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STRONG MULTIPLICITY ONE THEOREMS FOR AFFINE HECKE ALGEBRAS OF TYPE A

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Abstract. Given an irreducible module for the affine Hecke algebra H_n of type A, we consider its restriction to H_{n-1}. We prove that the socle of restriction is multiplicity free and moreover that the summands lie in distinct blocks. This is true regardless of the characteristic of the field or of the order of the parameter q in the definition of H_n. The result generalizes and implies the classical "branching rules" that describe the restriction of an irreducible representation of the symmetric group S_n to S_{n-1}.

1. Introduction

This paper is concerned with understanding a generalization of the "branching rules" that describe the restriction of an irreducible representation of the symmetric group S_n to S_{n-1}.

The classical result is that when the module is in characteristic zero, this restriction is multiplicity free, and one can nicely label the summands. This labeling is essentially an edge in the lattice of partitions, and is a key link between the representation theory of the symmetric group and the combinatorics of partitions.

Now suppose that the module is an irreducible representation of the symmetric group in characteristic p. Then it is no longer true that the restricted module is multiplicity free; however Kleshchev in [Kv] showed that the socle of the restricted module is multiplicity free. It is thus a "socle of restriction" – which coincides with "restriction" in characteristic zero – that links modular representation theory with the combinatorics of p-regular partitions.

In this paper we show that this phenomenon is a special case of a more general result, and in particular we give a different proof of both the characteristic zero and characteristic p branching theorems.

Our results include the representations of the complex reflection groups $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ for any r, and of their "q-deformations", the algebras defined by Ariki and Koike in [AK].

We show that if M is an irreducible module for any of the above groups, in any characteristic, or more generally, an irreducible module for the Ariki–Koike algebra, then $\operatorname{Res} M$ has multiplicity free socle.

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The crucial observation is that all of these algebras are quotients or degenerations of a single algebra, the *affine* Hecke algebra, and it is this algebra that is responsible for the multiplicity one phenomena.

This is the first step in producing a uniform parameterization of irreducible modules for each of the above algebras, generalizing the classical description for the symmetric group. The full parameterization can be found in [G], which is a sequel to this paper, and which explains further the underlying rigid structure that is exploited here. What we do not do in this paper is to explicitly describe which irreducible representations occur in the socle of restriction. This is done in [G] using the rigid structure, generalizing [Kv], which describes the combinatorics of the branching rule for the symmetric group explicitly in terms of p-regular partitions.

The following theorems are the main results of this paper. Write H_n for the affine Hecke algebra (defined in the body of the paper), and Res^n_{n-1} for the restriction functor from H_n to H_{n-1}.

Theorem (A). *Let M be an irreducible H_n-module over an algebraically closed field R. Then the socle of $\mathrm{Res}^n_{n-1} M$ is multiplicity free, and is isomorphic to the cosocle of $\mathrm{Res}^n_{n-1} M$.*

This theorem immediately implies all of the results described above for the "q-deformations" of the group algebras. The analogue of Theorem (A) for the degenerate affine Hecke algebra implies the results for $q = 1$, for example, for the symmetric group. See Appendix A.

In order to prove this theorem, we introduce a refinement of the restriction functor, first introduced in the case of the symmetric group by G. Robinson, and introduced for the Hecke algebras by Ariki.

Namely, as $H_{n-1} \otimes H_1$ is a subalgebra of H_n, we can decompose $\mathrm{Res}^n_{n-1} M$ as an H_1-module. The indecomposable summands are labeled by elements $a \in R^\times$, and we denote the resulting functor on H_n-modules by e_a (see Section 3.1 for the precise definition). Then Theorem (A) is an immediate consequence of

Theorem (B). *Let M be an irreducible H_n-module over an algebraically closed field R, and $a \in R^\times$. Then $e_a M$ is zero or has an irreducible socle, and $\mathrm{cosoc}\, e_a M$ is isomorphic to $\mathrm{soc}\, e_a M$.*

This is the "strong" multiplicity one theorem of the title.

2. Preliminaries

In this section we collect well-known definitions and elementary properties of the affine Hecke algebra and its finite dimensional modules.

2.1. Definitions

Throughout the paper we fix an algebraically closed field R, and an invertible element $q \in R$, i.e., a homomorphism $\mathbb{Z}[q, q^{-1}] \to R$.

We further *assume that* $q \neq 1$. This is for clarity of exposition; the changes in the statements of definitions and theorems that must be made when $q = 1$ are sketched in the Appendix, which is taken verbatim from [G].

If A is an R-algebra, we write $\operatorname{Rep} A$ for the category of left A-modules which are *finite dimensional* as R-modules. We recall that the socle of a module M, denoted $\operatorname{soc}(M)$, is the largest semisimple submodule of M, and that the cosocle of M, denoted $\operatorname{cosoc}(M)$, is its largest semisimple quotient. We also write $Z(A)$ for the center of the algebra A.

Recall that if A and A' are two R-algebras, the *irreducible* representations of $A \otimes A'$ are of the form $M \boxtimes M'$, where M is an irreducible A-module and M' is an irreducible A'-module. (This is certainly *not* true for general representations of $A \otimes A'$.) Throughout the paper, $M \boxtimes M'$ denotes the $A \otimes A'$-module which is isomorphic to $M \otimes_R M'$ as an R-module and has an $A \otimes A'$ action given by $(a \otimes a') \cdot (m \otimes m') = am \otimes a'm'$.

We write S_n for the symmetric group on n letters, $\tau_i = (i \quad i+1)$, $1 \leq i \leq n-1$, for the simple transpositions. Denote the length by $\ell(w)$ and the Bruhat order by $\leq$.

If $P = (n_1, \dots, n_k)$ is an *ordered* tuple, $n_i \in \mathbb{Z}_{>0}, \sum n_i = n$, it is convenient to denote $S_P = S_{n_1} \times \cdots \times S_{n_k} \subseteq S_n$. For example, $S_{(n)} = S_n$, $S_{(1,\dots,1)} = \{1\}$. For such P, we write $W^P_{\min}$ for the set of minimal length left coset representatives of $S_P \subseteq S_n$,

2.2. H^{fin}_n, H_n and H^λ_n

The *finite Hecke algebra*, H^{fin}_n is the R-algebra with generators $T_1, \dots, T_{n-1}$ and relations

$$\textit{braid relations} \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i, \ |i-j| > 1, \tag{1}$$

$$\textit{quadratic relations} \quad (T_i + 1)(T_i - q) = 0. \tag{2}$$

The braid relations imply that if $w = \tau_{i_1} \cdots \tau_{i_k}$ and $\ell(w) = k$, then $T_{i_1} \cdots T_{i_k}$ depends only on $w \in S_n$. It is denoted T_w, and the $\{T_w \mid w \in S_n\}$ form a basis of H^{fin}_n over R.

