer entry $rn + k$ on level r where $-\infty < r < \infty$. We draw the abacus so that each runner is vertical, oriented with $-\infty$ at the top and ∞ at the bottom, and we always put runner 0 in the leftmost position, increasing to runner $n-1$ in the rightmost position. Entries in the abacus diagram may be circled; such circled elements are called *beads*. The level of a bead labelled by $rn + k$ is r and its runner is k. Entries which are not circled will be called *gaps*. We shall say two abaci are equivalent if they differ by adding a constant to all entries. (Note, in this case we must cyclically permute the runners so that runner 0 is leftmost.) See Example 5.9 below.

Given a partition λ its abacus is any abacus diagram equivalent to the one obtained by placing beads at entries $\beta_k = h^{\lambda}_{(k,1)}$ and all entries $j \in \mathbb{Z}_{<0}$.

Remark 5.8. It is well-known that λ is an n-core if and only if its abacus is *flush*, that is to say whenever there is a bead at entry j there is also a bead at $j - n$.

We define the *balance number* of an abacus to be the sum over all runners of the largest level of a bead in that runner. We say that an abacus is *balanced* if its balance number is zero. Note that there is a unique abacus which represents a given n-core λ for each balance number. In particular, there is a unique abacus of λ with balance number 0. The balance number for a set of β-numbers of λ will increase by exactly 1 when we increase each β-number by 1. On the abacus picture, this corresponds to shifting all of the beads forward one entry. (Equivalently, we could add 1 to each entry, leaving the beads in place, after which we cyclically permute the runners so that runner 0 is leftmost.)

Example 5.9. Both abaci below represent the 4-core $\lambda = (5, 2, 1, 1, 1)$. The first one is balanced, but the second has balance number 1. The boxes of λ have been filled with their hooklengths.



$\lambda =$

9	5	3	2	1
5	1			
3				
2				
1				

FIGURE 3. Both abaci represent the 4-core λ.

Definition 5.10. Given an n-core λ, we use the flush, balanced abacus associated to λ to define the integer vector $\vec{n}(\lambda)$. The i^{th} component of $\vec{n}(\lambda)$ is the largest level of a bead on runner $i - 1$ of the abacus.

We will call this vector $\vec{n}(\lambda)$, in keeping with the notation of [9]. We note that the sum of the components of $\vec{n}(\lambda)$ is zero, by definition of balanced, so that $\vec{n}(\lambda) \in Q$.

Example 5.11. For the 4-core $\lambda = (5, 2, 1, 1, 1)$ discussed above, $\vec{n}(\lambda) = (2, 0, 0, -2)$, and the vector for the unbalanced abacus is $(-1, 2, 0, 0)$.

We recall the following Lemma, which can be found in [7].

Lemma 5.12. The map $\lambda \mapsto \vec{n}(\lambda)$ is an $\widehat{\mathfrak{S}}_n$-equivariant bijection

$$\{n\text{-cores}\} \to Q.$$

We recall here results of Anderson [1], which describe the abacus of an n-core that is also a t-core, for t relatively prime to n. When $t = mn + 1$, this takes a particularly nice form.

Proposition 5.13 (Anderson)**.** Let λ be an n-core. Suppose t is relatively prime to n. Let $M = nt - n - t$. Consider the grid of points $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with $0 \leq x \leq n - 1$, $0 \leq y$ labelled by $M - xt - yn$. Circle a point in this grid if and only if its label is obtained from the first column hooklengths of λ. Then λ is a t-core if and only if

(1) All beads in the abacus of λ are at entries $\leq M$, in other words at (x, y) with $0 \leq x \leq n - 1$, $0 \leq y$;
(2) The circled points in the grid are upwards flush, in other words if (x, y) is circled, so is $(x, y - 1)$;
(3) The circled points in the grid are flush to the right, in other words if (x, y) is circled and $x \leq n - 2$, so is $(x + 1, y)$.

Note that the columns of this grid are exactly the runners of λ's abacus, written out of order, with each runner shifted up or down relative to its new left neighbor. This shifting is performed exactly so labels in the same row are congruent $\bmod t$. This explains why the circles must be flush to the right as well as upwards flush.