The *affine Hecke algebra* [BZ] H_n (or H^{aff}_n) is the R-algebra, which, as an R-module, is isomorphic to $H^{\text{fin}}_n \otimes_R R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. The algebra structure is given by requiring that H^{fin}_n and $R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ are subalgebras, and that

$$T_iX_iT_i = qX_{i+1}. \tag{3}$$

Equivalently, if $f \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, then

$$T_if - {}^{\tau_i}fT_i = (q-1)\frac{f - {}^{\tau_i}f}{1 - X_iX_{i+1}^{-1}} \tag{4}$$

where $\tau_i \in S_n$ acts on $R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ by permuting X_i and X_{i+1}.

The *cyclotomic Hecke algebra* or *Ariki–Koike algebra* [AK] H^λ_n is the quotient H_n/I_λ, where I_λ is the ideal generated by the polynomial in X_1, $\prod_{i=1}^r (X_1 - \lambda_i)$, and $\lambda = (\lambda_1, \dots, \lambda_r)$ is an r-tuple $\in (R^\times)^r$.

The surjective algebra homomorphism $\operatorname{ev}_\lambda : H_n \to H_n/I_\lambda$ (first discovered by Cherednik [Ch]) induces a functor $\operatorname{ev}^*_\lambda : \operatorname{Rep} H^\lambda_n \to \operatorname{Rep} H_n$.

We warn the reader that given an exact sequence of H_n-modules $0 \to A \to B \to C \to 0$, where A and C are H^λ_n-modules, B need not be an H^λ_n-module.

Remark. Observe that any irreducible H_n-module (being finite-dimensional by Proposition 2.2 below) is an irreducible H^λ_n-module if we take r, λ large enough. Just let $\prod_{i=1}^r (X_1 - \lambda_i)$ be the characteristic polynomial of X_1 acting on the module. Conversely, any irreducible H^λ_n-module is an irreducible H_n-module (on which I_λ acts as 0).

The statements and proofs throughout this paper will be made for $\operatorname{Rep} H_n$. However, by the remark above, all results on irreducible modules apply to $\operatorname{Rep} H^\lambda_n$ as well.

2.3. Bernstein's center

Proposition 2.1. (Bernstein) *The center of H_n, $Z(H_n)$, is isomorphic to symmetric Laurent polynomials: $Z(H_n) \simeq R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$.*

Proof. The relation (4) makes it clear that $R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$ is contained in the center. Suppose $h = \sum f_w T_w \in Z(H_n)$ where $f_w \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Let $u \in S_n$ be maximal with respect to Bruhat order such that $f_u \neq 0$. If $u \neq 1$, then there exists some i such that $u(i) \neq i$. As $T_w X_i = X_{w(i)} T_w + \sum_{w' < w} g_{w'} T_{w'}$, for some $g_{w'} \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, comparing leading terms of $X_i h$ and $h X_i$ we reach a contradiction: the coefficient of T_u in $X_i h - h X_i$ is $f_u(X_i - X_{u(i)}) \neq 0$. Hence $h = f_1 \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Because h commutes with all the T_i, by the relation (4), we see $h \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$. $\square$

Proposition 2.2. *If M is an irreducible H_n-module, then M is finite dimensional with* $\dim_R M \leq n!$.

Proof. As $R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ is a free $R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$-module of rank $n!$, H_n is a free module over $Z(H_n)$ of rank $(n!)^2$. Dixmier's version of Schur's lemma implies that the center of H_n acts by scalars on absolutely irreducible modules, and hence M is an irreducible module for a finite dimensional algebra of dimension $(n!)^2$. $\square$

2.4. Characters

Write $S = R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ and let M be a finite dimensional H_n-module. Define the generalized S-eigenspace

$$M[s] = \{v \in M \mid (X_i - s_i)^m v = 0,\ m \gg 0,\ \text{for all } 1 \leq i \leq n\}$$

where $s = (s_1, \dots, s_n) \in R^n$.

Because the X_i commute, we can decompose M into a direct sum of generalized eigenspaces $\operatorname{Res}_S^{H_n} M = \bigoplus_{s \in (R^\times)^n} M[s]$.

We define the character of M to be the formal sum

$$\operatorname{ch} M = \sum_{s \in (R^\times)^n} (\dim M[s]) s.$$

This is just the class of $\operatorname{Res}_S M$ in the Grothendieck group of finite dimensional representations of S. Here we identify the irreducible representations of S with the elements of $(R^\times)^n$.

As $S = R[X_1^{\pm 1}, \dots, X_n^{\pm 1}] = R[X_1^{\pm 1}] \otimes_R \cdots \otimes_R R[X_n^{\pm 1}]$, we will also write $s = s_1 \boxtimes s_2 \boxtimes \cdots \boxtimes s_n$ for the 1-dimensional S-module with character $s = (s_1, \dots, s_n)$.

Write $M_s = \{v \in M \mid X_i v = s_i v, 1 \leq i \leq n\}$ for the *actual* simultaneous eigenspace of the elements X_i. Then $M_s \neq 0 \iff M[s] \neq 0$.

Lemma 2.3. *If M is indecomposable and $M[s] \neq 0$, $M[t] \neq 0$, then there exists $w \in S_n$ such that $w \cdot s = (s_{w^{-1}(1)}, \dots, s_{w^{-1}(n)}) = t$.*

Proof. Any module for H_n decomposes into a direct sum of generalized eigenspaces for $Z(H_n)$. Because M is indecomposable, if $f = f(X_1, \dots, X_n) \in Z(H_n) = R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$, then f acts with the same eigenvalue on all of M (perhaps not semisimply). Hence, if $v \in M_s$ is nonzero, then $fv = f(s_1, \dots, s_n) v$ and likewise $fu = f(t_1, \dots, t_n) u$ for $0 \neq u \in M_t$. Thus $f(s_1, \dots, s_n) = f(t_1, \dots, t_n)$ for any $f \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$ yielding s and t are in the same S_n-orbit. $\square$

Let $s \in (R^\times)^n$ and let $S_n \cdot s$ be the orbit of S_n on s. If M is a module with the property that $M[t] \neq 0$ only if $t \in S_n \cdot s$, we say that M has *central character* $S_n \cdot s$. So the lemma shows that every indecomposable module has a central character. If M is irreducible, then Dixmier's Schur lemma shows that $Z(H_n)$ acts by scalars.

2.5. Induction and restriction

Recall that if $A \subset B$ are R-algebras such that B is finitely generated both as a left and right A-module, the exact functor of restriction $\operatorname{Res}_A^B\colon \operatorname{Rep} B \to \operatorname{Rep} A$ has left and right adjoints, Ind and $\widehat{\operatorname{Ind}}$ defined by $\operatorname{Ind}_A^B\colon \operatorname{Rep} A \to \operatorname{Rep} B$, $M \mapsto B \otimes_A M$, and $\widehat{\operatorname{Ind}}_A^B\colon \operatorname{Rep} A \to \operatorname{Rep} B$, $M \mapsto \operatorname{Hom}_A(B, M)$; i.e., $\operatorname{Hom}_B(\operatorname{Ind} M, N) = \operatorname{Hom}_A(M, \operatorname{Res} N)$, $\operatorname{Hom}_B(N, \widehat{\operatorname{Ind}} M) = \operatorname{Hom}_A(\operatorname{Res} N, M)$. We will also refer to the adjointness of Ind (respectively $\widehat{\operatorname{Ind}}$) and Res as Frobenius reciprocity. Explicit isomorphisms are given by the maps $\alpha \mapsto (v \mapsto \alpha(1 \otimes v))$, $\beta \mapsto (w \mapsto \beta(w)(1))$. We will repeatedly make use of this when discussing maps $\operatorname{Ind}_A^B M \xrightarrow{\alpha} N$.

If B is a free A-module, then Ind and $\widehat{\operatorname{Ind}}$ are also exact functors. Further, if $A \subset B \subset C$ are inclusions of R-algebras, we have transitivity of induction and restriction: $\operatorname{Res}_A^B \operatorname{Res}_B^C = \operatorname{Res}_A^C$, $\operatorname{Ind}_B^C \operatorname{Ind}_A^B = \operatorname{Ind}_A^C$, $\widehat{\operatorname{Ind}}_B^C \widehat{\operatorname{Ind}}_A^B = \widehat{\operatorname{Ind}}_A^C$.

Now apply these remarks to the affine Hecke algebra. Given a sequence $P = (n_1, \dots, n_k)$ of nonnegative integers summing to n, write $H_P = H_{n_1} \otimes \cdots \otimes H_{n_k}$. We have an obvious embedding $H_P = H_{n_1} \otimes \cdots \otimes H_{n_k} \hookrightarrow H_n$, which makes H_n a free H_P-module. Applying the previous remarks we get exact functors $\operatorname{Res}, \operatorname{Ind}, \widehat{\operatorname{Ind}}$. When unambiguous from context, we write Res_P^n or Res_{H_P} for $\operatorname{Res}_{H_P}^{H_n}$, and similarly for Ind to lighten notation. These functors depend on the order $(n_1, \dots, n_k)$ and not just on the underlying set!

2.6. Shuffle lemma

The following results are a special case of the "Mackey formula" relating the composite of induction from H_P to H_n with the restriction from H_n to $H_{P'}$ for various P and P'. In particular, in Lemma 2.4, $P = (1, 1, \dots, 1)$ so $H_P = S$.

Lemma 2.4. *Let $s = (s_1, \dots, s_n) \in (R^\times)^n$. Then $\operatorname{ch}(\operatorname{Ind}_S^{H_n} s_1 \boxtimes \cdots \boxtimes s_n) = \sum_{w \in S_n} w \cdot s$.*

Proof. First, $\operatorname{Ind}_S^{H_n} s = H_n \otimes_S s$, which as an R-module is $H_n^{\mathrm{fin}} \otimes_R s$, and so $\dim \operatorname{Ind}_S^{H_n} s = \dim H_n^{\mathrm{fin}} = n!$. Further, if we write $Rv = s = s_1 \boxtimes \cdots \boxtimes s_n$, then a basis of $\operatorname{Ind}_S^{H_n} s$ is given by $\{T_w \otimes v \mid w \in S_n\}$. If we order the basis by Bruhat order, then by the relation (4) the action of all the X_i is upper triangular, and $X_i(T_w \otimes v) = s_{w^{-1}(i)}(T_w \otimes v) + \sum_{y<w} a_y (T_y \otimes v)$, for some coefficients $a_y \in R$. Hence the lemma. $\square$

Lemma 2.5. (1) $X_1\left(T_1 \left(\frac{1 - X_1/X_2}{1-q}\right) + 1\right) = \left(T_1 \left(\frac{1 - X_1/X_2}{1-q}\right) + 1\right) X_2$.

(2) *There is an H_2-map $\operatorname{Ind}_S^{H_2} N \xrightarrow{\tau} \operatorname{Ind}_S^{H_2} \mathrm{rev}^* N$ given by $1 \otimes u \mapsto \left(T_1 \left(\frac{1-X_1/X_2}{1-q}\right) + 1\right) \otimes u$, where for any S-module N, $rev^* N$ denotes the module with underlying set N and S-action induced by the automorphism $rev\colon S \to S$, $X_1 \mapsto X_2$, $X_2 \mapsto X_1$.*

(3) *The map τ^2: $\operatorname{Ind}_S^{H_2} s_1 \boxtimes s_2 \to \operatorname{Ind}_S^{H_2} s_1 \boxtimes s_2$ is multiplication by the scalar $(1 - q s_1/s_2)(1 - q s_2/s_1)/(1-q)^2$.*

(4) *When* $s_2 \notin \{qs_1, q^{-1}s_1\}$, τ *is an isomorphism. When* $s_2 = q^{\pm 1}s_1$, *the sequence* $\dots \xrightarrow{\tau} \operatorname{Ind}_S^{H_2} s_1 \boxtimes s_2 \xrightarrow{\tau} \operatorname{Ind}_S^{H_2} s_2 \boxtimes s_1 \xrightarrow{\tau} \operatorname{Ind}_S^{H_2} s_1 \boxtimes s_2 \xrightarrow{\tau} \dots$ *is exact, and so* $\operatorname{Ind}_S^{H_2} s_1 \boxtimes s_2$ *and* $\operatorname{Ind}_S^{H_2} s_2 \boxtimes s_1$ *have the same composition factors.*

Proof. The proof of (1) is a straightforward computation, and (2) follows from (1).

One may compute $(T_1\left(\frac{1-X_1/X_2}{1-q}\right)+1)^2 = \left(\frac{1-qX_1/X_2}{1-q}\right)\left(\frac{1-qX_2/X_1}{1-q}\right)$, which in particular is central in H_2, yielding (3) when $N = s_1 \boxtimes s_2$. For (4), because τ^2 is multiplication by a nonzero scalar when $s_2 \notin \{qs_1, q^{-1}s_1\}$, τ itself must be an isomorphism. (Observe that when $s_2 = s_1$, τ is just the identity.)

In the case $s_2 = qs_1$, write Rv and Rw for $s_1 \boxtimes s_2$ and $s_2 \boxtimes s_1$, respectively. Then $\operatorname{Ind}_S^{H_2} s_1 \boxtimes qs_1$ has a submodule $R(T_1 - q) \otimes v$ with character (qs_1, s_1), which is the image of one application of τ and the kernel of the next application. The quotient is the module $R(T_1+1) \otimes w$, which is a submodule of $\operatorname{Ind}_S^{H_2} qs_1 \boxtimes s_1$ and has the character (s_1, qs_1).

The case $s_2 = q^{-1}s_1$ is analogous. □

Corollary 2.6. *All irreducible modules with central character* $S_n \cdot s = S_n \cdot (s_1, \dots, s_n)$ *occur as composition factors of* $\operatorname{Ind}_S^{H_n} s_1 \boxtimes \cdots \boxtimes s_n$. *In particular, all of the modules* $\operatorname{Ind}_S^{H_n} w \cdot s$ *for* $w \in S_n$ *have the same composition factors.*

Proof. First we prove the second statement. We can extend the map of Lemma 2.5 to $\operatorname{Ind}_S^{H_n} s_1 \boxtimes \cdots \boxtimes s_i \boxtimes s_{i+1} \boxtimes \cdots \boxtimes s_n \xrightarrow{\tau} \operatorname{Ind}_S^{H_n} s_1 \boxtimes \cdots \boxtimes s_{i+1} \boxtimes s_i \boxtimes \cdots \boxtimes s_n$. By transitivity of induction and Lemma 2.5 statement (4), τ is either an isomorphism or gives an exact sequence as in (4); in either case, the above induced modules have the same composition factors. As the simple transpositions generate S_n, we see all the $\operatorname{Ind}_S^{H_n} w \cdot s$ have the same composition factors.