Corollary 5.14. Let λ be an n-core. Then λ is an $(mn+1)$-core if and only if $\langle \vec{n}(\lambda) \mid \alpha_i \rangle \geq -m$ for $0 < i < n$ and $\langle \vec{n}(\lambda) \mid \theta \rangle \leq m + 1$.

Proof. In the notation of Proposition 5.13, in the special case $t = mn + 1$, the columns of the grid are the runners of λ's abacus, written in reverse order. Furthermore, each runner has been shifted m units down relative to its new left neighbor. So the condition of being flush to the right on Anderson's grid is given by requiring on the abacus that if the largest circled entry on runner $i + 1$ is at level r then runner i must have a circled entry at level $r - m$. In other words, if $(a_1, \ldots, a_n) = \vec{n}(\lambda)$, then we require $a_i + m - a_{i+1} \geq 0$, i.e. $\langle \vec{n}(\lambda) \mid \alpha_i \rangle \geq -m$ for $0 < i < n$. Recall the 0^{th} and

$(n-1)^{st}$ and runners must also have this relationship (adding a constant to all entries in the abacus cyclically permutes the runners). This condition becomes $a_n + 1 + m - a_1 \geq 0$, i.e. $\langle \vec{n}(\lambda) \mid \theta \rangle \leq m+1$. $\square$

6. THE BIJECTION BETWEEN CORES AND ALCOVES

In this section, we prove the main result, Theorem 6.1, of the paper. We describe a bijection between the set of partitions that are both n-cores and $(mn+1)$-cores and the connected components of the m-Shi hyperplane arrangement complement in the subspace V of $\mathbb{R}^n$ that lie in the dominant chamber, or more specifically, the dominant m-minimal alcoves. Furthermore, this bijection commutes with the action of $\widehat{\mathfrak{S}}_n$. (We note the minor technicality that the action on cores is a left action, but we take the right action on alcoves when discussing the Shi arrangement.)

In particular, this map is just the restriction of the $\widehat{\mathfrak{S}}_n$-equivariant map

$$\begin{aligned} \{n\text{-cores}\} &\to \{\text{alcoves in the dominant chamber}\} \\ w\emptyset &\mapsto w^{-1}\mathcal{A}_0. \end{aligned}$$

Theorem 6.1. The map $\Phi : w\emptyset \mapsto w^{-1}\mathcal{A}_0$ for w a minimal length left coset representative of $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ induces a bijection between the set of n-cores that are also $(mn+1)$-cores and the set of m-minimal alcoves, which are in the dominant chamber of V.

Proof. Let λ be an n-core and write $\lambda = w\emptyset$ for $w \in \widehat{\mathfrak{S}}_n$ a minimal length left coset representative for $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$. Recall that $\vec{n}(\lambda) = w(0,0,\ldots,0) \in Q$. By Remark 3.4, $w^{-1}\mathcal{A}_0$ is in the dominant chamber. Recall by Corollary 4.3 that in this case $w^{-1}\mathcal{A}_0$ is m-minimal if and only if $\langle \beta \mid \alpha_i \rangle \geq -m$ for $0 < i < n$ and $\langle \beta \mid \theta \rangle \leq m+1$, where $\beta = w(0,\ldots,0) = \vec{n}(\lambda)$.

In Corollary 5.14 above, we have λ is an $(mn+1)$-core iff the conditions above hold for $\beta = \vec{n}(\lambda)$. $\square$

The bijection is pictured in Figure 4.

7. A BIJECTION ON ALCOVES

Although it is not an ingredient in the main theorem of this paper, the following theorem builds on the work of Sections 4 and 3, so is worth including here, but in a separate section. We thank Mark Haiman for pointing it out to us.

Define $\mathfrak{A}_m$ to be the m-dilation of $\mathcal{A}_0$:

$$\mathfrak{A}_m = \{v \in V \mid \langle v \mid \alpha_i \rangle \geq -m,\ \langle v \mid \theta \rangle \leq m+1\}.$$

Note that the set of alcoves in $\mathfrak{A}_m$ is in bijection with $Q/(mn+1)Q$. Furthermore, it is easy to see by translating (by $m\rho = \frac{m}{2}\sum_{\alpha\in\Delta^+}\alpha$) that $Q \cap \mathfrak{A}_m$