If M is irreducible with central character $S_n \cdot s$, then $M[w \cdot s] \neq 0$ for some $w \in S_n$. Hence by Frobenius reciprocity there is a surjection $\operatorname{Ind}_S^{H_n} w \cdot s \to M$, and so M is a composition factor of $\operatorname{Ind}_S^{H_n} s$. □

Note that we have just shown that any irreducible H_n-module occurs in the cosocle of $\operatorname{Ind}_S^{H_n} s$ for appropriate s.

We now describe the character of $\operatorname{Ind}_P^n M$ in terms of $\operatorname{ch} M$. We say a string $s = s_1 s_2 \cdots s_k$ is a shuffle of t and u if t is a subword of s and u is its complementary subword. The shuffle of t and u, denoted $t \sqcup\!\sqcup u$, is then the formal sum of all shuffles of t and u, with multiplicity.

The permutations of $\{1, 2, \dots, n+m\}$ that keep $1, \dots, m$ in order and $m+1, \dots, m+n$ in order are exactly the minimal length left coset representatives $W_{\min}^{(m,n)}$. It follows that if we write tu for the concatenation of t and u, which have length m and n, respectively, then $t \sqcup\!\sqcup u = \sum_{w \in W_{\min}^{(m,n)}} w \cdot tu$. For example, $1q \sqcup\!\sqcup qq^2 = 1qqq^2 + 1qqq^2 + q1qq^2 + 1qq^2q + q1q^2q + qq^21q$ $1 \sqcup\!\sqcup 23 = 123 + 213 + 231 = (1 + \tau_1 + \tau_2\tau_1) \cdot 123$.

We extend $\sqcup\!\sqcup$ linearly to sums of words.

Lemma 2.7. *Let* $M \in \operatorname{Rep} H_m$, $N \in \operatorname{Rep} H_n$. *Then*

$$\operatorname{ch} \operatorname{Ind}_{(m,n)}^{m+n} M \boxtimes N = \sum_{\substack{s \text{ is a shuffle} \\ \text{of } s' \text{ and } s''}} (\dim M[s'] \dim N[s''])s = \operatorname{ch} M \sqcup\!\sqcup \operatorname{ch} N.$$

Proof. Choose $\{u_j\}$ and $\{v_k\}$ bases of M and N on which S acts upper triangularly. Then $\mathrm{Ind}_{(m,n)}^{m+n} M \boxtimes N$ has basis $\{T_w \otimes (u_j \otimes v_k) \mid w \in W_{\min}^{(m,n)}, j, k\}$, and the action of S on this basis is upper triangular with eigenvalues as stated. □

3. Strong multiplicity one result

This section contains the proofs of Theorems (A) and (B) of the introduction. We begin by defining the functors e_a, $a \in R^\times$, which refine restriction. Throughout this section, we let M be a finite dimensional H_n-module.

3.1. Definitions

First let $\Delta_{\mathrm{a}} M$ denote the generalized eigenspace of $X_n - a$ in $\mathrm{Res}_{H_{n-1}\otimes H_1}^{H_n} M$. As $X_n - a \subseteq Z(H_{n-1} \otimes H_1)$, left multiplication by $(X_n - a)^m$ induces an endomorphism of $\mathrm{Res}_{H_{n-1}\otimes H_1}^{H_n} M$ and $\Delta_{\mathrm{a}} M$ is the kernel of this map for $m \gg 0$. We can take $m \leq \dim M$. In particular, $\Delta_{\mathrm{a}} M$ is an $H_{n-1} \otimes H_1$-module. As restriction and taking generalized eigenspaces are exact functors, Δ_{a} is exact. Furthermore $\mathrm{Res}_{H_{n-1}\otimes H_1}^{H_n} M = \bigoplus_{a\in R^\times} \Delta_{\mathrm{a}} M$, as X_n acts on M with eigenvalues in $R^\times$. If $s = S_n \cdot (s_1, s_2, \ldots, s_n)$, write $s + a = S_{n+1} \cdot (s_1, \ldots, s_n, a)$, and if $s_j = a$, we may assume it is s_n and write $s \setminus a = S_{n-1} \cdot (s_1, \ldots, s_{n-1})$. If M is finite dimensional and indecomposable with central character s, then for $m \gg 0$, $(\sum_1^n X_j - s_j)^m M = 0$. If $\Delta_{\mathrm{a}} M \neq 0$, the fact that $\sum_1^n (X_j - s_j) = (X_n - a) + \sum_1^{n-1}(X_j - s_j)$ yields $\ker(X_n - a)^m = \ker(\sum_1^{n-1}(X_j - s_j))^m$ for $m \gg 0$. Thus, $\Delta_{\mathrm{a}} M$ is also the generalized eigenspace of $\sum_1^{n-1}(X_j - s_j) \in Z(H_{n-1})$.

Define

$$e_a M = \mathrm{Res}_{H_{n-1}}^{H_{n-1}\otimes H_1}(\Delta_{\mathrm{a}} M).$$

Alternatively, $e_a M$ is the generalized eigenspace of $\sum_1^{n-1}(X_j - s_j)$ acting on $\mathrm{Res}_{H_{n-1}}^{H_n} M$. The above discussion makes the following claim clear.

Claim 3.1. *e_a is an exact functor $\mathrm{Rep}\, H_n \to \mathrm{Rep}\, H_{n-1}$ and $\mathrm{Res}_{H_{n-1}}^{H_n} M = \bigoplus_{a\in R^\times} e_a M$.*

Notice that if M has central character s, $e_a M$ has central character $s \setminus a$, and if $a \neq b$, then $s \setminus a$ is distinct from $s \setminus b$. If M, N are irreducible, then

$$N \subseteq e_a M \iff N \boxtimes a \subseteq \mathrm{Res}_{H_{n-1}\otimes H_1}^{H_n} M, \tag{5}$$

which fails if N is reducible.

Define

$$\varepsilon_a(M) = \max\{m \geq 0 \mid e_a^m M \neq 0\}.$$

We also define

$$\widetilde{e}_a M = \mathrm{soc}\, e_a M.$$

Then (5) implies that for M, N irreducible $N \subseteq \widetilde{e}_a^k M \iff N \boxtimes a \boxtimes \cdots \boxtimes a \subseteq \mathrm{Res}_{n-k,1,1,\ldots,1}^n M$. Later we shall also see that $\widetilde{e}_a M$ is isomorphic to $\mathrm{cosoc}\, e_a M$.

We remark that $\widetilde{e}_a M = 0 \iff e_a M = 0 \iff \varepsilon_a(M) = 0$. The major result of this section is that if M is irreducible, then $\widetilde{e}_a M$ is either zero or also irreducible.