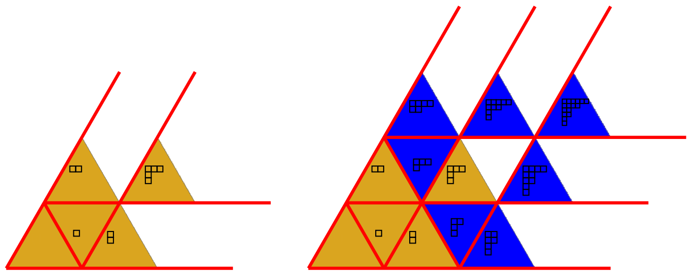


FIGURE 4. m-minimal alcoves $w^{-1}\mathcal{A}_0$ in the dominant chamber of the 1-Shi and 2-Shi arrangements of type A_2 filled with the 3-core partition $w\emptyset$. In the first, they are also 4-cores, and in the second are also 7-cores.

is in bijection with $Q \cap \overline{(mn+1)\mathcal{A}_0}$. It is the latter that is discussed in Lemma 7.4.1 of [10] and studied in [3] (technically for the co-root lattice $Q^\vee$). Taking the latter bijection into account, the second statement of Theorem 7.1 below appears in Theorem 4.2 of [3].

Theorem 7.1. (1) The map $w\mathcal{A}_0 \mapsto w^{-1}\mathcal{A}_0$ restricts to a bijection between alcoves in the region $\mathfrak{A}_m$ and m-minimal alcoves.
(2) The map $w(0,\ldots,0) \mapsto w^{-1}\mathcal{A}_0$ restricts to a bijection between $Q \cap \mathfrak{A}_m$ and m-minimal alcoves in the dominant chamber.

Proof. Observe $\mathfrak{A}_m = {H_{\theta,m+1}}^- \cap \bigcap_{i=1}^{n-1} {H_{\alpha_i,-m}}^+$ can be viewed as an m-dilation of (the closure of) $\mathcal{A}_0 \subseteq {H_{\theta,1}}^- \cap \bigcap_{i=1}^{n-1} {H_{\alpha_i,0}}^+$.

The second statement follows directly from Corollary 4.3.

A proof of the first statement can be given that is very similar to that of Propositions 4.1 and 3.2. Instead we shall use those propositions to prove it.

Write $w^{-1}(\alpha_i) = \alpha - k\delta$ with $\alpha \in \Delta$.

Suppose $k < 0$. Then $\alpha_i \notin \mathrm{Inv}(w^{-1})$ so by Proposition 3.2 $w\mathcal{A}_0 \subseteq {H_{-\alpha_i,0}}^- = {H_{\alpha_i,0}}^+ \subseteq {H_{\alpha_i,-m}}^+$.

Suppose $k \geq 0$. Then the m-minimality of $w^{-1}\mathcal{A}_0$ implies $k \leq m$ when $\alpha > 0$ and $k \leq m-1$ in the case $\alpha < 0$ by Propositions 3.2 and 4.1 and Remark 4.2. When $\alpha > 0$, $w^{-1}(\alpha_i + k\delta) = \alpha$ so $\alpha_i + k\delta \notin \mathrm{Inv}(w^{-1})$. By Proposition 3.2, $w\mathcal{A}_0 \subseteq {H_{-\alpha_i,k}}^- = {H_{\alpha_i,-k}}^+ \subseteq {H_{\alpha_i,-m}}^+$ since $-k \geq -m$. When $\alpha < 0$, $w^{-1}(\alpha_i + (k+1)\delta) = \alpha + \delta > 0$ so $w\mathcal{A}_0 \subseteq {H_{-\alpha_i,k+1}}^- = {H_{\alpha_i,-k-1}}^+ \subseteq {H_{\alpha_i,-m}}^+$ since $-k-1 \geq -m$. Hence $w\mathcal{A}_0 \subseteq \bigcap_{i=1}^{n-1} {H_{\alpha_i,-m}}^+$.

In the case $i = 0$ since $\alpha_i = \delta - \theta$, a similar argument gives $w\mathcal{A}_0 \subseteq {H_{\theta,m+1}}^-$. Hence we get $w\mathcal{A}_0 \subseteq \mathfrak{A}_m$.