3.2. Preliminary results

We will need the following two propositions in the proof of the main theorem. The first is another special case of the Mackey formula (the general case can be found in [G], and in many other places).

Proposition 3.2. *Let $M \in \operatorname{Rep} H_m$, $K \in \operatorname{Rep} H_k$. Then the following are exact sequences of $H_{n-1} \otimes H_1$-modules, where $n = m + k$ and the twist by y is defined below:*

$$0 \to \operatorname{Ind}_{m,k-1,1}^{n-1,1} M \boxtimes \operatorname{Res}_{k-1,1}^{k} K \xrightarrow{\alpha} \operatorname{Res}_{n-1,1}^{n} \left(\operatorname{Ind}_{m,k}^{n} M \boxtimes K\right)$$
$$\xrightarrow{\beta} \operatorname{Ind}_{m-1,k,1}^{n-1,1} {}^{y}\left((\operatorname{Res}_{m-1,1}^{m} M) \boxtimes K\right) \to 0,$$
$$0 \to \operatorname{Ind}_{m,k-1,1}^{n-1,1} M \boxtimes \Delta_{\mathrm{a}} K \to \Delta_{\mathrm{a}} \left(\operatorname{Ind}_{m,k}^{n} M \boxtimes K\right) \to \operatorname{Ind}_{m-1,k,1}^{n-1,1} {}^{y}(\Delta_{\mathrm{a}} M \boxtimes K) \to 0.$$

Proof. We describe the maps α and β on a basis, and leave to the reader to check that these define $H_{n-1} \otimes H_1$-module maps, and that the sequence is exact.

Let $y = \tau_{n-1}\tau_{n-2} \cdots \tau_m$. If N is an $H_{m-1} \otimes H_1 \otimes H_k$-module, let ${}^{y}N$ denote the $H_{m-1} \otimes H_k \otimes H_1$-module $\sigma_y^*(N)$, where σ_y is the algebra isomorphism $H_{m-1} \otimes H_k \otimes H_1 \xrightarrow{\sigma_y} H_{m-1} \otimes H_1 \otimes H_k$ given by $\sigma_y(T_w) = T_{y^{-1}wy}$, $\sigma_y(X_j) = X_{y^{-1}(j)}$. For $u \in N$, let $\overline{u}$ denote the corresponding element in ${}^{y}N$.

Observe that $\{1, y\}$ are minimal length double coset representatives of $S_{n-1} \backslash S_n / S_m \times S_k$. This is because the shuffles of $12 \cdots m$ and $m+1 \cdots n$ consist of those fixing n, namely $W_{(m-1,k)}$, and those that do not and hence send m to the n^{th} position; this gives us a decomposition $W_{\min}^{(m,k)} = \{xy \mid x \in W_{\min}^{(m-1,k)}\} \bigcup W_{\min}^{(m,k-1)}$.

For $u \in M, v \in K$, $\alpha(T_w \otimes (u \otimes v)) = T_w \otimes (u \otimes v)$, where $w \in W_{\min}^{(m,k-1)}$. We define $\beta(T_w \otimes (u \otimes v)) = 0$ and $\beta(T_x T_y \otimes (u \otimes v)) = T_x \otimes (\overline{u \otimes v})$ for $x \in W_{(m-1,k)}$.

The second exact sequence is obtained from the first by taking the generalized eigenspace of X_n with eigenvalue a. $\square$

We remark that the following are exact sequences of H_{n-1}-modules, via the same maps, except that we need not twist by y if we view $H_{m-1} \otimes H_k$ as a subalgebra of H_{n-1} (as opposed to a subalgebra of $H_{m-1} \otimes H_1 \otimes H_k$):

$$0 \to \operatorname{Ind}_{m,k-1}^{n-1} M \boxtimes \operatorname{Res}_{k-1}^{k} K \xrightarrow{\alpha} \operatorname{Res}_{n-1}^{n} \left(\operatorname{Ind}_{m,k}^{n} M \boxtimes K\right) \xrightarrow{\beta}$$
$$\operatorname{Ind}_{m-1,k}^{n-1} (\operatorname{Res}_{m-1}^{m} M) \boxtimes K \to 0,$$
$$0 \to \operatorname{Ind}_{m,k-1}^{n-1} M \boxtimes e_a K \to e_a \left(\operatorname{Ind}_{m,k}^{n} M \boxtimes K\right) \to \operatorname{Ind}_{m-1,k}^{n-1} e_a M \boxtimes K \to 0.$$

Part (1) of the following result was first proved by Kato [Ka]. There are now many proofs of this result in the literature; here is yet another. (We would like to acknowledge useful conversations with Arun Ram.)

Denote by aK_n the module $\operatorname{Ind}_S^{H_n} a \boxtimes \cdots \boxtimes a$.

Proposition 3.3. (1)(*Kato's Theorem*) *$aK_n = \operatorname{Ind}_S^{H_n} a \boxtimes \cdots \boxtimes a$ is irreducible. It is the unique H_n-irreducible with central character $s = (a, \ldots, a)$.*

(2) *$\Delta_{\mathrm{a}}(aK_n)$ has an n-step socle filtration that coincides with its cosocle filtration and each factor is $aK_{n-1} \boxtimes a$.*

(3) *$e_a(aK_n)$ has an n-step socle filtration that coincides with its cosocle filtration and each factor is aK_{n-1}.*

Proof. Any submodule of aK_n must contain an S eigenvector. We will show that there is only one, up to a scalar multiple, such eigenvector and that it generates all of aK_n, yielding the irreducibility.

Write Rv for the S-module $a \boxtimes \cdots \boxtimes a$; then aK_n has basis $\{T_w \otimes v \mid w \in S_n\}$. Order the basis by Bruhat order.

We first compute the eigenspace of X_1. Any T_w may be written as $T_u T_1 T_2 \cdots T_j$ for some $u \in S_{\{2,\dots,n\}} = S_{n-1}$. Observe $X_1 T_u = T_u X_1$ by relation (4), and so $T_u \otimes v$ is an eigenvector for X_1 with eigenvalue a. We show that these are the only such eigenvectors. Observe

$$(X_1 - a)T_1 T_2 \cdots T_j \otimes v = (1-q)a(T_1 \cdots T_{j-1} + qT_j^{-1}T_1 \cdots T_{j-2} + \dots \\ + q^{k-2}T_3^{-1} \cdots T_j^{-1}T_1 + q^{k-1}T_2^{-1} \cdots T_j^{-1}) \otimes v. \quad (6)$$

Suppose $(X_1 - a)\sum_{u,j} \alpha_{u,j} T_u T_1 T_2 \cdots T_j \otimes v = 0$. Pick j maximal such that $\alpha_{u,j} \neq 0$ for some u, and for such j, consider a maximal u with $\alpha_{u,j} \neq 0$. Then (6) shows that if $j \geq 1$, the coefficient of $T_u T_1 \cdots T_{j-1} \otimes v$ is nonzero: a contradiction. Thus the $\{T_u \otimes v \mid u \in S_{\{2,\dots,n\}}\}$ span the $X_1 - a$ eigenspace. The computation (6) also shows that X_1 acts as $(n-1)!$ many Jordan blocks of size n with eigenvalue a. Each Jordan block is generated over $X_1 - a$ by $T_u T_1 \cdots T_{n-1}$ for $u \in S_{n-1}$.