For the converse, suppose $w\mathcal{A}_0 \subseteq \mathfrak{A}_m$. Letting $\beta = w(0,0,\ldots,0) \in \mathfrak{A}_m$ we get $\langle\beta \mid \theta\rangle \leq m+1, \langle\beta \mid \alpha_i\rangle \geq -m$, $1 \leq i < n$ because $\mathfrak{A}_m$ is closed. Assume $w^{-1}\mathcal{A}_0$ is not m-minimal. Then $\exists i, \alpha, k$ with $0 \leq i < n, \alpha \in$

$\Delta, k \in \mathbb{Z}$ such that $w^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,k}}^+$ but $w^{-1}s_i\mathcal{A}_0 \subseteq {H_{\alpha,k}}^-$ with $k > m$ if $\alpha > 0$, but $k > m-1$ if $\alpha < 0$. Note by Proposition 4.1 $w^{-1}(\alpha_i) = \alpha - k\delta$ where $-k = \langle \beta \mid \alpha_i \rangle$. First consider $\alpha > 0$. Then $\langle \beta \mid \alpha_i \rangle = -k < -m$ which is a contradiction. Next consider $\alpha < 0$. Then $\alpha_i + k\delta \in \mathrm{Inv}(w^{-1})$ so by Proposition 3.2 $w\mathcal{A}_0 \subseteq {H_{-\alpha_i,k}}^+ = {H_{\alpha_i,-k}}^- \subseteq {H_{\alpha_i,-m}}^-$ since $k \geq m > m-1$, contradicting $w\mathcal{A}_0 \subseteq \mathfrak{A}_m \subseteq {H_{\alpha_i,-m}}^+$. $\square$

The bijection is illustrated in Figures 6, 5, and 7, the first part comparing Figure 6 to Figure 5, and the second part in Figure 7.

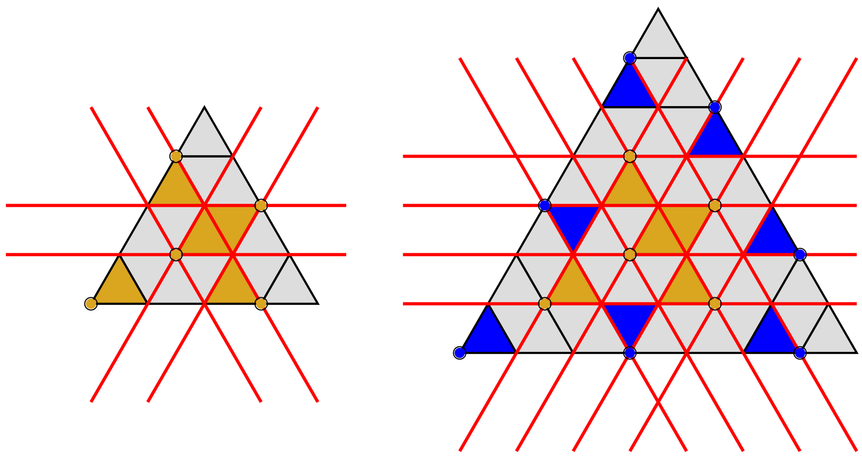


FIGURE 5. $w\mathcal{A}_0$ for the m-minimal alcoves $w^{-1}\mathcal{A}_0$ in Figure 6 below, $m = 1, 2$. Note $w\mathcal{A}_0 \subseteq \mathfrak{A}_m$. Each $\gamma \in Q$ is in precisely one yellow/blue alcove, so this illustrates the second statement of Theorem 7.1.

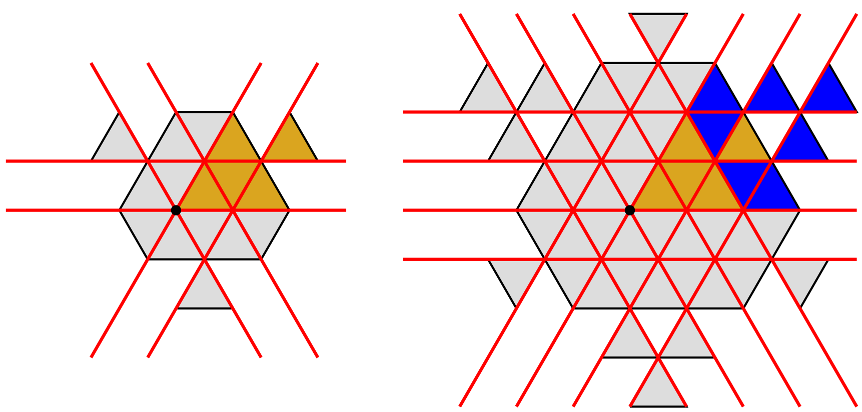


FIGURE 6. m-minimal alcoves in the m-Shi arrangement for $m = 1$ ($m = 2$). Dominant alcoves are shaded yellow (and/or blue, respectively).