Replacing X_1 with X_2 and T_k with T_{k+1} in the paragraph above, the same computation shows that the intersection of the $X_1 - a$ and $X_2 - a$ eigenspaces is the span of $\{T_u \otimes v \mid u \in S_{\{3,\dots,n\}}\}$. Continuing, the simultaneous eigenspace for $X_1 - a, \dots, X_{n-1} - a$ is the span of $\{1 \otimes v\}$ which is 1-dimensional, is automatically also an $X_n - a$ eigenvector, and most importantly, it generates all of aK_n. As it must lie in any submodule, aK_n is irreducible.

Given any irreducible M with central character $(a, \dots, a)$, Frobenius reciprocity gives a nonzero map $aK_n = \mathrm{Ind}_S^{H_n} a \boxtimes \cdots \boxtimes a \to M$, which is necessarily an isomorphism as each module is irreducible. This completes the proof of the first statement.

We have also shown that $1 \otimes v$ spans the a-eigenspace for $X_1, \dots, X_{n-1}$ (i.e., that we may omit X_n), and hence we conclude that $\widetilde{e}_a aK_n = aK_{n-1}$.

A similar computation to that for X_1 above gives that $X_n - a$ acts on aK_n with $(n-1)!$ many Jordan blocks of size n. Consider the associated filtration of $\mathrm{Res}^{H_n}_{H_{n-1}\otimes H_1} aK_n$ whose j^{th} term is $\ker(X_n - a)^j = \mathrm{Im}(X_n - a)^{n-j}$. As X_n must act semisimply on the subquotients of the socle filtration of $\mathrm{Res}^{H_n}_{H_{n-1}\otimes H_1} aK_n = \Delta_{\mathrm{a}}\, aK_n$, the socle filtration must be a refinement of this filtration. On the other hand, each subquotient of this filtration is the simple module $aK_{n-1} \boxtimes a$. Hence this filtration is the socle filtration. The same reasoning applies equally to the cosocle filtration, and hence the two coincide.

We have now shown statement (2). Statement (3) is proved by exactly the same argument, where we replace $X_n - a$ in the above discussion with the central element $X_1 + \ldots + X_{n-1} - (n-1)a$ of H_{n-1}. So restricting the filtration above to H_{n-1} gives the socle (and cosocle) filtration of $e_a aK_n$. □

3.3. Proof of Theorems (A) and (B)

Lemma 3.4. *Let $n_1, \dots, n_k \geq 0$ satisfy $\sum n_i = n$. Then* $\operatorname{soc} \mathrm{Res}^n_{n_1,\dots,n_k} aK_n = aK_{n_1} \boxtimes \cdots \cdots \boxtimes aK_{n_k}$ *is irreducible.*

Proof. If N is in the socle, then it must contain an S eigenvector, but there is only one, up to a scalar multiple, such eigenvector in aK_n, and it generates aK_{n_i} over H_{n_i}. □

From Proposition 3.3 and the lemma above, if M is irreducible, then $\mathrm{soc}((\Delta_{\mathrm{a}})^k M) = (\mathrm{soc} \circ \Delta_{\mathrm{a}})^k M$, which shows tthat

$$\widetilde{e}_a^k M = \mathrm{Res}_{n-k}^{n-k,1,\dots,1} \mathrm{soc}(\Delta_{\mathrm{a}})^k M. \tag{7}$$

We will use this observation in the proof of the main theorem.

Recall that $\varepsilon_a(M) = \max\{m \geq 0 \mid e_a^m M \neq 0\}$.

Lemma 3.5. *Let N be an irreducible H_n-module.*

(1) *Suppose $\varepsilon_a(N) = 0$. Then $M = \mathrm{cosoc}(\mathrm{Ind} N \boxtimes aK_k)$ is irreducible. Further, $\varepsilon_a(M) = k$ and all other composition factors of $\mathrm{Ind} N \boxtimes aK_k$ have $\varepsilon_a < k$. Similarly, $M = \mathrm{soc}(\widehat{\mathrm{Ind}} N \boxtimes aK_k)$ and all other composition factors of $\widehat{\mathrm{Ind}} N \boxtimes aK_k$ have $\varepsilon_a < k$.*

(2) *For any irreducible M, there exists an irreducible N with $\varepsilon_a(N) = 0$ such that $M = \mathrm{cosoc}(\mathrm{Ind} N \boxtimes aK_{\varepsilon_a(M)})$. Further, we may take $N \subseteq \widetilde{e}_a^{\varepsilon_a(M)} M$.*

(3) *For any irreducible L, $M = \mathrm{cosoc}(\mathrm{Ind} L \boxtimes aK_k)$ is irreducible with $\varepsilon_a(M) = \varepsilon_a(L) + k$, and all other composition factors of $\mathrm{Ind} L \boxtimes aK_k$ have strictly smaller ε_a.*

Proof. By Frobenius reciprocity, $N \boxtimes aK_k$ must occur in the restriction of any quotient of $\mathrm{Ind}_{H_n \otimes H_k}^{H_{n+k}} N \boxtimes aK_k$, in particular in its cosocle M. On the other hand, examining characters, by the shuffle Lemma 2.7 and the fact $\varepsilon_a(N) = 0$, we see that $N \boxtimes aK_k$ occurs in $\mathrm{Res}_{n,k}^{n+k} \mathrm{Ind}_{n,k}^{n+k} N \boxtimes aK_k$ with multiplicity one, yielding the cosocle is irreducible.

By the shuffle lemma, we see $\varepsilon_a(M) = k$ and any other composition factor has strictly smaller ε_a.

We conclude that we have a direct sum decomposition

$$\mathrm{Res}_{n,k}^{n+k} M = (N \boxtimes aK_k) \oplus A \tag{8}$$

for some module A with $e_a^k A = 0$, the sum being direct as the center of $H_{n-k} \otimes H_k$ acts on the terms with distinct eigenvalues. In particular, $N \boxtimes aK_k$ is a quotient of $\mathrm{Res}_{n,k}^{n+k} M$, yielding $M \hookrightarrow \widehat{\mathrm{Ind}}_{n,k}^{n+k} N \boxtimes aK_k$. Arguing as above, M is the socle, and all other composition factors have smaller ε_a.

For (2), given an irreducible H_m-module M, if $\varepsilon_a(M) = 0$, we are done. If not, write $\varepsilon = \varepsilon_a(M)$ and pick an irreducible $N \subseteq \widetilde{e}_a^{\varepsilon} M$. Then $\varepsilon_a(N) = 0$. By (5), $N \boxtimes a \boxtimes \cdots \boxtimes a \subseteq \mathrm{Res}_{m-\varepsilon,1,\dots,1}^m M$ and so by Frobenius reciprocity and the irreducibility of M, there is a surjection $\mathrm{Ind}_{m-\varepsilon,1,\dots,1}^m N \boxtimes a \boxtimes \cdots \boxtimes a \to M$. By (1) of the lemma, $M = \mathrm{cosoc}\,\mathrm{Ind}_{m-\varepsilon,1,\dots,1}^m N \boxtimes a \boxtimes \cdots \boxtimes a = \mathrm{cosoc}\,\mathrm{Ind} N_{m-\varepsilon,\varepsilon}^m \boxtimes aK_{\varepsilon}$.