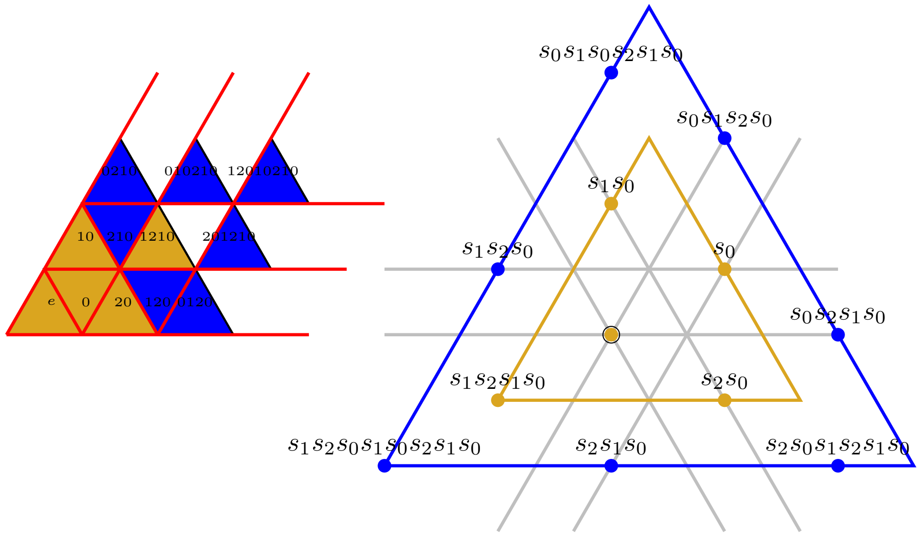


FIGURE 7. w in the alcove $w^{-1}\mathcal{A}_0$; and the corresponding $\beta = w(0, \ldots, 0)$

8. NARAYANA NUMBERS

In this section, we add another set to Athanasiades' list in Theorem 1.2 of [3] of combinatorial objects counted by generalized Narayana numbers. We further refine the enumeration of n-cores λ which are also $(mn+1)$-cores. This refinement produces the m-Narayana numbers, or generalized Narayana numbers, $N_n^m(k)$, which are defined in Definition 8.6 below. Recall Definition 5.3: the (k,l)-box of the n-core λ is referred to as an i-box if it has residue $i = l - k \bmod n$. Our refinement here is to count the number of n-cores λ which are also $(mn+1)$-cores by the number of residues i such that λ has exactly m removable i-boxes.

8.1. **Proposition 4.1 revisited.** Recall Equation (5.1) in Remark 5.5 that a (n-regular) partition has weight

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{(x,y)\in\lambda} \alpha_{(y-x) \bmod n}.$$

It is well-known that $s_i\lambda = \mu$ iff $s_i\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$ where the action of $\widehat{\mathfrak{S}}_n$ on the weight lattice is given by

$$s_i(\gamma) = \gamma - \langle \gamma \mid \alpha_i \rangle \alpha_i.$$

We refer the reader to [15, Chapters 5, 6] for details on the affine weight lattice, definition of Λ_0, and so on. For computational purposes, all we need remind the reader of is that $\langle \Lambda_0 \mid \alpha_i \rangle = \delta_{i,0}$ and $\langle \alpha_0 \mid \alpha_i \rangle = 2\delta_{i,0} - \delta_{i,1} - \delta_{i,n-1}$.

Remark 8.1. Equation (5.1) says that if s_i removes k boxes (of residue i) from λ, or adds $-k$ boxes to λ to obtain μ, then $\mathrm{wt}(\mu) = s_i(\mathrm{wt}(\lambda)) = \mathrm{wt}(\lambda) - k\alpha_i$.