For a given L, by the exactness of induction and by part (2), there exists an irreducible N with $\varepsilon_a(N) = 0$ such that $\mathrm{Ind} N \boxtimes aK_{\varepsilon_a(L)+k} = \mathrm{Ind} N \boxtimes aK_{\varepsilon_a(L)} \boxtimes aK_k \twoheadrightarrow \mathrm{Ind} L \boxtimes aK_k$. As the above map is a surjection, $\mathrm{cosoc}\,\mathrm{Ind} N \boxtimes aK_{\varepsilon_a(L)+k}$ surjects onto $\mathrm{cosoc}\,\mathrm{Ind} L \boxtimes aK_k = M$, which is thus irreducible by part (1). Again applying (1) to N yields the rest of (3). $\square$

We now prove the main theorems of the paper. We begin with Theorem (B) of the introduction, which is the following theorem and part (2) of Corollary 3.6.

Theorem. *Let M be an irreducible H_n-module. Then $\widetilde{e}_a M = \mathrm{soc}\, e_a M$ is irreducible or is zero.*

Proof. Recall $\widetilde{e}_a M = 0 \iff e_a M = 0 \iff \varepsilon_a(M) = 0$, so we may assume $\varepsilon = \varepsilon_a(M) \neq 0$. We begin by showing it is sufficient to prove $\widetilde{e}_a^{\varepsilon} M$ is irreducible. Let $L \subseteq \operatorname{soc} e_a M$. We will show that $\widetilde{e}_a^{\varepsilon-1} L \neq 0$ and in fact that $\varepsilon_a(L) = \varepsilon - 1$. As $e_a^{\varepsilon} L \subseteq e_a^{\varepsilon} e_a M = 0$, $\varepsilon_a(L) \leq \varepsilon - 1$. By (5) we have a map $L \boxtimes a \to \operatorname{Res}^{H_n}_{H_{n-1}\otimes H_1} M$ and thus $\operatorname{Ind}^{H_n}_{H_{n-1}\otimes H_1} L \boxtimes a \twoheadrightarrow M$. The shuffle lemma implies that $\varepsilon_a(\operatorname{Ind}^n_{n-1,1} L \boxtimes a) = \varepsilon_a(L) + 1$. By the exactness of e_a, $e_a^{\varepsilon}\operatorname{Ind}^n_{n-1,1} L \boxtimes a \twoheadrightarrow e_a^{\varepsilon} M \neq 0$, so $\varepsilon_a(L) + 1 = \varepsilon_a(\operatorname{Ind}^n_{n-1,1} L \boxtimes a) \geq \varepsilon_a(M) = \varepsilon$. Hence if $\widetilde{e}_a^{\varepsilon} M = \widetilde{e}_a^{\varepsilon-1}\widetilde{e}_a M$ is irreducible, then $\widetilde{e}_a M$ must also have been irreducible.

Now pick an irreducible submodule $N \subseteq \widetilde{e}_a^{\varepsilon} M$. We will show $N = \widetilde{e}_a^{\varepsilon} M$. By definition, $\varepsilon_a(N) = 0$. By Lemma 3.5, part (2), $M = \operatorname{cosoc}(\operatorname{Ind}^n_{n-\varepsilon,\varepsilon} N \boxtimes aK_\varepsilon)$, and so by the exactness of Δ_{a} we have a surjection ${\Delta_{\mathrm{a}}}^{\varepsilon}(\operatorname{Ind} N \boxtimes aK_\varepsilon) \twoheadrightarrow {\Delta_{\mathrm{a}}}^{\varepsilon} M$. As $\varepsilon_a(N) = 0$, Proposition 3.2 implies ${\Delta_{\mathrm{a}}}^{\varepsilon}(\operatorname{Ind} N \boxtimes aK_\varepsilon) = N \boxtimes \operatorname{Res}_S aK_\varepsilon$. So we have a surjection $N \boxtimes \operatorname{Res}_S aK_\varepsilon \twoheadrightarrow {\Delta_{\mathrm{a}}}^{\varepsilon} M$. On the other hand, applying Frobenius reciprocity to $\operatorname{Ind}^n_{n-\varepsilon,\varepsilon} N \boxtimes aK_\varepsilon \to M$ gives $N \boxtimes aK_\varepsilon \to \operatorname{Res}^n_{n-\varepsilon,\varepsilon} M$, which is necessarily an injection as $N \boxtimes aK_\varepsilon$ is irreducible. Restricting further, $N \boxtimes \operatorname{Res}_S aK_\varepsilon \to {\Delta_{\mathrm{a}}}^{\varepsilon} M$ is an injection. This map is the same as the surjection above, and hence ${\Delta_{\mathrm{a}}}^{\varepsilon} M = N \boxtimes \operatorname{Res}_S aK_\varepsilon$.

Finally, by Proposition 3.3, $\operatorname{soc} N \boxtimes \operatorname{Res}_S aK_\varepsilon = N \boxtimes a \boxtimes \cdots \boxtimes a$. Then by (7), $\widetilde{e}_a^{\varepsilon} M = \operatorname{Res}^{n-\varepsilon,1,1,\ldots,1}_{n-\varepsilon} \operatorname{soc}({\Delta_{\mathrm{a}}}^{\varepsilon} M) = \operatorname{Res}^{n-\varepsilon,1,1,\ldots,1}_{n-\varepsilon} N \boxtimes a \boxtimes \cdots \boxtimes a = N$. □

In light of what we have just proved, we could have defined $\widetilde{e}_a M$ as the socle of the *actual* eigenspace of $X_n - a$. Further, note that in the proof above, $\operatorname{Res}_{H_{n-\varepsilon}} \operatorname{cosoc}(\Delta_{\mathrm{a}})^{\varepsilon}(M) = N$ as well, so that we could repeat the argument above with Ind replaced by $\widehat{\operatorname{Ind}}$ and reversing arrows. Thus $\widetilde{e}_a^{\varepsilon} M = (\operatorname{cosoc} \circ e_a)^{\varepsilon} M$. We will use this observation below to conclude that $\widetilde{e}_a M = \operatorname{cosoc} e_a M$.