A straightforward rephrasing of Proposition 4.1 is then:

Proposition 8.2. Let λ be an n-core, $k \in \mathbb{Z}_{>0}$, and $w \in \widehat{\mathfrak{S}}_n$ of minimal length such that $w\emptyset = \lambda$. Fix $0 \leq i < n$. The following are equivalent:

(1) λ has k many removable i-boxes; in particular $|s_i\lambda| = |\lambda| - k$ as the action of s_i removes those i-boxes.
(2) $\langle \vec{n}(\lambda) \mid \alpha_i \rangle = -k$ for $i \neq 0$, $\quad \langle \vec{n}(\lambda) \mid \theta \rangle = k + 1$ for $i = 0$,
(3) $w^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,k}}^{+}$, $\quad w^{-1}s_i\mathcal{A}_0 \subseteq {H_{\alpha,k}}^{-}$ where $w^{-1}(\alpha_i) = \alpha - k\delta$.

When we rephrase Corollary 4.3 in this context, it says:

Proposition 8.3 (Corollary 4.3 restated)**.** Suppose $\lambda = w\emptyset$ is the n-core associated to the dominant alcove $\mathcal{A} = w^{-1}\mathcal{A}_0$ via the bijection Φ of Section 6. Then $\mathcal{A}$ is m-minimal if and only if whenever λ has exactly k removable boxes of residue i then $k \leq m$. (And in this case, λ is also an $(mn+1)$-core.)

Example 8.4 (Example 3.3 continued)**.** Let us again consider the 3-core $\lambda = (5, 3, 2, 2, 1, 1)$, where $\lambda = w\emptyset$ for $w = s_1s_2s_0s_1s_2s_1s_0 = s_2t_{(-2,0,2)} = t_{(-2,2,0)}s_2$. In the figure below λ is pictured with each box marked with its residue $\bmod 3$. Let $\mathcal{A} = w^{-1}\mathcal{A}_0$, which is pictured in Figure 1. Proposition 8.2 tells us the 4 removable 1-boxes of λ correspond to the 4 half-spaces $\mathcal{A} \subseteq {H_{\theta,k}}^{+}$ for $1 \leq k \leq 4$, and also that $\langle (-2, 2, 0) \mid \alpha_1 \rangle = -4$.

$$\lambda = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} 0 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

We encode some related information in the following chart. Recall that in place of $\langle \vec{n}(\lambda) \mid \alpha_0 \rangle$ we instead calculate $\langle \vec{n}(\lambda) \mid \theta \rangle$.

α_i	$w^{-1}(\alpha_i)$	$\frac{1}{2}$-space bounded by wall of $w^{-1}\mathcal{A}_0$	i-boxes	$\langle \vec{n}(\lambda) \mid \alpha_i \rangle$
α_0	$-\alpha_1 + 3\delta$	${H_{\alpha_1,3}}^{-}$	3 addable 0-boxes	$-3 + 1$
α_1	$\theta - 4\delta$	${H_{\theta,4}}^{+}$	4 removable 1-boxes	-4
α_2	$-\alpha_2 + 2\delta$	${H_{\alpha_2,2}}^{-}$	2 addable 2-boxes	2

Example 8.5. Consider the 4-core $\mu = (5,2,2,1,1,1) = w\emptyset$ for $w = s_3s_0s_1s_2s_3s_2s_1s_0$. In the figure below μ is pictured with each box marked with its residue $\bmod 4$. Let $\mathcal{A} = w^{-1}\mathcal{A}_0$. Note $w^{-1} = s_3t_{(2,0,-2,0)}$ and $\vec{n}(\mu) = (2,0,-2,0)$. Then

$$\mathcal{A} \subseteq {H_{\alpha_3,1}}^+ \cap {H_{\alpha_2+\alpha_3,2}}^- \cap {H_{\alpha_1,2}}^- \cap {H_{\alpha_1+\alpha_2,2}}^+,$$

which corresponds to μ's 1 removable 0-box, 2 addable 1-boxes, 2 addable 2-boxes, 2 removable 3-boxes. Note $w^{-1}(\alpha_0) = \alpha_3 - \delta$, $w^{-1}(\alpha_1) = -(\alpha_2 + \alpha_3) + 2\delta$, $w^{-1}(\alpha_2) = -\alpha_1 + 2\delta$, $w^{-1}(\alpha_3) = (\alpha_1 + \alpha_2) - 2\delta$.

$$\mu = \begin{array}{|c|c|c|c|c|}
\hline
0 & 1 & 2 & 3 & 0 \\ \hline
3 & 0 \\ \cline{1-2}
2 & 3 \\ \cline{1-2}
1 \\ \cline{1-1}
0 \\ \cline{1-1}
3 \\ \cline{1-1}
\end{array}$$

This concludes Example 8.5.

We could conversely construct λ from the number of removable/addable boxes of each residue, since this data describes $\mathrm{wt}(\lambda)$ and hence determines λ. However, in practice executing the bijection of Section 6, one may as well find w, for instance, as follows. Given an n-core λ, determine $\vec{n}(\lambda)$ as described in Section 5. Let $u \in \mathfrak{S}_n$ be of minimal length such that $u(-\vec{n}(\lambda))$ is dominant. Then $w = u^{-1}t_{\vec{n}(\lambda)}$ and

$$\begin{aligned} w^{-1}\mathcal{A}_0 &= t_{-\vec{n}(\lambda)}u\mathcal{A}_0 \\ &\subseteq {H_{u^{-1}(\theta),1-\langle \vec{n}(\lambda)|\theta\rangle}}^- \cap \bigcap_{i=1}^{n-1} {H_{u^{-1}(\alpha_i),-\langle \vec{n}(\lambda)|\alpha_i\rangle}}^+ \end{aligned}$$

which describes where $w^{-1}\mathcal{A}_0$ is located in V. This process gives a slightly more constructive version of the bijection in Theorem 6.1.

We also note that one can easily read off the coordinates $k(w^{-1},\alpha)$ that Shi uses in [21] from the data above, describing where $w^{-1}\mathcal{A}_0$ is located in V with respect to the (affine) root hyperplanes.

8.2. **A refinement.** Proposition 8.2 thus gives us another combinatorial interpretation of the m-Narayana numbers, as in [3].

Definition 8.6. The k^{th} m-Narayana number of type A is

$$N_n^m(k) = \frac{1}{nm+1}\binom{n-1}{n-k-1}\binom{mn+1}{n-k}.$$

Recall from [3] that $N_n^m(k)$ counts how many dominant regions of the m-Shi arrangement have exactly k hyperplanes $H_{\alpha,m}$ separating them from $\mathcal{A}_0$ such that $H_{\alpha,m}$ contains a wall of the region. The m-Narayana numbers have many other combinatorial interpretations.

In other words, for fixed k, we count how many m-minimal alcoves $\mathcal{A} = w^{-1}\mathcal{A}_0$ satisfy that for exactly k positive roots α, there exists an i such that $w^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,m}}^+$ but $w^{-1}s_i\mathcal{A}_0 \subseteq {H_{\alpha,m}}^-$. It is clear that

$$\sum_{k\geq 0} N_n^m(k) = m\text{-Catalan number}$$

since each dominant m-minimal alcove gets counted once.

By Proposition 8.2 above, $N_n^m(k)$ equivalently counts how many n-cores λ that are also $(mn+1)$-cores have exactly k distinct residues i such that λ has precisely m removable i-boxes. See Example 8.8 below.

Corollary 8.7. Let $N_n^m(k)$ denote the m-Narayana number of type A_{n-1}. Then

$$\begin{aligned} N_n^m(k) = |\{\lambda \mid \lambda \text{ is an } n\text{-core and } (mn+1)\text{-core and } \exists K \subseteq \mathbb{Z}/n\mathbb{Z} \\ \text{with } |K| = k \text{ such that } \lambda \text{ has exactly } m \text{ removable boxes} \\ \text{of residue } i \text{ iff } i \in K\}|. \end{aligned}$$

Example 8.8. For $n = 3$, $m = 2$, the m-Catalan number is $12 = 5 + 6 + 1$.

$$\begin{aligned} N_3^2(0) =& 5 = |\{\emptyset, (1), (2), (1,1), (3,1,1)\}| \\ N_3^2(1) =& 6 = |\{(3,1), (2,1,1), (2,2,1,1), (4,2), (5,3,1,1), (4,2,2,1,1)\}| \\ N_3^2(2) =& 1 = |\{(6,4,2,2,1,1)\}| \end{aligned}$$

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