Corollary 3.6. *Let M be an irreducible H_n-module.*

(1) *Let $\varepsilon = \varepsilon_a(M)$ and $N = \widetilde{e}_a^{\varepsilon} M$. Then for $0 \leq k \leq \varepsilon$, $\widetilde{e}_a^{k} M = \operatorname{cosoc} \operatorname{Ind} N \boxtimes aK_{\varepsilon-k}$.*

(2) *$\widetilde{e}_a M \simeq \operatorname{cosoc} e_a M$.*

(3) *If L is also irreducible and $\widetilde{e}_a M \simeq \widetilde{e}_a L \neq 0$, then $M \simeq L$.*

Proof. By the previous theorem, $N = \widetilde{e}_a^{\varepsilon} M$ is irreducible. Further $\varepsilon_a(N) = 0$ and $\operatorname{cosoc}(\operatorname{Ind}^n_{n-\varepsilon,\varepsilon} N \boxtimes aK_\varepsilon) = M$. Replacing M by $\widetilde{e}_a^{k} M$, which is again irreducible, finishes the proof of (1) because $N = \widetilde{e}_a^{\varepsilon-k}(\widetilde{e}_a^{k} M)$.

By Lemma 3.5, $M = \operatorname{soc} \widehat{\operatorname{Ind}}^n_{n-\varepsilon,\varepsilon} N \boxtimes aK_\varepsilon$, so that dualizing the proof of Theorem (B), and (1) above, $\operatorname{cosoc} e_a M = \operatorname{soc} \widehat{\operatorname{Ind}}^{n-1}_{n-\varepsilon,\varepsilon-1} N \boxtimes aK_{\varepsilon-1}$. By Lemma 3.5 this is isomorphic to $\operatorname{cosoc} \operatorname{Ind}^{n-1}_{n-\varepsilon,\varepsilon-1} N \boxtimes aK_{\varepsilon-1} = \widetilde{e}_a M$. If $\widetilde{e}_a M \simeq \widetilde{e}_a L$, then $\widetilde{e}_a^{\varepsilon} M \simeq \widetilde{e}_a^{\varepsilon} L = N$, and so M and L are $\operatorname{cosoc} \operatorname{Ind} N \boxtimes aK_\varepsilon$ which is irreducible. □

Theorem (B) has the following simple consequence which includes Theorem (A) of the introduction. The special case of the finite Hecke algebra of type A, i.e., when $\lambda = (1)$, was first shown by Kleshchev [Kv] for $q = 1$, and Brundan [Br] for $q \neq 1$. (Brundan's proof is an extension of Kleshchev's, in which quantum $\mathfrak{gl}_n$ replaces Kleshchev's use of the enveloping algebra of $\mathfrak{gl}_n$.) Both the result and proof below, which are for the affine Hecke algebra of type A and hence arbitrary cyclotomic Hecke algebras, are new. We remind the reader here of the Appendix, which covers the case $q = 1$, i.e., $H_n = RS_n$, even in prime characteristic.

Theorem. (1) (Theorem (A)) *Let M be an irreducible H_n-module. Then* $\operatorname{soc}\operatorname{Res}^{H_n}_{H_{n-1}}M$ *is multiplicity free and isomorphic to* $\operatorname{cosoc}\operatorname{Res}^{H_n}_{H_{n-1}}M$.

(2) *Let M be an irreducible H^λ_n-module. Then* $\operatorname{soc}\operatorname{Res}^{H^\lambda_n}_{H^\lambda_{n-1}}M$ *is multiplicity free and is isomorphic to* $\operatorname{cosoc}\operatorname{Res}^{H^\lambda_n}_{H^\lambda_{n-1}}M$.

(3) *Let M be an irreducible H^λ_n-module. Then* $\operatorname{cosoc}\operatorname{Ind}^{H^\lambda_{n+1}}_{H^\lambda_n}M$ *is multiplicity free.*

Proof. The first statement follows from Theorem (B), Claim 3.1, and the observation that the central characters of e_aM and e_bM differ if $a \neq b$.

The second statement follows from the first, by the remark in Section 2.2, because ev^* commutes with restriction, and because the restriction of an H^λ_n-module is an H^λ_{n-1}-module. As $\operatorname{soc}\circ\operatorname{Res}$ is adjoint to $\operatorname{cosoc}\circ\operatorname{Ind}$ when acting on semisimple modules, the final statement follows. In fact, by adjointness, $\operatorname{soc}\widehat{\operatorname{Ind}}^{H^\lambda_{n+1}}_{H^\lambda_n}M$ is also multiplicity free. We omit the easy proof that $\operatorname{Ind}^{H^\lambda_{n+1}}_{H^\lambda_n}M \simeq \widehat{\operatorname{Ind}}^{H^\lambda_{n+1}}_{H^\lambda_n}M$ which yields $\operatorname{soc}\operatorname{Ind}^{H^\lambda_{n+1}}_{H^\lambda_n}M$ is multiplicity free and isomorphic to $\operatorname{cosoc}\operatorname{Ind}^{H^\lambda_{n+1}}_{H^\lambda_n}M$ as well. □

We point out that $\operatorname{Ind}^{H^\lambda_{n+1}}_{H^\lambda_n}$ is quite different from the induction $\operatorname{Ind}^{n+1}_{n,1}$ used throughout the rest of the paper. (See [G] for a detailed comparison.)

The techniques of this paper easily yield various other interpretations of $\varepsilon_a(M)$. It is the largest size of a Jordan block of X_n acting on M with generalized eigenvalue a. It is also both the multiplicity with which $\widetilde{e}_aM$ occurs in a composition series of e_aM, and the dimension of $\operatorname{End}(e_aM)$. For the proofs of these statements, see [G], and for related results, see [V].

A. Modifications when $q = 1$.

The following appendix is taken from [G].

If $q = 1$, the above theorems and constructions go through, without change, once we make the appropriate definitions.

Let R be a field of characteristic ℓ, $\ell \in \mathbb{N}\cup\{\infty\}$. Instead of the affine Hecke algebra, we work with the *degenerate* or *graded* affine Hecke algebra $\overline{H}^{\mathrm{gr}}_n$, defined by Drinfel'd [D] and Lusztig [L]. This is isomorphic as an R-module to $R[S_n]\otimes R[x_1,\dots,x_n]$ with algebra structure defined by requiring that $R[S_n]$ and $R[x_1,\dots,x_n]$ are subalgebras, and that $\tau_i\cdot f - {}^{\tau_i}f\cdot\tau_i = (f - {}^{\tau_i}f)/(x_{i+1}-x_i)$.

Given $\lambda \in R^r$ we define following [G] the *degenerate cyclotomic algebra* $\overline{H}^\lambda_n = \overline{H}^{\mathrm{gr}}_n/I_\lambda$, where $I_\lambda = \overline{H}^{\mathrm{gr}}_n\prod_{i=1}^r(x_1-\lambda_i)\overline{H}^{\mathrm{gr}}_n$. All definitions and theorems are as before, once we agree to write the group law in $\operatorname{Im}\mathbb{Z}$ multiplicatively, so qi denotes what is usually written $i+1$. No changes are necessary in the proofs.

In particular we have:

Corollary. *Let M be an irreducible RS_n-module. Then* $\operatorname{soc} Res^{S_n}_{S_{n-1}}M$ *is multiplicity free and is isomorphic to* $\operatorname{cosoc} Res^{S_n}_{S_{n-1}}M$.

This is immediate from our theorem, as when $\lambda = (1)$, the degenerate cyclotomic algebra $\overline{H}^{(1)}_n = RS_n$, the group algebra of S_n.

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